PRINCIPLES OF ELECTRICAL ENGINEERING

THE COMPANION VOLUME TO THIS BOOK

By the Same Author

"TEST PAPERS AND SOLUTIONS ON ELECTRICAL ENGINEERING"

The above book, although issued separately for the convenience of the reader, is an integral part and an extension of "Principles of Electrical Engineering". It contains the full solutions to the 203 Test Questions included at the end of the present volume, and constitutes a valuable guide to the actual working out of numerical and other examples, without which a real grasp of the principles of electrical engineering cannot be obtained.

The examples are all chosen and arranged with a view to emphasizing the different aspects of the principles dealt with in the individual chapters of the present volume and to make clear any points which might otherwise be obscure. Wherever possible, the questions and answers have been taken from actual practice.

Publication of "Test Papers and Solutions" as a separate volume allows the reader to have before him the answer to the problem he is studying and, at the same time, to be able to refer to various sections of the text which may have a bearing on it.

PRINCIPLES OF ELECTRICAL ENGINEERING

A Comprehensive Work covering the Principles of Heavy current and Light current Engineering Practice: also covering the requirements of the B.Sc. (Engineering), A.M.I.E.E., and Higher Examinations in this Subject.

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PREFACE

THE aim of this book is to present as comprehensively and in as limited a space as may be possible, an account of the basic principles of the science of electrical engineering, a leading idea throughout the book being to place emphasis on the identity of the principles relating to both heavy current and light-current engineering practice. The scope of modern electrical engineering is immense and it is only when a clear understanding of the fundamental principles has been obtained that sound progress can be hoped for in the study of any branch of the subject.

Chapter I includes a brief historical survey of the development of standards for units of measurement from the earliest days of the industry, and it is hoped that students and others will find this useful as evidence of the vast amount of work which has been done in electrical engineering to make precision measurements possible. It is of some importance to note that in 1933 the "General Conference on Weights and Measures" resolved that from January 1, 1940, the technical units were to be based on the absolute C.G.S. electrical units, and this resolution would normally have come into force throughout the civilized world on that date: the immediate practical consequences of this change over are indicated on page 15. No account of units would be complete without including the M.K.S. system, the technical significance of which has become of rapidly increasing importance in recent years.

Chapters II and IV include an account of recent developments in insulation technique, and Chapter V gives a treatment of networks which it is hoped will be found useful for the solution of a wide range of technical problems.

In Chapter VII a necessarily brief account is given of some of the more important recently developed magnetic materials for both high-frequency and low frequency purposes, as well as for the manufacture of permanent magnets.

In Chapters IX and X the use of complex quantities is explained as well as their applications to a wide range of problems, and for this purpose it has been thought advisable to make use of German script letters to denote such complex quantities. The extreme simplification to which many otherwise complicated problems can be reduced by this method of treatment, should make it more widely used than appears to be the case at present.

Bridge methods of measurement in both heavy and light current work have now become of great technical significance, and several representative types of such bridges have been selected for description.

Oscillatory systems have necessarily received treatment in consider-

able detail in Chapters IX and X in view of the multitude of the varieti of electrical engineering machines and apparatus with which such problem are now associated.

In Chapter XII a number of graphical methods for the solutions o technical problems are explained, and these methods will often be foun to provide a simple, accurate, and powerful means for solving problem for which mathematical treatment would be impracticable. The free an confident use of such methods, when appropriate, will frequently lea not only to a simple and exact solution, but will also provide a valuabl insight into the physical significance of the intermediate stages by whic the solution has been reached.

Chapter XIV contains a detailed treatment of "skin effect", and th attractive prospects of applying this phenomenon to a variety of industrial purposes as are now in prospect will give rapidly increasing importance to a study of the substance of this chapter.

The principles of the long distance high tension transmission of electric energy are considered in Chapter XV and should provide a sufficient foundation for those who wish to pursue in greater detail investigations in this branch of electrical engineering.

The basic facts and fundamental mathematical equations relating to the propagation of electromagnetic waves through space are given in Chapter XVI, and a knowledge of these is essential for any serious scientific discussion of radio transmission and reception.

It is realised that only by the actual working out of numerical and other examples can a real grasp of the subject be obtained, and for this purpose special attention has been given to the preparation of a set of examples associated with the contents of each chapter. The full solution for each example is given, for the reader's convenience, in a separate volume. In the case of a few of these examples the opportunity has been taken to include with the solution additional information to supplement the book work.

It is desired to make acknowledgment here of the courtesy of the Editor of *Engineering* in permitting the inclusion of material taken from a number of articles by the author and which have been published recently in that journal: references are given in the text of the book to the dates of issue of the articles concerned. Messrs. Methuen have also kindly agreed to the reproduction of some diagrams and other matter taken from the author's book, *Electrical Engineering*, which is now out of print.

T. F. W.

SHEFFIELD, 1946.

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PRINCIPLES OF ELECTRICAL ENGINEERING

Chapter I

FUNDAMENTAL UNITS : TECHNICAL UNITS

THE fundamental units of length, mass, and time, which are now universally adopted for scientific purposes, are those which were agreed upon by a Congress in Paris in 1881, viz.

NIT OF LENGTH : One Centimetre (very closely equal to 10^{9} times a quadrant of the earth measured from the Equator to the Pole).
NIT OF MASS : One Gram (very closely equal to the mass of one cubic centimetre of water at the temperature of its maximum density).
NIT OF TIME : One Second ($\frac{1}{86400}$ part of a mean solar day).

It was agreed that units founded on these quantities should be called ntimetre-Gram-Second absolute units or more briefly, C.G.S. units. nese units are sometimes much smaller, and, in other cases, much rger than the quantities which they are required to define, and in order avoid the necessity of having to use very large numerical multipliers divisors, the Congress agreed that multiplication by one million ould be expressed by the prefix " mega- ", and division by one million the prefix " micro- ".

echanical Units

RCE : The c.g.s. unit is the *dyne*, and is the force which is required to give an acceleration of 1 cm. per sec. per sec. to a mass of one *gram*.

K AND ENERGY : The c.g.s. unit is the *erg*, and is the work done when the point of application of a force of one dyne is moved through one centimetre.

OWER : The c.g.s. unit is one erg per second.

: The c.g.s. unit is the *gram-calorie* and is the heat necessary to raise the temperature of one gram of water through 1° C. (actually from 4° C. to 5° C., that is, at the temperature of the maximum density).

Some useful approximate relationships between the magnitudes of units are given in the following list :

LENGTH	1 inch = 2·54 cm.: 13 in. = 33 cm.: 1 metre = 39·37 in.: 1 km. = $\frac{5}{8}$ mile: 1 mil = $\frac{1}{1000}$ in. = 0·0254 mm.
FORCE	1 gm.-weight = 981 dynes: 1 mgm.-weight ≏ 1 dyne: 1 mega-dyne = 1,019 gm.: 1 kgm.-weight = 0·981 mega-dynes: 1 kgm.-weight ≏ 2·2 lbs.-weight. 1 lb.-weight = 0·445 mega-dynes.
WORK OR ENERGY	1 joule = 10^7 ergs = 0·102 kg.m. = 0·737 ft.-lb. = 0·24 gm.-calories: 1 m.kg. = 9·81 joules = 7·24 ft.-lb. = 2·35 gm.-calories: 1 gm. cm. = 981 ergs: 1 ft.-lb. = 1·36 joules: 1 gm.-calorie = 4·2 joules = 3·08 ft.-lb.: 1 kWh. = 1 B.o.T. unit = 3,412 B.Th.U. = 860 × 10^3 gm.-calories = 860 kg.-calories.
POWER	1 kW. = 737 ft.-lb. per sec. = 1,000 joules per sec. = 102 m.kg. per sec. = 1·34 horse-power: 1 ft.-lb. per sec. = 1·36 watts. One Continental horse-power (PS) = 736 watts = 75 m.kg. per sec.
VELOCITY	1 km. per hour = 27·8 cm per sec. = 0·623 miles per hour: 1 mile per hour = 1·6 km. per hour = 0·45 m. per sec. = 1·46 ft. per sec.

C.G.S. Electric and Magnetic Units

The c.g.s. electrical and magnetic units are derived from the following Laws, viz.

(i) Coulomb's Law for the force between two quantities of electricity, q and q', which are respectively concentrated at points r cm. apart and situated in a medium of which the dielectric constant is ε,

$$\text{Force} = \frac{q \cdot q'}{\varepsilon . r^2} \qquad (1)$$

If this force is 1 dyne when $q = q'$: $r = 1$ cm. and $\varepsilon = 1$, then the magnitude of each of the concentrated quantities of electricity will be 1 *electrostatic c.g.s. unit.*

(ii) Coulomb's Law for the force between two quantities of magnetism (or magnetic poles), m and m', which are respectively concentrated at points distant r cm. apart and situated in a medium of magnetic permeability μ,

$$\text{Force} = \frac{m . m'}{\mu . r^2} \qquad (2)$$

If this force is 1 dyne when $m = m' : r = 1$ cm. and $\mu = 1$, then the magnitude of each of the concentrated magnetic quantities will be *one electromagnet unit.*

(iii) Laplace's Formula defining the force on an elementary length of a current-carrying conductor when placed in a magnetic field,

$$\text{Force} = \frac{m \, . \, C \, . \, \delta s \, . \, \sin\theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and is independent of the medium in which the action takes place.

The interpretation of Laplace's Formula will be clear from a reference to Fig. 1, in which a magnetic quantity m is assumed to be concentrated at the point P, distant r cm. from the elementary length $\delta s = ab$ of a conductor which is carrying a current of C units. The angle θ defines the inclination of the elementary length δs of the conductor, and the line of action PO of the magnetic force at O and due to the magnetic

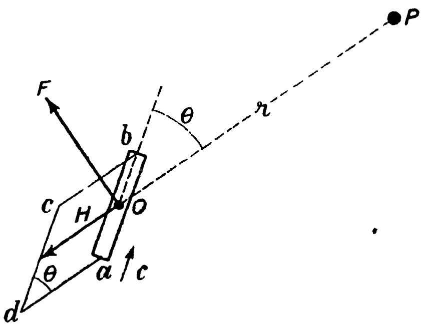


Fig. 1.

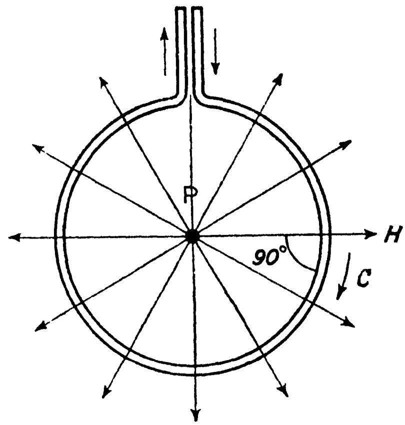


Fig. 2.

quantity m at P. From the expression (2) the magnetic force at O will be $H = \frac{m}{r^2}$, and the expression (3) states that the force on the element δs will be given by the product of the current and the area of the parallelogram $abcd$ where $ab = \delta s$: $ad = H$ and θ is the angle between ab and ad.

The direction of the mechanical force F will be perpendicular to the plane of the parallelogram $abcd$. If the conductor is bent into the form of a circle, as shown in Fig. 2, of radius 1 cm., and if a unit magnetic pole is placed at the centre P of the circle, then the magnetic intensity will be $H = 1$ at every point of the conductor and its direction will be everywhere at right angles to the conductor, that is to say, the value of θ in the expression (3) will be 90°. If the current C which flows in the conductor is of such a magnitude that the total force on the whole length of the conductor is 2π dynes, then the force on each centimetre

length of the conductor will be 1 dyne, and the magnitude of the current is then defined as *one electromagnetic unit*, that is:

The c.g.s. electromagnetic unit of current is that current which, when flowing in a circular conductor of 1 cm. radius, will act on a unit magnetic pole at the centre of the circle with a force of 2π dynes.

The direction in which the force acts will then be perpendicular to the plane of the circle (see also Chapter VIII, Fig. 234, Fleming's "Left-Hand Rule").

From the three expressions for electric force which have already been considered in the foregoing, it will be seen that combining (1) and (2) gives,

$$\mu.\epsilon = \frac{q.q'.m.m'}{r^4(\text{Force})^2} \quad . \quad . \quad . \quad . \quad (4)$$

and substituting (3) for (4), gives

$$\mu.\epsilon = \frac{q.q'.m.m'}{m^2.i^2\sin^2\theta(\delta s)^2} = \left(\frac{\delta t}{\delta s}\right)^2 = \frac{1}{(\text{Velocity})^2},$$

since: Quantity = Current × Time, i.e. $q = C \times \text{Time}$,

from which it is clear that the dimensions of the product $\mu.\epsilon$ are those of the reciprocal of a velocity squared, that is,

$$\frac{1}{\mu.\epsilon} = (\text{Velocity})^2.$$

Direct experiment shows that this velocity in open space is

$$c = 2{\cdot}9979 \times 10^{10} \text{ cm per second},$$

and this is the velocity of electromagnetic waves in open space, and consequently also the velocity of light in open space (see Chapter XVI).

The Dimensions of Units

It is important to consider the dimensions of the various electric and magnetic units, and these may be conveniently expressed in the following way:

(*i*) The *velocity* of a body is measured by its time rate of displacement, that is to say, it is given by the ratio $\frac{\text{Length}}{\text{Time}}$, so that the dimensions of a velocity are said to be $(\text{Length})^1 : (\text{Time})^{-1}$, or, more briefly stated,

$$LT^{-1}.$$

(*ii*) The *acceleration* of a body is the time rate of change of its velocity and is measured by the ratio $\frac{\text{Velocity change}}{\text{Time}}$. The dimensions of an acceleration are therefore said to be,

$$LT^{-2}.$$

(*iii*) *Force* is measured by the product

$$\text{Mass} \times \text{Acceleration},$$

or, alternatively stated,

Force = Rate of change of (mv), i.e. rate of change of momentum, so that its dimensions are,

$$MLT^{-2}.$$

(*iv*) This procedure may be applied to Coulomb's "Inverse Square Law" for two electric quantities, viz.

$$\text{Force}: F = \frac{q \cdot q'}{\varepsilon \cdot r^2}.$$

If the electric quantities are equal to one another, then

$$F = \frac{q^2}{\varepsilon \cdot r^2} \text{ or } q = r \cdot \sqrt{\varepsilon \cdot F}.$$

The dimensions of an electric quantity are thus:

$$\varepsilon^{1/2} M^{1/2} L^{3/2} T^{-1}.$$

(*v*) Similarly from Coulomb's Law for two magnetic quantities, viz.

$$F = \frac{m \cdot m'}{\mu \cdot r^2},$$

it follows that the dimensions of a magnetic quantity (e.g. a magnetic pole) are

$$\mu^{1/2} M^{1/2} L^{3/2} T^{-1}.$$

Proceeding from the foregoing results, it is possible to write down the dimensions of all the electric and magnetic quantities, and some of the more important of these are given in Table I.

TABLE I

Item	*Dimensions in Terms of* L, M, T, ϵ	L, M, T, μ
Dielectric Constant	ϵ	$L^{-2}T^{2}\mu^{-1}$
Magnetic Permeability	$L^{-2}T^{2}\epsilon^{-1}$	μ
Electric Quantity	$L^{3/2}M^{1/2}T^{-1}\epsilon^{1/2}$	$L^{1/2}M^{1/2}\mu^{-1/2}$
Magnetic Quantity (Pole)	$L^{1/2}M^{1/2}\epsilon^{-1/2}$	$L^{3/2}M^{1/2}T^{-1}\mu^{1/2}$
Electric Potential or E.M.F.	$L^{1/2}M^{1/2}T^{-1}\epsilon^{-1/2}$	$L^{3/2}M^{1/2}T^{-2}\mu^{1/2}$
Magnetic Potential	$L^{3/2}M^{1/2}T^{-2}\epsilon^{1/2}$	$L^{1/2}M^{1/2}T^{-1}\mu^{-1/2}$
Electric Force (Strength of Electric Field)	$L^{-1/2}M^{1/2}T^{-1}\epsilon^{-1/2}$	$L^{1/2}M^{1/2}T^{-2}\mu^{1/2}$
Magnetic Force (Strength of Magnetic Field)	$L^{1/2}M^{1/2}T^{-2}\epsilon^{1/2}$	$L^{-1/2}M^{1/2}T^{-1}\mu^{-1/2}$
Electric Displacement (Surface Density)	$L^{-1/2}M^{1/2}T^{-1}\epsilon^{1/2}$	$L^{-3/2}M^{1/2}\mu^{-1/2}$
Magnetic Induction	$L^{-3/2}M^{1/2}\epsilon^{-1/2}$	$L^{-1/2}M^{1/2}T^{-1}\mu^{1/2}$
Electric Current	$L^{3/2}M^{1/2}T^{-2}\epsilon^{1/2}$	$L^{1/2}M^{1/2}T^{-1}\mu^{1/2}$
Conductivity	$LT^{-1}\epsilon$	$L^{-1}T\mu^{-1}$
Resistance	$L^{-1}T\epsilon^{-1}$	$LT^{-1}\mu$
Capacitance	$L\epsilon$	$L^{-1}T^{2}\mu^{-1}$
Inductance	$L^{-1}T^{2}\epsilon^{-1}$	$L\mu$
Magnetic Moment	$L^{3/2}M^{1/2}\epsilon^{-1/2}$	$LM^{1/2}T^{-1}\mu^{1/2}$

It is to be observed that the pairs of items which are bracketed together in Table I are reciprocally analogous, the dimensions of one in terms of ε being the same as the dimensions of the other in terms of μ. It is further to be noticed that the dimensions of any one item must be the same, whether expressed in terms of ε or μ, so that, for example, in the first item in the Table (the dielectric constant),

$$\varepsilon = L^{-2}T^{2}\mu^{-1}: \text{ that is, } \varepsilon.\mu = \frac{T^2}{L^2},$$

and since the dimensions of a velocity are $\frac{L}{T}$, the dimensions of $\sqrt{\varepsilon.\mu} \times$ (velocity) = a pure number. This gives again the very important result already pointed out in the foregoing, viz. that the dimensions of the product $\varepsilon.\mu$ are those of the reciprocal of a (velocity)^2. That is to say, although the dimensions of ε and μ are not separately known, the dimensions of their product is known. As already stated in the foregoing, the value of the velocity c in the expression,

$$c = \frac{1}{\sqrt{\varepsilon.\mu}} = 2{\cdot}9979 \times 10^{10} \text{ cm. per second,}$$

which is the velocity of light in open space.

That is to say, in any system of units, if μ is the magnetic permeability and ε is the dielectric constant, then

$$\varepsilon.\mu = \frac{1}{(\text{velocity of light in that system of units})^2}$$

see also, expression (14), page 22.

Maxwell made a remarkable observation relative to this result. It had already been shown by Rowland's experiments that an infinitely long straight wire having a charge of q electrostatic units per unit length and moving in its own direction with a velocity v cm. per second would be equivalent to a current $\frac{q.v}{c}$ in electromagnetic units. If two such wires were to be placed parallel to each other at a distance apart d cm. and moving in the same direction with the same velocity, there would be a force of *attraction* between them as defined by the expression given in Chapter VIII, page 217, viz.

$$\text{Force of attraction} = \frac{2}{d}\left(\frac{q.v}{c}\right)^2 \text{ dynes per centimetre length.}$$

Simultaneously, there will be a force of *repulsion* between the two wires due to the electric charges, of an amount as defined by Coulomb's Law (page 83), viz.

$$\text{Force of repulsion} = \frac{2q^2}{d} \text{ dynes per centimetre length.}$$

The force of attraction will be equal to the force of repulsion if

$$c = v.$$

that is to say, the velocity c is that with which each wire must move in the direction of its length in order that the two wires shall have no action on each other.

The Electrostatic C.G.S. Absolute System and the Electromagnetic C.G.S. Absolute System of Units

If, for open space, it be assumed that the dielectric constant is $\epsilon = 1$ and the units of length, mass, and time are, respectively, the centimeter, gram, and second, then the electric and magnetic quantities when expressed in these units are said to be defined in *electrostatic c.g.s. absolute units.* Thus, the electrostatic c.g.s. absolute unity of quantity is that quantity which, when concentrated at a point distant 1 cm. in air, from an equal concentrated quantity, is repelled with a force of 1 dyne.

Similarly, if for open space the value of the magnetic permeability is $\mu = 1$ and the units of length, mass, and time are the c.g.s. units, then the electrical and magnetic quantities expressed in terms of these units are said to be defined in *electromagnetic c.g.s absolute units.*

For most technical purposes it is convenient to use multiples of the absolute units, that is, powers of ten, and in Table II are given the technical units and their relationships to the c.g.s. absolute units for some of the more important electric and magnetic technical quantities. It is of interest to note that the names *Ohm*, *Volt*, and *Farad*, were proposed by the British Association and officially accepted by the Paris Congress in 1881, this Congress also having settled the names *Ampere* and *Coulomb* for the technical units of current and electric quantity, respectively. The name *Henry* for the technical unit of inductance was determined by the Chicago Conference in 1893. Previously, the unit of inductance had been termed the *Quadrant*, since, in the electromagnetic c.g s. absolute system, it has the value 10^9 cm., and this is also very approximately, the length of a quadrant of the earth measured from the Equator to the Pole. Another term which had been used for the unit of inductance was the *Secohm*, since the dimensions of the unit of inductance are

$$\text{Henry} = \frac{\text{Volt} \times \text{Second}}{\text{Ampere}} = \text{Second} \times \text{Ohm}.$$

It is of interest to note from Table I that electric resistance in the electromagnetic system of units has the dimensions of a velocity, whilst the dimensions of (electric current)^2 are those of a force. It will also be seen from Table II that the ohm is equal to 10^9 electromagnetic c.g.s. units, that is to say, the unit of velocity will be 10^9 of the c.g.s. unit. In other words, if the resistance of 1 ohm were to be taken as the absolute unit of resistance in place of the electromagnetic c.g.s. unit, the corresponding unit of velocity would be 10^9 times that of the c.g.s. unit.

Suppose, then, that the resistance of 1 ohm is taken as the new

TABLE II

Item	*Technical Unit*	*Number of Electromagnetic C.G.S. Absolute Units in the Technical Unit*	*Number of Electrostatic C.G.S. Absolute Units in the Technical Unit*
Electric Quantity	Coulomb	10^{-1}	3×10^{9}
Electric Current	Ampere	10^{-1}	3×10^{9}
Electric Capacitance	Farad	10^{-9}	9×10^{11}
	Micro-farad	10^{-15}	9×10^{5}
	Picofarad	10^{-21}	9×10^{-1}
Electric Potential Difference	Volt	10^{8}	$\frac{1}{3 \times 10^{2}}$
Electric Resistance	Ohm	10^{9}	$\frac{1}{9 \times 10^{11}}$
Electric Conductivity	Siemens	10^{-9}	9×10^{11}
Inductance	Henry	10^{9}	$\frac{1}{9 \times 10^{11}}$
Work or Energy	Joule	10^{7}	
Power	Watt	10^{7}	
	Kilowatt	10^{10}	
	Megawatt	10^{13}	

absolute unit of resistance, what must be the corresponding new units of length, mass, and time? Assuming that the second is still retained as the unit of time, the new unit of length must be 10^9 cm. in order that the new unit of velocity shall be 10^9 time the c.g.s. unit. Further, reference to Table II shows that the new absolute unit of current will be $\frac{1}{10^{10}}$ times the electromagnetic c.g.s. unit, and since (as stated above) the dimensions of force are those of (electric current)^2 the new absolute unit of force will be $\frac{1}{10^{20}}$ times the c.g.s. unit. But

$$\text{Force} = \text{Mass} \times \text{Acceleration},$$

and since the new unit of acceleration will be 10^9 times the c.g.s. unit, the new unit of mass will be,

$$\frac{1}{10^{20}} \times 10^9 = 10^{-11} \text{ gm.}$$

It is seen, therefore, that the new absolute system of units so obtained will be:

Unit of length $= 10^9$ cm. : Unit of mass $= 10^{-11}$ gm. :
Unit of time $= 1$ sec.

Absolute and Technical Systems of Measurement

In the foregoing considerations the abstract definitions of the absolute c.g.s. electric and magnetic units have been explained. The measurement of quantities as defined by such an absolute system may be achieved in two different ways, viz.

(*i*) The value of the quantity under consideration can be obtained solely by means of the units of mass, length, and time, and one electric or magnetic quantity, such, for example, as the calculated inductance of a coil of known dimensions.

(*ii*) A standard of reference may first of all be prepared, which shall be as nearly as is practicable of the same magnitude as defined by the corresponding absolute unit, and the quantity concerned can then be compared with this standard of reference. Such a system of measurement may be termed a *relative*, or *technical*, or *practical* system.

Until the last few years, the procedure (*i*) has presented such formidable difficulties as regards the measurement of a quantity in absolute units that it was not practicable to apply it, and, in consequence, the method (*ii*) has always been employed in practice. As will be pointed out in what follows, however, the precision with which the required measurement can now be made in absolute units is such that the Conférence générale des Poids et Mesures which, before the War, met once every six years at Sèvres, resolved that on January 1, 1940, measurement in absolute units was to be the legal method in all those countries which had sent representatives to the "Metre Conference", and this comprised practically the whole civilised world.

For the *technical* system of measurement as hitherto carried out, it is necessary to have some concrete standards of reference which embody the respective absolute units as nearly as it is practicable to do so. Even before any absolute system of units was evolved, the rapidly developing electrical engineering industry felt the need for some authoritative standard of resistance, and those firms which were concerned with the manufacture of machines and apparatus had constructed arbitrary standards of reference, although none of these satisfied the required conditions of reliable stability. For example, about 1841 Jacobi proposed the first concrete standard for the unit of resistance, viz. a coil of wire of known length and cross-sectional area. This, however, was not satisfactory as it could not be duplicated with any degree of precision.

In 1860 Sir W. Siemens pointed out that, in order to overcome the difficulties due to non-homogeneity of solid metal, *mercury*, which is the only metal which is liquid at normal temperatures, should be used, and, owing to its homogeneous characteristic, stands alone as being of the same and unalterable specific resistance, so that in duplicating such a standard, it was only necessary to define precisely its geometrical dimensions (length and cross-section). This proposal was recognised as such an outstanding advance in comparison with all previous proposals

that it soon displaced them all. The proposal therefore made it possible to place the practical unit of resistance on a firm and reliable foundation. Siemens' proposal for the dimensions of such a mercury column was that the length should be 1 metre and the cross-section 1 square millimetre, and for a long period of years this was known as the "Siemens Unit".

In 1863 (Newcastle) the Appendix "C" of the Second Report of the Electrical Standards Committee by Clerk Maxwell and Fleeming Jenkin was issued. This Appendix dealt with the elementary relationship between the electrical quantities and is the historic achievement of Maxwell and the founders of the c.g.s. system. During the period 1862-75 a Board of Trade Committee made efforts to produce a more reliable standard of resistance from various alloys, of which the platinum-silver alloy was the most successful, and this was used for a long time and was known as the B.A. Unit. It was found, however, that this did not maintain its constancy over a long period of time and could not be duplicated with sufficient accuracy.

During 1888-90 E. Weston and Dr. Heusler collaborated in a systematic search for a suitable alloy, and eventually produced "manganin", which satisfied the most rigid requirements, and in consequence of which it is, even to-day, in an unassailable position throughout the world as a material for precision and standard resistances.

The position with regard to the evaluation of the resistance unit was not so satisfactory. Up to the year 1881 of the various methods for measuring the value of the material standard in absolute units, only four results were available, viz.

Year	*Observer*	*Magnitude of the Ohm expressed as the Length of a Column of Mercury of 1 sq. mm. Cross-section*
1873	Lorenz	1·0710 metres
1877	H. F. Weber	1·0590 ,,
1878	Rowland	1·0616 ,,
1881	Rayleigh and Schuster	1·0598 ,,

As regards the second electrical unit (i.e. current), up to the year 1881 no absloute measurements had been reported which fulfilled the required conditions as they were then understood. Whereas up to that date efforts had been concentrated on the definition of (absolute) independent units, a new departure was now made. The aim now was not only the creation of units based on reliable foundations, but also to provide for their introduction into practice and, with this in view, the first Paris Congress of 1881 established the Ohm, the Ampere, and the Volt, in place of the absolute units. The experience which had been

accumulated with regard to the choice of suitable resistance material was now crystallised by the resolution of the Congress that a mercury column of stated dimensions should be chosen as the unit of resistance, but a statement as to what those dimensions should be was postponed and a commission was charged with the task of preparing recommendations regarding these dimensions. (It is of interest to note here that the Congress of 1881 introduced the prefixes "mega-" and "micro-".)

The Paris Congress of 1884 fixed the value of the technical ohm, in accordance with the existing state of knowledge, as 1·06 Siemens units. A definition of the ampere was also given, and in order that the technical (empirical) unit could be easily and satisfactorily duplicated, it was defined by the amount of silver deposited in a silver voltameter by a quantity of electricity of 1 coulomb in 1 second. A statement as to what that amount of silver should be was deferred, since at that time sufficiently rigorous measurements of the electrochemical equivalent of silver had not been made. In addition to the two units of ohm and ampere, the joule and the watt were defined by the Congress, as well as the unit of inductance for which the term "Quadrant" had been introduced by Maxwell. This was replaced by the term "henry" at the Chicago Conference of 1893. Whilst at the time of the Paris Congress there was not sufficient experimental data available of the required precision, in the few years following, the number of exact determinations of the ohm and the ampere had accumulated to such an extent that adequate material was available for stating the absolute values.

The assessment in terms of the mercury column of the absolute ohm on the basis of the many investigations which had been made for this purpose, was a matter of some difficulty, and it seemed to be desirable to institute an expert critical survey of the results of the individual measurements in order to arrive at the most probable value, and the governing body of the P.T.R. (Physicalische technische Reichsanstalt) entrusted this work to Dr. Dorn, who was particularly well qualified to undertake it. The result of his examination of the fifteen investigations which came under review gave the probable value of the absolute ohm as equal to the resistance of a mercury column between 1·06274 and 1·06292 metres long, and one sq. mm. cross-section at 0° C. In view of the degree of uncertainty which these margins of values defined, Dorn recommended that the value should be fixed with an accuracy of 1 part in 1,000 and for this reason he proposed the value of 1·063 metres as the German Legal Ohm, it being observed that this value had already been proposed as the British Legal Ohm.

As regards the absolute value of the ampere, the matter was much simpler. For the purpose of fixing the legal value the P.T.R. selected the determinations of Lord Rayleigh and Mrs. Sidgwick for one basis of measurement, and that of F. and W. Kohlrausch for the other. In each case the determination was made by measuring the amount of

silver deposited in a silver voltameter by one ampere, the former value being 1·11794 milligrams and the latter 1·11826 mgm. As in the case of the ohm, it was not considered desirable to assume an accuracy greater than 1 part in 1,000, and the value was fixed as 1·118 mgm., it being observed that this value had also been decided upon by the British Board of Trade Committee. It is noteworthy that the mean value of all the reliable measurements which were available at that time was 1·118 mgm., and of these measurements, five were made by means of the "current balance" and one by each of the following instruments, viz. Tangent Galvanometer, Sine Galvanometer and the Dynamometer, and although the value of 1·118 mgm. was specified as having an accuracy of 1 part in 1,000, it was realised that, actually, the value was correct to within 1 part in 10,000.

Having settled in this way the values of the absolute ohm and the absolute ampere with an accuracy of 1 part in 1,000, and decided that this degree of accuracy could be attained by technical measurements, it appeared that the time had arrived at which international agreement could be established with regard to the legal definitions of the technical units.

The International Conference on Electrical Units in Charlottenburg, 1905

At the suggestion of the Bureau of Standards, Washington, the P.T.R. issued invitations to those States which were concerned with the supervision and solution of the problems associated with the establishment of the electric units. Invitations were also issued to those specialists who were, by reason of their experience in this kind of work, particularly qualified to advise on this problem. The State Laboratories which were represented were P.T.R. (Germany): N.P.L. (Great Britain): Vienna: and Brussels, and the technical specialists were Professors F. Kohlrausch, E. Mascart and B. Carhart. The Chicago Conference of 1893 had fixed the respective values of the three quantites which are related by Ohm's Law, viz.:

(*i*) The Ohm. The unit of resistance shall be what is known as the "international ohm", which is substantially equal to one thousand million units of resistance of the c.g.s. system of electromagnetic units, and is represented by the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 gm. in mass, and of a constant cross-sectional area, and of the length of 106·3 cm.

(*ii*) The Ampere. - The unit of current shall be what is known as the "international ampere", which is one-tenth of the unit of current of the c.g.s. system of electromagnetic units, and is the practical equivalent of the unvarying current which, when passed through a solution of silver nitrate in water in accordance with standard specifications, deposits silver at the rate of 0·001118 gm. per second.

(*iii*) The Volt.—The unit of electromotive force shall be what is known as the "international volt", which is the e.m.f. that, steadily applied to a conductor whose resistance is one international ohm, will produce a current of one international ampere, and is practically equivalent to 1/1·434 of the e.m.f. between the poles or electrodes of the voltaic cell known as Clark's cell, at a temperature of 15° C., and prepared in the manner described in the standard specifications.

At the meeting of the Charlottenberg Conference the most urgent matter under discussion was the lack of uniformity of electrical units in the different countries, which was largely due to the fact that the Chicago Conference of 1893 had fixed the respective values of the *three* quantities which are related by Ohm's Law, whereas not more than *two* of these quantities should have been fixed by definition, the third being then determined by Ohm's Law. The matter was dealt with by resolving that the value of the ohm and the ampere should be specified by resolution. As a standard of reference for e.m.f. the Cadmium cell was chosen instead of the Clark's cell as previously used.

The question was then considered as to what values were to be assigned to the technical ohm and the technical ampere, that is to say, what were to be the dimensions of the mercury column which would represent the resistance of 1 ohm and what the amount of the silver voltameter deposit which would be produced by a current of 1 ampere in each case, with a margin of error not greater than 1 part in 1,000.

International Conference in London, 1908

The definitions issued by the Chicago Conference of 1893 did not make the International Units distinct from the absolute values expressed by the c.g.s. electromagnetic system. The London Conference made this distinction and adopted as the two basic units: (*i*) the ohm defined by reference to the mercury column, and (*ii*) the ampere defined by reference to the silver voltameter.

Washington Conference, 1910

At this date only the N.P.L. and the P.T.R. possessed a mercury column resistance, and in order to compare these with wire resistances, the values of which had been measured by comparison, with the respective mercury columns were then compared with the following result:

$$(\text{Ohm})_{\text{N P L}} - (\text{Ohm})_{\text{P T R}} = 1 \times 10^{-5}.$$

The mean value of both units in combination with the results of the silver voltameter tests was made the basis of measurement for the e.m.f. of the Weston cell, and this value was then specified as

1·01830 volts at 20° C.

The International Conference of the Advisory Committee for Electric Units and the International Committee of Weights and Measures, Paris, 1928

The first matter which this Conference had to decide was whether the empirical international units defined by the International Conference of 1908 in London should be maintained, or whether they should be replaced by the so-called absolute units. In favour of such a change-over was the fact that the degree of precision with which the absolute units could then be measured had reached such a stage that it was about of the same order as that with which the empirical units could be measured, that is to say, about the same degree of precision with which it was possible to measure the resistance of a column of mercury of given dimensions and the weight of silver deposit in a silver voltameter. The single objection to the change-over which gave rise to hesitation was the fact that a difference of 5 parts in 10,000 was known to exist between the empirical ohm and the absolute ohm, and this was sufficiently large to involve a widespread readjustment of calibration and other consequences arising from the relatively long period of time for which the international ohm had been the legal standard throughout the world. The Conference fully appreciated this argument, but held the view that the great advantages which would accompany the change-over were sufficient to outweigh the objections, and accordingly they resolved to recommend that the change-over should take place.

It was decided, however, that the change-over should not take effect before the relationship which existed between the international and the absolute units had been determined with the necessary and attainable precision. For this purpose it was agreed that those State Laboratories which were adequately equipped should carry out the necessary experimental work in accordance with a programme devised by the Advisory Committee. As a result of these decisions, the Bureau of Standards at Washington has obtained the following results, viz.

(*i*) Current Balance Method (*Journal of Research*, 1934):

> One Bureau of Standards International Ampere = 0·999928 Absolute Ampere.
>
> The authors are of the opinion that this result differs from the true value by less than 20 parts in one million.

(*ii*) Calculated Inductance of a Single-Layer Solenoid (*Journal of Research*, 1938):

> One Bureau of Standards International Ohm = 1·000468 absolute ohms.
>
> The authors are of the opinion that this result differs from the true value by less than 20 parts in one million.

In the year 1933 the "General Conference for Weights and Measures" resolved that the technical units—ampere, ohm, volt, watt, farad, henry from a specified date (i.e. January 1, 1940) shall not be based on measurements with the silver voltameter and the mercury column, but shall be derived from the corresponding absolute c.g.s. electromagnetic units by multiplication with appropriate powers of 10.

According to modern views of electrical theory there are *four* independent dimensions, whereas in the domain of mechanics there are only *three*, viz. length, mass, and time. If, then, as is the case with the international units, the system involves *current*, *resistance*, *length*, and *time*, the unit of mass becomes a dependent unit, that is to say, it must be derived from the four accepted independent units. In accordance with this requirement, the international ampere, ohm, meter, and second, lead to an international unit of mass which is 1·0003 times the kilogram —in other words, the kilogram is only an approximate unit. Since, however, it is not practicable to change the definition of the kilogram as established by statute and involving all its decimal varieties, the objection to the international system of units is that it has one unit too many.

The system of absolute technical units as established by the Conference of 1933 is free from this objection, since it is based on the gram, centimetre, second, and the inductance constant μ_0, that is, four independent units inclusive of the kilogram (or gram). The following table gives the respective values of the international units in terms of the absolute unit, viz.:

TABLE III

One	International	Ampere	0 9999	Absolute	Ampere
One	,,	Coulomb	0 9999	,,	Coulomb
One	,,	Ohm	1 0005	,,	Ohms
One	,,	Volt	1 0004	,,	Volts
One	,,	Henry	1 0005	,,	Henry
One	,,	Farad	0 9995	,,	Farad
One	,,	Weber	1 0004	,,	Webers
One	,,	Watt	1 0003	,,	Watts

The absolute value of the Inductance Constant is (page 19)

$$\mu_0 = 4\pi \times 10^{-9} = 1{\cdot}256637 \times 10^{-8} \frac{\text{henry}}{\text{centimetre}} \text{ units},$$

and the value of the international inductance constant is

$$\mu_0 = 1{\cdot}25606 \times 10^{-8} \text{ henry}.$$

Hence:

$$\text{Absolute } \mu_0 = 4\pi 10^{-9} \frac{\text{Absolute volt} \times \text{Second}}{\text{Absolute ampere} \times \text{Second}} \text{ units} \quad . \quad (5)$$

The introduction of the absolute units will involve a certain practical simplification. For example, the measurement of the international ampere by means of the silver voltameter to the required degree of

precision requires from 60 to 80 voltameter tests each involving the observance of rigorous conditions of procedure, whilst the measurement of the mercury column which defines the technical ohm involves, if anything, even more exacting experimental skill. For the new system the current will be measured by means of a current-balance and the inductance of a coil of known geometrical dimensions, and winding data, will be measured by the Maxwell Bridge, involving the measurement of a capacitance and the time of vibration of a tuning-fork.

The Experimental Determination of the Absolute Value of the Ohm

The most recent absolute determination of the ohm is that carried out by the Bureau of Standards, Washington, the details of the measurement being given in the *Journal of Research*, 1938, Vol. 21, pages 375–423, and the following is a brief outline of the procedure.

(*i*) A single-layer solenoid was constructed, and from the dimensions and winding data the value of the inductance L could be calculated with an error of not more than one part in one million.

(*ii*) The magnitude of this inductance was then measured in terms of a calibrated capacitance C.

(*iii*) The capacitance C was calibrated by the Maxwell Bridge in terms of a resistance R.

The difference between the measured value of L and the calculated value was a measure of the error in the determination of C. The corresponding corrected value of C was therefore a measure of the resistance R in terms of the accuracy of the Maxwell Bridge, and this bridge can be constructed to have an error of not more than 1 part in 10^5.

The principle of the Maxwell Capacitance Bridge will be understood from the following considerations and by reference to Fig. 3. The capacitance of a condenser is defined by the relationship,

$$C = \frac{Q}{V}$$

where C is in farads when Q is in coulombs and V in volts.

If the condenser is successively charged with Q coulomb, then discharged, and re charged with Q coulomb, the sequence being repeated at the rate of n times per second, the mean value of the current in the condenser will be,

$$I - nQ = nCV \quad . \quad . \quad . \quad . \quad . \quad (6)$$

That is to say, for a given value of the applied pressure V, the mean value of the current so produced will be proportional to the capacitance of the condenser. If, now, the expression (6) is compared with the Ohm's Law relationship, viz.

$$I = \frac{V}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

it will be seen that the condenser system is equivalent to the resistance R, where

$$R = \frac{1}{nC} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A condenser which is being charged and discharged periodically and arranged in one arm of a Wheatstone Bridge as shown in Fig. 3, acts therefore like a resistance, and, by the use of a change-over switch, the magnitude of the resistance R is easily found, since it is that resistance which when substituted for the condenser system in the arm of the bridge gives the same balance. This principle for the measurement of a capacitance was first proposed by Maxwell and is of an accuracy which appears to be limited only by that of the vibrating contact, e.g. an accuracy of one part in 100,000.

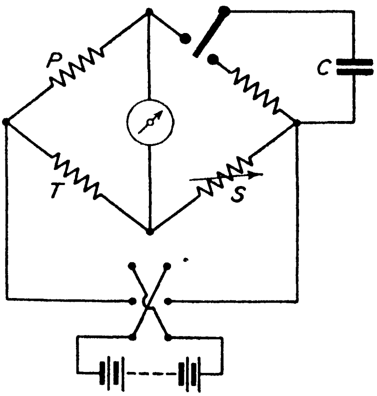


Fig. 3.

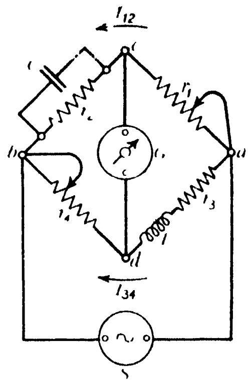


Fig. 4.

The principle of the measurement of the inductance L in terms of the capacitance C is shown in Fig. 4, from which the relationship is found between C and L to be,

$$L = Cr_1r_4$$

and

$$\frac{r_1}{r_2} = \frac{r_3}{r_4}$$

and the derivations of these equations will be found in Chapter IX, page 309.

The Production of Accurate One-second Time Intervals (*Bureau of Standards Journal of Research*, Vol. 21, 1938, page 367)

The quartz-crystal oscillator, when operated under the most favourable conditions, is now recognised as the most accurate known marker of time intervals. Pendulums and tuning-forks could possibly be operated with comparable precision, although greater difficulty would

probably be experienced in their use. Usually, standard piezo-oscillators operate at frequencies near 100 kHz., and lower frequencies are obtained by multi-vibrators. Frequencies down to about 1,000 hz. are usually obtained in this way.

Frequencies below 1,000 hz. are usually obtained by causing a 1,000-hz. synchronous motor to rotate a beam of light or drive a generator having the appropriate number of poles, or to drive reduction gears and contactors. With these methods the main sources of error are the hunting of the synchronous motor and mechanical imperfections. Time intervals obtained from a 1,000-hz. synchronous motor differ by something like 1/2000th part of a second.

The Bureau of Standards Research which is detailed in the *Journal* used multi-vibrators for stepping down the frequency from 1,000 hz. to 1 hz. in three stages of 10 : 1 for each stage, and a special circuit was devised by means of which time intervals of 1 sec. were obtained with an accuracy of not less than one part in a million.

Formerly, the difficulty of measuring frequencies of very high values was the measurement of the time. The best standard clocks could not develop a greater accuracy than about one or two thousandths of a second per day. Consequently, the idea was conceived of measuring time by frequency, and the first published account of such a " clock " was made in 1929. As a result of many years of research, a quartz clock has now been constructed which for long periods maintains an accuracy of one part in 100,000,000, and for short periods an accuracy of one part in 1,000,000,000, which is about 0·032 sec. per year. Such a quartz clock is now the standard for time and frequency to which all measurements can be referred. The quartz oscillator generates 60,000 hz., the temperature of the crystal being automatically maintained constant to $\frac{1}{1000}$ ° C.

The Inductance Constant μ_0

In Fig. 5 is shown a closed ring provided with a uniformly distributed winding of w turns, of constant cross section and the mean length of the magnetic path in the ring being l cm., and the cross-sectional area of the ring being S sq. cm.

If a current of I electromagnetic units flows in the winding of w turns, then

$$4\pi I w = Hl,$$

also

$$B = \mu H : \text{ and Flux } \phi = B.S.$$

The inductance of the winding, that is, the number of flux-linkages per unit of current, is

$$L = \frac{B.S.w}{I} = \mu\frac{4\pi S.w}{l}\frac{S.w}{I} \text{ electromagnetic units,}$$

that is

$$L = \left(\frac{4\pi\mu}{10^9}\right) w^2 \frac{s}{l} \text{ henry,}$$

or

$$L = \mu_0 w^2 \frac{s}{l} \text{ henry,}$$

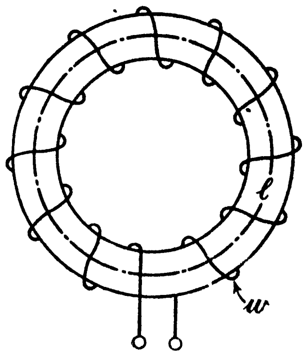


Fig. 5.

where μ_0 is the *inductance constant.* For a non-magnetic ring the inductance constant is

$$\mu_0 = \frac{4\pi}{10^9} \frac{\text{henry}}{\text{centimetre}} \text{ units,}$$

so that

$$\mu_0 = 4\pi \times 10^{-9} = 1{\cdot}256637 \times 10^{-8} \quad . \quad . \quad (9)$$

or, expressed in international units, the value of the inductance constant is

$$\mu_0 = 1{\cdot}25606\ 10^{-8} \frac{\text{henry}}{\text{centimetre}} \text{ units} \quad . \quad . \quad (10)$$

The M.K.S. System of Units

An absolute system of units based on the metre, kilogram, and second, can, of course, be formulated, and for electrical engineering purposes such a system has many advantages over the c.g.s. system, and was first proposed by Professor Giorgi about the year 1901. The *four* fundamental units (see page 1) for the m.k.s. system are the metre, the kilogram, the second, and the inductance constant.

The following Table IV summarises the mechanical units referred to the m.k.s. system and the c.g.s. system:

TABLE IV

Length	1 metre	10^2 c.g.s. units	
Mass	1 kilogram	10^3 ,, ,,	
Time	1 second	1 ,, ,,	
Velocity	1 m.p.s.	10^2 ,, ,,	
Momentum = mv . . .	1 kg. at a velocity of 1 m.p.s.	10^5 ,, ,,	
Force	1 vis or newton	10^5 ,, ,,	(dynes)
Energy.	1 vis acting through 1 m. (1 joule)	10^7 ,, ,,	(ergs)
Power	1 watt	10^7 ,, ,,	

In Table V are shown the relationships between the m.k.s. and c.g.s. systems of units for the electrical and magnetic quantities, viz.

TABLE V

Item	*M.K.S. Unit*	*No. of C.G.S. Units in 1 M.K.S. Unit*
Electric Quantity	1 coulomb	10^{-1}
Current	1 ampere	10^{-1}
P.D. and E.M.F.	1 volt	10^{8}
Resistance	1 ohm	10^{9}
Inductance	1 henry	10^{9}
*Capacitance	1 farad	10^{-9}
Magnetic Flux	1 weber	10^{8}

$$* \; 1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{10^{-1} \text{ c.g.s. units}}{10^{8} \text{ c.g.s. units}} = 10^{-9} \text{ c.g.s. units.}$$

The m.k.s. unit of magnetic flux is 1 weber = 10^8 c.g.s. lines, and the m.k.s. unit of magnetic flux density is

1 weber per sq. m. = 10^4 c.g.s. lines per sq. cm. = 10^4 gauss.

One m.k.s. unit magnetic pole gives rise to a flux of 4π webers, that is,

1 m.k.s. unit pole = 10^8 c.g.s. unit poles.

If a long straight air-core solenoid is uniformly wound, the m.m.f. at the central part of the core, that is, at a distance from the ends which is large in comparison with the diameter of the solenoid (see also Chapter VII), will be :

(*i*) In the c.g.s. electromagnetic system,

$$\text{m.m.f.} = \frac{4\pi}{10}\{\text{ampere turns per cm. length of the solenoid}\}.$$

(ii) In the m.k.s. system,

$$(\text{m.m.f.})' = 4\pi\{\text{ampere-turns per } metre \text{ length of the solenoid}\}.$$

The magnetic force at the central part of the solenoid core will be

$$H = \frac{4\pi}{10}\{\text{ampere-turns per cm. length}\} \quad . \quad . \quad \text{oersted}$$

that is to say, since $\frac{1}{10}$ ampere = 1 electromagnetic c.g.s. unit of current :

One {c.g.s. unit current-turn} per cm. length

will produce a magnetic field strength of 4π oersted, so that

One {ampere-turn} per metre length

will produce a magnetic field strength of $4\pi \times 10^{-3}$ oersted so that

1 m.k.s. unit magnetic field strength = 10^{-3} oersted . (11)

If two m.k.s. unit magnetic poles are placed at a distance apart of

1 metre in open space, the force with which each will act on the other is

$$F = \frac{1}{\mu} \frac{m' \times m_1'}{(r')^2} = \frac{10^8 \times 10^8}{(100)^2} = 10^{12} \text{ dynes,}$$

since in the c.g.s. system the permeability is $\mu = 1$ for open space.

If now the m.k.s. system of units is so chosen that when $m' = m_1' = 1$ and $r' = 1$, the force between them is 1 vis or newton, that is, 10^5 dynes, when the permeability is given the value of unity, then the permeability of open space must be given the value μ' in order that the force between these two poles shall be 10^{12} dynes when in open space. That is to say,

$$F = \frac{1}{\mu'}\left[\frac{m' \times m_1'}{(r')^2}\right] = 10^{12} \text{ dynes}$$

and since, as already stated,

$$\frac{m' \times m_1'}{(r')^2} \text{ is to be equal to } 10^5 \text{ dynes}$$

it follows that,

$$\frac{1}{\mu'} 10^5 = 10^{12} \text{ dynes}$$

that is,

$$\boxed{\mu' = 10^{-7}} \qquad . \quad . \quad . \quad . \quad . \quad (12)$$

Example. In the c.g.s. system the energy stored in a magnetic field is (see Chapter VIII, page 242)

$$U = \frac{\mu H^2}{8\pi} \text{ ergs per c.cm.,}$$

where μ is the magnetic permeability, so that μ for air $= 1$.

In the c.g.s. system

$$H = \frac{4\pi}{10} \frac{Iw}{l};$$

where l is in centimetres, and I is in amperes.

In the m.k.s. system

$$H' = \frac{4\pi I w}{l'}.$$

where H' is the magnetic force in m.k.s. units
l' ,, ,, length of path ,, ,, ,, (metres)

so that

$$\frac{H'}{H} = 10\frac{l}{l'} = 10 \times 10^2 = 10^3 (\text{for } H \text{ in oersted}),$$

that is to say, 1,000 m.k.s. units of magnetic force = 1 oersted.

The energy stored in an electromagnetic field in air is

$$U' = \frac{\mu'(H')^2}{8\pi} \text{ joules per cubic metre,}$$

where μ' is the magnetic permeability of air in the m.k.s. system of unit.

From the results derived in the foregoing it is known that

$$\mu' = 10^{-7} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Further, if ϵ' is the specific inductive capacity of air, then in order that the relationship

$$\varepsilon.\mu = \frac{1}{\begin{pmatrix}\text{Velocity of light}\\ \text{in c.p.s.}\end{pmatrix}^2} = \left(\frac{1}{3 \times 10^{10}}\right)^2$$

as given on page 2 for the c.g.s. system, shall also hold for the m.k.s. system, viz.

$$\epsilon'\mu' = \frac{1}{\begin{pmatrix}\text{Velocity of light}\\ \text{in m.p.s.}\end{pmatrix}^2} = \left(\frac{1}{3 \times 10^{8}}\right)^2 \quad . \quad . \quad (14)$$

it is seen that the following relationship must also hold

$$\epsilon' = \frac{1}{9 \times 10^9} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Rationalised Units

In Chapter VII (see also page 175) it is shown that a consequence of Coulomb's Law of the force between two magnetic poles, the magnetic intensity at a point in open space distant r cm. from a magnetic pole of strength m electromagnetic c.g.s. units, is

$$H = \frac{m}{\mu.r^2} \text{ oersted.}$$

If, however, in accordance with the suggestion of Oliver Heaviside, the magnetic force is expressed as,

$$H = \frac{m}{4\pi\mu.r^2} \text{ units,}$$

where the permeability of open space is taken to be 4π, then this convention leads to a new system of units known as the "rationalised" system, in which the constant 4π disappears from a number of formulae and equations. In the following Table VI is shown the relationship between some of the unrationalised and the rationalised units.

TABLE VI

Item	*Electromagnetic C.G.S. Unrationalised Unit*	*Electromagnetic C.G.S. Rationalised Unit*
Pole strength	m	$m^* = 4\pi m$
Magnetic intensity	H	$H^* = \frac{H}{4\pi}$
Magnetic permeability	μ	$\mu^* = 4\pi\mu$
Dielectric constant	ϵ	$\epsilon^* = \frac{\epsilon}{4\pi}$
Electric quantity	q	$q^* = \frac{q}{4\pi}$
Magnetomotive force	m.m.f.	(m.m.f.)* $= \frac{1}{4\pi}$ m.m.f.

It will be seen from the foregoing table that the use of rationalised units leads to the disappearance of the constant from many formulae and equations. Thus, in Chapter VIII it is shown that the magnetomotive force round a closed path which is linked once with an electric circuit, when measured in unrationalised units, is given by the equation

$$\text{m.m.f.} = 4\pi I w,$$

when I is the current in electromagnetic units, and w is the number of turns in the electric circuit.

When expressed in rationalised units the equation becomes

$$(\text{m.m.f.})^* = Iw.$$

Again, the capacitance of a plate condenser in which air is the dielectric is known to be (see Chapter IV, page 102)

$$C = \frac{\varepsilon S}{4\pi d} \text{ c.g.s. electrostatic units,}$$

when expressed in the unrationalised system of units, whereas

$$C^* = \varepsilon^* \frac{S}{d} \text{ c.g.s. electrostatic units}$$

when expressed in the rationalised system.

Further, on page 19, it has been seen that the inductance constant for open space, when expressed in unrationalised electromagnetic c.g.s. units, is

$$\mu_0 = \frac{4\pi}{10^9},$$

whereas when expressed in rationalised units this constant becomes

$$\mu_0^* = 10^{-9}.$$

It will also be clear, however, that the constant 4π will appear in some of the rationalised units, whilst it is absent from the corresponding unrationalised units Thus, in both systems the relationship for open space, viz.

$$(\text{Permeability}) \times (\text{Dielectric constant}) = \frac{1}{c^2}$$

must hold, where c is the velocity of light, and for c.g.s. units, $c = 2{\cdot}9979 \times 10^{10}$ cm. per sec. and $\mu = 1$.

In the unrationalised electromagnetic c.g.s. system

$$\epsilon = \frac{1}{c^2} = \frac{1}{(3 \times 10^{10})^2},$$

whilst in the rationalised electromagnetic c g.s. system $\mu^* = 4\pi$, so that

$$\epsilon^* = \frac{1}{\mu^* c^2} = \frac{1}{4\pi(3 \times 10^{10})^2}.$$

On the whole, therefore, it would appear that the rationalised system of units does not have any outstanding practical advantages as compared with the unrationalised system.

Chapter II

STRUCTURE OF THE ATOM: CONDUCTORS AND INSULATORS: THE ELECTRIC CURRENT: ELECTRIC RESISTANCE

The Structure of the Atom

EACH atom of a material substance consists of a core or " nucleus " and one or more electrons which travel in orbits round the nucleus and are separated from it by empty space. In the normal unelectrified condition, the positive charge of the nucleus is equal in magnitude to the total negative charges of the electrons. The nucleus itself consists of protons, that is, positively charged particles, and neutrons which are uncharged particles of the same mass as the proton.

The magnitude of the charge of the electron is

$$e = \begin{cases} 1{\cdot}59 \times 10^{-19} \text{ coulombs} \\ 1{\cdot}59 \times 10^{-20} \text{ electromagnetic units} \\ 4{\cdot}77 \times 10^{-10} \text{ electrostatic units} \end{cases} \quad . \quad . \quad (1)$$

Cathode rays (see page 36) consist of a stream of electrons, and by measuring the deflection of such rays in a magnetic field and in an electric field, the ratio of the electric charge to the mass can be found, viz.

$$\frac{e}{m} = 1{\cdot}76 \times 10^{8} \text{ coulomb per gram} \quad . \quad . \quad . \quad (2)$$

so that the mass of the electron is

$$m = \frac{e}{1{\cdot}76 \times 10^{8}} = \frac{1{\cdot}59 \times 10^{-19}}{1{\cdot}76 \times 10^{8}} = 9{\cdot}02 \times 10^{-28} \text{ gm.} \quad . \quad (3)$$

From the results of electrolysis measurements (see page 56) it is known that 1 gm. of hydrogen ions have a total charge of 96,540 coulombs, so that 1 gm. of electrons will carry $\frac{1{\cdot}76 \times 10^{8}}{96{,}540} = 1{,}845$ times as much electricity as 1 gm. of hydrogen, from which it follows that the hydrogen nucleus has a mass 1,845 times that of an electron, that is to say, the mass of the hydrogen nucleus is

$$1{,}845 \times 9 \times 10^{-28} = 1{\cdot}66 \times 10^{-24} \text{ gm.} \quad . \quad . \quad (4)$$

It is seen from the foregoing that the mass of the electron is so minute in comparison with that of the hydrogen nucleus that it may be looked upon as relatively of negligible amount and in consequence the mass of an atom is determined by the number of protons and neutrons in the nucleus. For example, the oxygen atom in its normal uncharged con-

ditions has 8 electrons moving in orbits round the nucleus, so that there are 8 protons in the nucleus. Since the atomic weight of oxygen is 16, the nucleus also contains 8 neutrons. It is seen, then, that the number which defines the atomic weight of an atom is exactly equal to the sum of the number of protons and neutrons which its nucleus contains. For a large number of elements this condition does in fact hold good, but for many elements there is a considerable discrepancy which can be accounted for as follows.

The chemist determines the atomic weight of an element by means of a sufficiently large mass of the chemically pure substance to enable him to make a precision measurement of the weight, and such a measurement of course gives the average of the atomic weight for an immense number of atoms. By means of the "mass spectrograph", however, Aston caused a stream of ions of the chemically pure element to be deflected, first by means of an electric field, and then by a magnetic field, finally allowing them to impinge on a photographic plate. A measurement of the differing amounts of deflection of the individual ions then showed that elements of identical chemical properties could have different atomic weights. This phenomenon is now accounted for by a corresponding difference in the number of neutrons present in the nucleus. Thus there is the so-called "light hydrogen", ^{1}H, of which the nucleus consists entirely of a single proton: and there are two so-called "heavy hydrogen" atoms, ^{2}H . ^{3}H, which have the respective atomic weights of 2 and 3. According to the process by which the hydrogen is prepared, the element may appear as of two kinds, viz.:

(*i*) a mixture of ^{1}H and ^{2}H deuterons in the ratio of 5,000 : 1

(*ii*) A mixture of ^{2}H and ^{3}H in the ratio of 200,000 : 1.

Atoms of identical chemical properties but of different atomic weights are known as "isotopes".

There is also known a mixture ^{16}O and ^{18}O of chemically pure oxygen in the ratio 630 : 1, and in addition to the normal or "light" water there is a "heavy" water. Heavy water is highly deleterious to living organisms, and in contrast to the normal light water gives indications of a higher temperature for the freezing-point and a higher temperature for its maximum density.

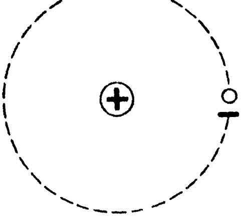


Fig. 1.

Another prominent example of this phenomenon of nucleus structure is the element chloride, for which the atomic weight is found by chemical measurements to be 35·46, whereas it is now known that there are two types of chlorine atoms, viz. ^{35}Cl and ^{37}Cl.

The atom of "light hydrogen" in the normal uncharged condition contains only one electron in orbital motion round the nucleus, so that the nucleus only contains one proton, and such an

atom therefore has the simplest structure of all elements, viz. 1 proton of charge $+1{\cdot}59 \times 10^{-19}$ coulomb and 1 electron of charge $-1{\cdot}59 \times 10^{-19}$ coulomb revolving in an orbit round the nucleus, as is shown diagrammatically in Fig. 1. The atomic number of lithium is 3, that is to say, there are normally 3 electrons moving in orbits round the nucleus, and consequently the nucleus contains 3 protons. The atomic weight of lithium, however, is 7, so that in addition to the 3 protons the nucleus of the lithium atom contains 4 neutrons, the diagrammatical representation being shown in Fig. 2.

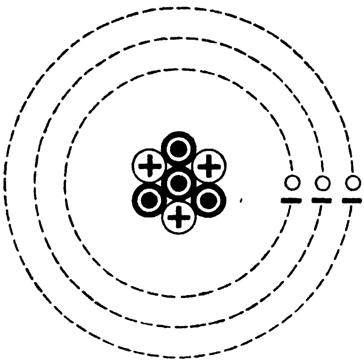


Fig. 2.

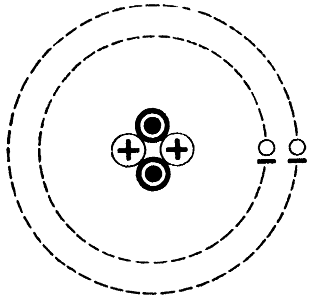


Fig. 3.

Another atom of simple structure which is of great technical significance (see page 29) is the helium atom ^{2}He, of which the atomic number is 2, so that there are 2 electrons normally moving in orbits round the nucleus. The atomic weight of helium, however, is 4 times that of hydrogen, so that in addition to the 2 protons the nucleus of this atom contains 2 neutrons, as shown in Fig. 3.

The Photon

The foregoing account of the Bohr-Rutherford structure of the atom as a series of electrons revolving round the nucleus in a similar fashion to that of the planets travelling in orbits round the sun does not suffice to account for the energy transformations which take place within the atom, and it is necessary to combine the Bohr view of the atom with the quantum theory, in accordance with which, of all the paths which are mathematically possible, only a few are actually available for the electron orbit.

In the normal condition of the atom, each of the revolving electrons will occupy the orbit of lowest energy of all its possible paths, and this is its basic orbit. It is only by exciting the atom by conveying energy to it from outside that the electron can be forced into an orbit of higher

energy quantum, and when the electron falls back into its normal orbit it emits a "photon" of energy which is defined by the equation

$$\varepsilon = h \times f \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where h is Planck's constant and equals $6{\cdot}55 \times 10^{-27}$ erg-sec.,
f is the frequency in hertz,
ε is the energy in ergs.
For example, if a body is radiating violet light, that is light waves of frequency $f = 758 \times 10^{12}$ hertz, then

$$\varepsilon = (6{\cdot}55 \times 10^{-27}) \times (758 \times 10^{12}) = 496 \times 10^{-14} \text{ erg},$$

and this energy ε is only emitted in very minute time intervals, with a relatively long pause between successive emissions of energy. It can therefore be said that each atom of the radiating body must acquire an amount of energy $\varepsilon = hf = 496 \times 10^{-14}$ erg before it can emit violet light. Further, from the relativity theory it is known that mass is equivalent to energy as defined by the following equation, viz.

$$\text{Energy in ergs} = \text{mass in grams} \times c^2 \quad . \quad . \quad (6)$$

where c is the velocity of light in open space and is equal to $2{\cdot}9987 \times 10^{10}$ cm. per second. From the numerical data given in the foregoing, it is seen that the equivalent mass of the photon of violet light is,

$$m = \frac{496 \times 10^{-14}}{(2{\cdot}9987 \times 10^{10})^2} = 55{\cdot}1 \times 10^{-31} \text{ gm.}$$

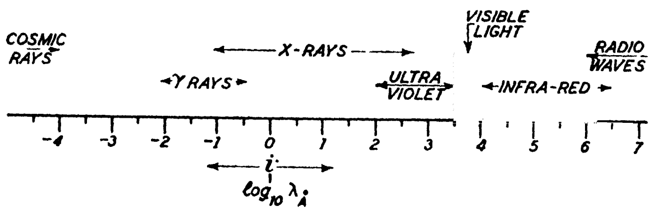


Fig. 4.

In Fig. 4 is given a chart showing the ranges of electromagnetic wavelengths and frequencies which are of special technical interest, the relationship between wavelength and frequency being

$$\begin{aligned}(\text{wave-length } \lambda \text{ in cm.}) \times (\text{frequency } f \text{ in hertz}) &= \text{velocity of light} \\ &= 3 \times 10^{10} \text{ cm. per second.}\end{aligned}$$

The scale given on this chart is the logarithm to base 10 of the wavelength $\lambda_Å$ in Ångstrom (Å) units, viz. $1 Å = 10^{-8}$ cm. Thus, for a wavelength of 10^{-4} cm.,

$$\lambda_Å = 10^4 \text{ Å}: \text{ so that } \log_{10} \lambda_Å = 4.$$

Disintegration of the Atom

Examples of atomic disintegration are found in the natural radio-activity of uranium and radium and, in a lesser degree, thorium, potassium, rubidium and many other elements. The atoms of these substances must be considered as having unstable nuclei which tend to explosively disintegrate, and in consequence to eject portions of the atomic structure. Such natural disintegration gives rise to the so-called α-rays, β-rays, and γ-rays. Of these the α-rays are simply helium nuclei (see page 27) which are ejected at a high velocity and are consequently positively charged and will be deflected by an electric field. The β-rays are a stream of electrons which are emitted as a consequence of the transformation of the nucleus protons and neutrons and are of the same nature as "cathode rays" (see page 37), so that they will be deflected by either an electric or a magnetic field. Since the β-rays are negatively charged the deflection in an electric field will be in the opposite direction to the deflection of the α-rays, as is shown in Fig. 5. The γ-rays are electromagnetic waves and are of the same nature as "Rontgen rays" (see page 37), from which they are distinguished by having a shorter wave-length. These rays are not deflected by either an electric or a magnetic field.

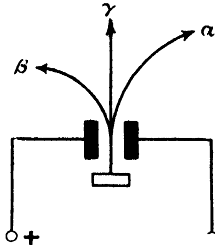


Fig. 5.

Such natural radiation is an example of the transformation of mass into energy in accordance with the previously considered equation (6) (see page 28), viz.

$$\text{Energy in ergs} = \text{mass in grams} \times c^2,$$

where c is the velocity of light, i.e. $c = 3 \times 10^{10}$ cm. per second. The law of the rate of decay of radio-activity for radium was derived by Rutherford and Soddy from experimental measurements, and it is known, for example, that radium will be reduced to one-half its original mass after about 1,600 years of radiation. The final product of the radiation of uranium and of thorium is lead and, in the case of earth containing uranium or thorium, it is possible, from a knowledge of the amount of lead found associated with these metals, to establish the time for which the radiation must have been taking place. In this way an estimate of the age of geological layers in which these substances are found can be made.

In recent years great developments have taken place in methods for artificially disintegrating the atom, the most notable and powerful of these being the "cyclotron" of Professor Lawrence of California. In this appliance a target of the material to be disintegrated is bombarded with a stream of deuterons, that is the nuclei of atoms of heavy hydrogen, and as a result of such artificial methods of atomic disintegration it has

been found that, during the transformations of the protons and neutrons, secondary electric particles, viz. the neutrino and the positron appear, but which apparently have only a transient existence.

The magnetic characteristics of the ferromagnetic metals iron, cobalt and nickel can now be accounted for as a consequence of the "spins" which are superposed on the electrons revolving round the nucleus, that is a motion similar to the revolution of the planets on their axes. Further reference to this will be found in Chapter VII.

Conduction of Electricity through Gases

The flow of current in a metal conductor is defined by Ohm's Law, which relates the potential difference with the current, as is explained on page 44. This law, however, entirely fails when currents which are formed of free electrons or free ions are considered. For such currents in open space the controlling factor is the "space charge", and in place of the inexhaustible store of electrons which are available in a metal conductor, which in the case of currents formed by electrons in open space it is first of all necessary to provide the electrified particles of which the electronic and ionic currents are composed.

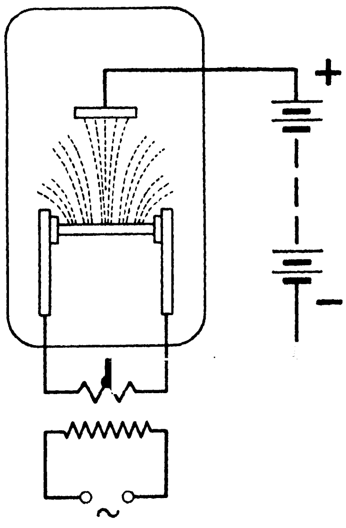


Fig. 6.

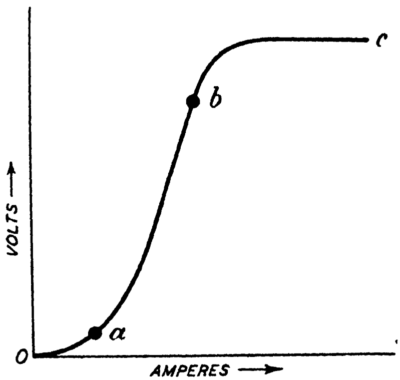


Fig. 7.

Suppose two electrodes as shown in Fig. 6 are respectively maintained at positive and negative potentials, and that the space surrounding them is completely exhausted of gas. If the temperature of the negative electrode, that is, the cathode, is raised to a sufficient degree, electrons will be ejected and will travel towards the anode under the influence of the electrostatic field which exists between the electrodes. At first sight

it might be supposed that all the electrons which are ejected from the cathode would reach the anode when only a small p.d. was applied to the electrodes: that is to say, the current strength would be very little affected by the magnitude of the p.d. Actually, however, the fact is quite otherwise. At any point in the evacuated container an electron will be under the control, not only of the field due to the charges on the respective electrodes, but also of the field due to all the other free electrons in the space between the electrodes, or otherwise expressed, the field due to the "space charge". The question then arises as to how the current value I will be related to the p.d. V between the electrodes. For small values of V the space charge will drive most of the electrons back to the cathode so that the rate of increase of current will be small as is shown by the portion Oa of the graph in Fig. 7 When the pressure V increases, the force exerted on the electrons by the field due to the electrodes will increase, and at the same time the controlling effect of the space charge will diminish owing to the increased speed of transport of the electrons across the space. By direct calculation it can be shown that the rise of current with the applied p.d. will then be given by the equation,

$$I = kV^{3/2} \quad . \quad . \quad . \quad . \quad (7)$$

where k is a constant. The approximate range for which this equation holds is shown by ab in Fig. 7. When the p.d. between the electrodes is still further increased, the rate of increase of the current diminishes and eventually reaches a saturation value, as is shown by the portion bc of the graph in Fig. 7. That is to say, as soon as all the electrons which are ejected by the cathode in unit time are actually transported to the anode, the current can no longer be increased by increasing the applied p.d.

Effect of Positively Charged Particles (Ions) on the Space Charge

It has already been seen that the deciding factor which determines the relationship between the p.d. and the current in an evacuated container, is the space charge. If it is desired to influence this relationship in order to obtain, say, an increased current for low values of the applied p.d., it is necessary that the effective space charge shall be reduced by the injection of positively charged particles. This effect can be obtained if a small quantity of gas is included in the evacuated chamber, since in this way the electrons will have the opportunity of "ionising" the gas, that is, dissociating each gas molecule into an electron, and a positively charged particle, that is, an "ion". The consequence of this will be seen from the following considerations. The magnitude of the space charge is determined by the net effect of the total number of free positive and negative particles which are released per unit volume, that is to say, it depends not only on the number of

positive and negatively charged particles which are released per second, but also on the time for which these charged particles remain in the space between the electrodes or, in other words, upon the speed with which the electrons and ions move in the electric field. For a given field strength this speed is inversely proportional to the square root of the mass of the particle, so that, for example, the ratio of the speed of the mercury ion of mass 332×10^{-24} gm. to the speed of an electron of mass $= 9{\cdot}02 \times 10^{-28}$ gm. will be

$$\sqrt{\frac{9{\cdot}02 \times 10^{24}}{332 \times 10^{28}}} = \frac{1}{607}.$$

It follows, therefore, that a mercury ion will move at a speed of only $\frac{1}{600}$th of that of an electron. This fact explains the astonishingly large effect on the current-pressure characteristic (Fig. 8) when a minute amount of gas or vapour is admitted to the evacuated space between the electrodes. Thus, if only sufficient mercury vapour is admitted to produce positive particles which will account for $\frac{1}{600}$th part of the total current, the space charge will become zero since the mercury ions will remain 600 times as long on their journey as the electrons on their journey across the space between the electrodes.

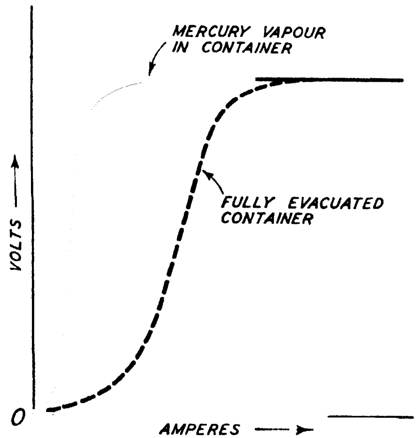


Fig. 8.

The reduction of the space charge by means of positive ions is of the greatest practical importance since the requisite p.d. for a given current will be reduced as the magnitude of the counteracting space charge is reduced, and this implies a corresponding reduction in the power loss in the appliance. That is to say, there will be a corresponding reduction of the cooling necessary to dissipate the heat loss. Reducing the magnitude of the space charge in this way makes it possible to construct containing vessels with gas or vapour filling which can be used for heavy currents. The use of highly evacuated containers must be confined to light current operation on account of the difficulty of dissipating the heat when heavy currents flow in such gas-free containers.

The Glow Discharge

The simplest type of glow discharge is that in which electrons are discharged from the cathode by raising its temperature as has been

considered already in the foregoing. These electrons are accelerated in the electric field and, by ionisation of the gas, produce positive ions. The ions then travel towards the cathode, neutralising the negative space charge on the way, and so release those electrons in the neighbourhood of the cathode which are bound by the space charge, a state of equilibrium being eventually reached which is dependent upon the magnitude of the p.d. between the electrodes.

There is another type of glow discharge which is of great practical significance and which does not depend upon a heated cathode for the production of the electrons. This employs a cold cathode which actually produces a glow discharge very similar to that which is obtained from a hot cathode. This may be explained by means of the following considerations. In a highly evacuated chamber there will always be individual free electrons present which may be due, for example, to radio-active effects from which no such space will ever be entirely free. Such free electrons will become accelerated in the electric field between the electrodes and will ionise the gas. The released positive ions will then move towards the cathode with ever-increasing speed in the electric field and will also ionise the gas in their path. In this way a cumulative or "avalanche" effect develops at a very high speed until a condition of equilibrium is reached. The difference between the glow discharge produced by a hot cathode and that produced by a cold cathode is not so great as might appear at first sight since, fundamentally, it is a matter of indifference whether the positive ions release the electrons from their bound condition in the space charge, or from their bound condition in the atomic structure of the material of the cathode.

Mechanism of the Glow Discharge

The addition of a minute amount of gas or vapour in the evacuated container vessel greatly complicates the process by which a glow discharge is produced. The electrified carriers newly released by ionisation partake in the further process of ionisation, and this process is the most important factor in the excitation of a glow discharge. In order that the electrons shall acquire sufficient energy to effect ionisation by impact with a molecule, they must travel across a definite potential difference in the electric field: that is to say, for a given field a definite distance must be traversed. In order to simplify the consideration as much as possible a highly idealised condition will be assumed in which the two determining factors are (*i*) The distance x which the electrons must travel when ejected from the cathode in order that they may acquire the requisite amount of ionising energy, and (*ii*) the so-called "mean free path" y, that is, the distance which the electrons must travel on the average before they encounter a molecule, this distance being dependent upon the gas pressure in the evacuated space.

Assuming that at the commencement of the process the potential

difference is more or less uniform, and that x is considerably greater than y, for example, $x = 2y$, so that only those electrons will be able to produce ionisation which have been able to travel the distance x before impact with a molecule. Due to this impact, however, a positive space charge will be formed at the distance x from the cathode, and the potential at this point will be raised relatively to the cathode. This implies an increase of the strength of the field in this neighbourhood and a consequent reduction of the distance x which an electron has to travel in order to acquire the requisite ionising energy. A little consideration will show that the general effect of this process will be that the centre of generation of the positive ions will draw gradually closer to the cathode until eventually the critical condition will be reached for which $x = y$. At this distance from the cathode a strong positive space charge will be developed, since at this place the electrons will most frequently collide with the molecule with adequate energy to ionise them. Further, the released ions will have their minimum speed in this space and will therefore linger in this region for the maximum time. A large p.d. will thus be developed between this region and the cathode: in most cases this p.d. forms the main portion of the total p.d. between the anode and the cathode. This p.d. is known as the "cathode drop" and both theoretically and technically is the most important potential factor in the formation of any glow discharge. This region can be seen as a dark space (Crookes' Dark Space) and is bounded by the cathode glow on one side and by the negative glow on the other, as is shown in Fig. 9.

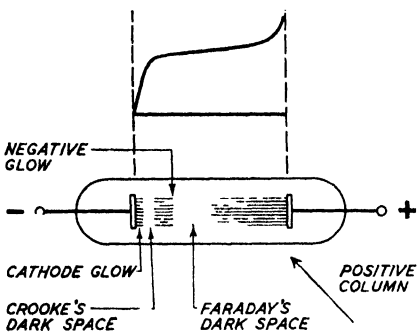


Fig. 9.

By means of considerations which involve not only ionisation phenomenon but also the losses at the walls of the containing vessel and the impact of the electrons on the anode, it is possible to account for the other main discharge formations shown in Fig. 9, such as the "positive column" and the "anode drop". The fundamental consideration which controls these phenomena is that a condition of equilibrium must be established between the various operating factors in the complicated sequence of processes.

Transition from Glow Discharge to Arc Discharge

As has been pointed out already, a complete statement of the conditions of equilibrium would involve highly complicated relationships, but it may be said that the space charge is locked with a definite current

density, and consequently, for a given current, the space charge is locked with a definite surface area of the cathode. The consequence of this relationship is that the amount of the cathode surface which is covered by the glow discharge will increase proportionally with the current strength so long as the total surface of the cathode remains uncovered. That is to say, so long as the total surface of the cathode remains uncovered, the cathode pressure drop will remain constant at its " normal " value as the current strength is varied. If, after the whole cathode surface has become covered by the glow discharge, the current is still further increased, an increase of the cathode pressure drop will take place, and since there will be, in consequence, a greater number of electrons ejected per unit area of cathode surface than will be the case with the normal glow discharge, the whole character of the discharge will be abruptly changed. The negative glow will contract to a single brilliant spot which will move restlessly over the surface of the cathode, thus the glow discharge will have become transformed into an *arc discharge*.

A general idea of the physical sequence of events may be obtained as follows: Suppose that the glow discharge has completely covered the cathode surface, or that the current strength has reached its maximum value consistent with the normal cathode pressure drop. If, now, at some spot x on the cathode surface the current density is increased, this will cause an increase in the number of ions at the corresponding point in the positive ion space-charge layer which is opposite to the spot x on the cathode surface. The temperature at this point on the cathode surface will therefore rise and the electron emission will automatically concentrate at this spot, so that the glow discharge will be transformed into the arc discharge.

Conduction of Electricity across a Spark Gap

In the case of air and other gases, the conduction of the current across the gap is effected by ionisation of the path and this may be briefly considered in the light of what has been said in the foregoing. Since the ionisation of the path is caused by the electric field, a certain time interval is required to produce the requisite conductivity, so that between the initiation of the ionisation and the subsequent spark discharge, a definite, though extremely small, time interval is required, and this interval forms the " discharge lag " of the spark. The duration of this time lag depends upon a complex system of relationships and may vary between wide limits such as 10^{-3} to 10^{-8} sec. In the case of homogeneous electric fields, the dissociation of the molecules and atoms of the gas into electrified particles will take place more or less simultaneously throughout the whole path of the subsequent spark, so the lag of the discharge in this case will be relatively small. In the case of non-uniform fields, however, the ionisation process will be initiated

at points of maximum field intensity and will spread from there outwards. When electrodes are used of which the radius of curvature is small in comparison with the length of the gap, the discharge lag will be greatly increased. For oil and other liquid insulators, the energy of ionisation is very large, and in such cases the discharge lag will be very much greater than in air for otherwise similar conditions. The same considerations also apply to solid insulators such as porcelain, paper, and mica. In the case of unsymmetrical electrodes such as are found in suspension insulators, the polarity has a great influence on the discharge. In general, the negative electrode may be said to exert the controlling influence since the electrons on which ionisation depends are supplied from this electrode.

The Photoelectric Effect

If light falls on a naked metal surface, particularly one of the alkali metals, a simultaneous emission of electrons takes place from the surface of the metal, and the intensity of this emission is directly proportional to the intensity of the incident light of a given wave-length. This proportionality of the intensity of the incident light and the intensity of the stream of emitted electrons finds a valuable application for photometric measurements and a photoelectric cell constructed for this purpose is illustrated in Fig. 10. The sensitive metal surface (e.g. potassium) on which the light falls is shown at C, and a battery B is connected in series with an ammeter so that the negative pole is joined to C and the positive pole to the anode A. The p.d. applied between the anode A and the cathode C causes the emitted electrons from the cathode surface to flow into the anode whilst an equal number pass from the battery into the cathode. The range of operation of this cell extends from ultraviolet wavelengths and through the complete spectrum (see Fig. 4) down to the short infra-red wavelengths and particularly important application of this type of appliance is found in the measurement of light for astronomical work. Such light-sensitive cells are also applied to an immense variety of important industrial purposes.

Cathode Rays

It has been seen on page 33 that an electric discharge through an exhausted tube produces a " cathode glow " at the negative electrode. In 1880 Crookes showed that when the gas pressure is reduced to a sufficiently small value, the glow in the tube disappears altogether and the passage of electricity is only shown by the fluorescence of the glass tube which becomes extremely brilliant at that part of the tube which faces the cathode. Crookes explained this by assuming that electrical particles were shot off the cathode at right angles to its surface and formed a beam or " ray " which he called in consequence the " cathode ray ". It was found that a particularly brilliant phosphorescent glow was

produced when the ray was directed on to a screen coated with zinc sulphide.

It is now known that the cathode ray consists of electrons travelling at a very high speed, and in the recent type of Coolidge tubes a velocity of about 90,000 miles per second is reached, that is, about one-half the speed of light. The cathode ray will be deflected if it crosses either a magnetic or an electric field, and an important application of this property is found in the cathode ray oscillograph illustrated in Fig. 11, which shows diagrammatically the principle of its operation.

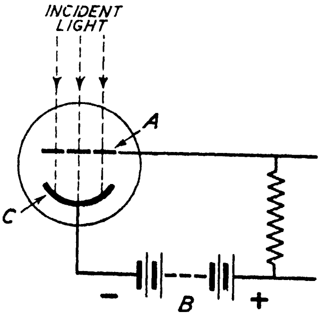


Fig. 10.

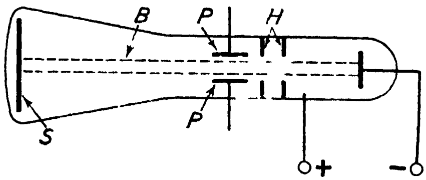


Fig. 11.

The ray passes through a hole in each of two metal discs *H*, and two plates *Q* are placed parallel to the beam so that when a potential difference is applied to the plates the direction of the electric field so produced is perpendicular to the ray and consequently deflects it by an amount which is proportional to the intensity of the electric field. The beam strikes a zinc sulphide screen *S* and the movement of the beam as the p.d. across the plates *Q* is varied, can be seen by the phosphorescent trace on the screen.

X-rays

Many years before Röntgen discovered "X rays" in 1895, J. J. Thomson had expressed the view that if rapidly moving charged particles were to be suddenly stopped, electromagnetic waves of the nature of light would be developed, and he also suggested that the light seen at the surface of the glass tube in which a cathode ray is produced, was in fact an example of the generation of electromagnetic waves due to the sudden stoppage of the electrons when the cathode ray strikes the wall of the tube. It is now known that X-rays to which the stoppage of high-speed electrons gives rise, are electromagnetic waves of the same nature as visible light, but of a much higher frequency.

In Fig. 12 is shown a diagram of the principle parts of an X-ray tube in which the cathode filament is caused to emit electrons by heat due to a current passed through it from the battery shown. The electrons are accelerated by the electric field due to the p.d. which is applied between the cathode and the anode, and consequently the electrons

strike the target at a very high speed, thus generating the X-rays. Such X-rays are shielded and absorbed in all directions except for a small narrow beam which is emitted through the tube and applied to the purpose for which it is intended.

The Dry Metal Plate Rectifier

This type of rectifier is used for charging accumulators from a source of alternating current, for exciting direct-current electromagnets, for operating cinema arc-lamps and for many other purposes. The theory of the "valve" action of this type of rectifier is not yet complete, but the general principle of its operation will be understood from what follows. This principle has been known for a long time, and the modern form of the rectifier is in fact a large-scale improved example of the crystal detector of the early days of wireless reception apparatus.

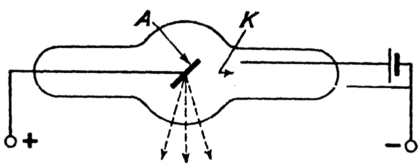


Fig 12

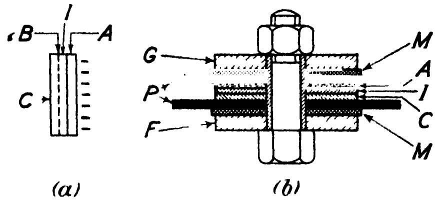


Fig. 13

There are three main types, viz. (*i*) the cuprous oxide, Cu_2O (Kupferoxydul), (*ii*) the selenium, (*iii*) the copper sulphide. All of these are characterised by the following components as illustrated in Fig. 13*a* and 13*b*.

(1) The emission electrode (cathode), C; (2) the "blocking boundary," B; (3) the layer of semi-conducting material, I; (4) the contact electrode (anode), A.

In the cuprous oxide type the emission electrode is a copper plate on which the semi-conducting layer of cuprous oxide (Cu_2O) is formed in such a way that there is an extremely intimate contact between the copper plate and the semi-conductor, the thin layer of oxide being produced by a chemical or thermal treatment, by means of which the oxide is caused to "grow" out of the copper-plate surface. This semi-conductor is of low electrical conductivity and of rough surface, and in order to ensure a good contact and hence a minimum contact resistance between the anode *A* and the Cu_2O surface, the latter is dusted with fine graphite powder (or sprayed with metal), and a malleable metal (lead) is then used as the anode plate, the advantage of lead being that it can be pressed on to the treated surface of the semi-conductor and ensure in this way the best possible contact between the anode and the uneven surface of the semi-conducting cuprous oxide.

The semi-conductor Cu_2O is characterised by the concentration of electrons at the surface in contact with the anode, so that when a p.d. is applied in such a way that the plate A is connected to the positive terminal of the supply whilst the cathode C is connected to the negative terminal, a continuous stream of electrons will flow into the positive terminal of the supply source, and consequently the valve action is then "open" and the rectifier then forms a conducting path. When the polarity of the supply terminals is reversed, the concentration of electrons at the anode surface blocks the passage of a current, and then the valve action is then, in effect, "closed". When the applied p.d. is such that the anode A is connected to the negative pole the valve effect remains closed without rupture even when an electric field of mean intensity of up to from 3 to 6 million volts per centimetre is applied. The effective thickness of the blocking boundary B in Fig. 13*a* is of the order of from 10^{-5} cm. to 10^{-6} cm. The effective capacitance of the activated surface of the semi-conductor is from 0·01 to 0·025 μF. Such rectifiers can be used for frequencies up to about 10,000 hz. without any appreciable effect on the operation being due to the self-capacitance. By suitably proportioning the dimensions of the component parts, this type of rectifier can also be used for high-frequency work.

In Fig. 13*b* is shown an assembled 2-volt unit of a cuprous oxide rectifier and corresponding parts of Fig. 13*a* and 13*b* have been given the same distinguishing letters. Cooling plates are shown at PP and layers of insulating material are shown at MM. By means of the clamping bolt the lead anode can be pressed with sufficient force to make a good electrical contact with the active surface of the layer of Cu_2O.

The Mercury Arc Rectifier

When a mercury arc is operated in a highly evacuated container, between a graphite (or metal) electrode and a pool of mercury, it is characterised by the property of being able to pass a current in one direction only, viz. the direction for which the graphite electrode is positive (the anode) and the mercury pool is the cathode or negative terminal. A diagram of the principle features of such a rectifier is shown in Fig. 14, in which the mercury pool is at C and two anodes are shown respectively at A_1 and A_2. The alternating-current supply is connected to the primary winding of a transformer, the secondary winding of which has a centre-point tapping which is connected to the mercury pool C. The ends of the secondary winding of the transformer, respectively, go to the anodes A_1 and A_2.

The rectifying action is obtained as follows. Since the a.c. supply pressure passes through successive half-cycles of opposite polarity, the two anodes A_1 and A_2 will, in turn, become positive with respect to the mercury pool C. From what has already been said in the foregoing however, the arc can only form between the pool and that anode which

is of positive polarity, so that the arc will pass from one anode to the other in turn. The consequence will be that the mercury pool will remain continuously the positive pole and the centre-tapping point of the transformer, the negative pole for the d.c. output terminals.

Conductors and Insulators : Ohm's Law

In the foregoing paragraphs of this chapter attention has been given to the special characteristic features of the passage of electricity through gases, and in what follows, the electrical properties of solid and liquid conductors and insulators will be considered. In general, all substances may be classified as either conductors or insulators of electricity, although there is no rigid boundary between these two classes : conductors, semi-

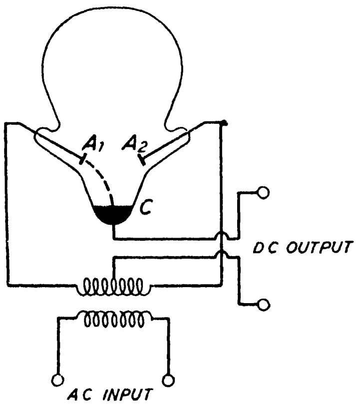


Fig. 14.

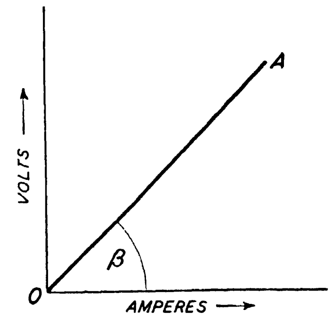


Fig. 15.

conductors, and insulators merge into one generalised group in the sense that to some degree, all substances are conductors of electricity. Broadly speaking, it may be said that the electrons in a conductor are free to move from molecule to molecule, whereas in an insulator they are only able to move with the molecule.

Electric conductors comprise substances in the solid, liquid, and gaseous states, and of these the pure metals have by far the greatest electrical conductivity. In the case of those liquid conductors which are chemical compounds, conduction takes place by means of " electrolysis ".

A current of 1 ampere flows in a conductor when 1 coulomb of quantity passes per second through any cross-section of the conductor. If the current is varying its magnitude at any moment is given by the expression

$$i = \frac{\delta q}{\delta t} \text{ amps.}$$

where δq coulomb is the quantity of electricity which passes through any cross-section of the conductor in δt second. In the notation of the calculus this relationship becomes

$$i = \frac{dq}{dt} \text{ amps.}$$

For solid and liquid conductors "Ohm's Law" applies, and this law may be stated as follows:

Provided that the physical condition (i.e. temperature and mechanical strain) remains unaltered, the ratio of the potential difference V between the ends of a wire in which a current I is flowing, and the magnitude of the current will be constant, that is:

$$\frac{V}{I} = R, \qquad . \quad . \quad . \quad . \quad . \quad (8)$$

where R is a constant and is termed the "resistance" of the wire. The constancy of this ratio is shown in Fig. 15 by the graph connecting the p.d. in volts and the current in amperes, so that the slope of this straight line gives the resistance in ohms, viz.

$$\tan \beta = R \text{ ohms } (\Omega).$$

The accuracy of Ohm's Law has been established by the most searching experimental tests in which a deviation of one part in 100,000 could have been detected, but no departure from the constancy of the ratio of V to I has been observed, provided that the conditions as to the constancy of physical state during the measurements have been observed. The great importance of Ohm's Law is its signification that the resistance of a conductor is independent of both the applied p.d. and the current in the conductor, and it forms a basis for the whole system of electrical measurements.

The unit of electrical resistance is the "ohm" and may be defined as follows:

The International Ohm is the resistance offered to an unvarying current by a column of mercury at the temperature of melting ice, the mass of the column being 14·45 *gm., the cross-sectional area being constant, and the length being* 106·3 *cm.* (see also Chapter I, page 12).

This specification of the ohm defines a resistance which is equal to 10^9 absolute electromagnetic units c.g.s., that is

$$1 \text{ ohm} = 10^9 \text{ electromagnetic units} \qquad . \quad . \quad (9)$$

Ohm also showed that the resistance of a conductor of uniform cross-section q and length l is given by the expression

$$R = \rho \frac{l}{q} \qquad . \quad . \quad . \quad . \quad . \quad (10)$$

where ρ is a characteristic constant of the material of the conductor and

is called the "specific resistance" or the "resistivity". Thus, in the expression (10), if $l = 1$ cm. and $q = 1$ sq. cm., then $R = \rho$, so that,

The specific resistance of a conducting material is the resistance between the opposite faces of a cube of the material of 1-cm. sides. More briefly the specific resistance is usually stated as the resistance per "centimetre cube", it being assumed that the current flows between the opposite faces of the cube.

The "conductivity" is the reciprocal of the specific resistance and is measured in reciprocal ohms $\left(\frac{1}{\Omega}\right)$: frequently, the name "siemens" is applied to this unit. Copper is of special importance in electrical engineering on account of its high conductivity and other valuable properties. Until recently, electrical conductors have been almost exclusively made of copper, although in recent years aluminium has achieved great importance for this purpose. The International Standard Specification for annealed copper is as follows:

(*i*) At a temperature of 20° C. the resistance of a wire of standard annealed copper 1 metre in length and of uniform section of 1 sq. mm. $\frac{1}{58}$ ohm.

(*ii*) At a temperature of 20° C. the density of standard annealed copper is 8·89 gm. per cubic centimetre.

(*iii*) At a temperature of 20° C. the "constant mass" temperature coefficient of resistance of standard annealed copper measured between two potential points rigidly fixed to the wire is 0·00393 per 1° C.

(*iv*) As a consequence, it follows from (*i*) and (*ii*) that at a temperature of 20° C. the resistance of a wire of standard annealed copper of uniform section 1 sq. mm., 1 metre in length and weighing 1 gm., is $\frac{1}{58} \times 8{\cdot}89 = 0{\cdot}15328$ ohm.

If the specific resistance of the material of a conductor is known, it is easy to calculate the resistance of a conductor of a given length and given cross section. For many practical purposes such as problems relating to the transmission and distribution of electricity, it is usually more convenient to express the specific resistance in terms of the metre as the unit of length and the square millimetre as the unit of cross-section, thus

$$R = \rho \frac{l}{q} \text{ m. mm.}^2 \qquad . \quad . \quad . \quad . \quad (11)$$

where ρ is the resistance of a conductor 1 metre long and 1 sq. mm. in cross section.

In general, the resistance of a conductor is notably affected by the temperature, and it is found that the resistance of all pure metals and of nearly all alloys increases with the rise of temperature. The resistance of carbon and non-metallic liquids, however (that is, electrolytes), *decreases* with rise of temperature. The variation of resistance with

temperature of metals and alloys is closely represented by the following formula for a wide range of temperatures.

$$R_{t_2} = R_{t_1}[1 + \alpha(t_2 - t_1)]\Omega \quad . \quad . \quad (12)$$

where R_{t_2} is the resistance at the temperature t_2° C.,
R_{t_1} ,, ,, ,, ,, t_1° C.,

and γ is termed the "temperature coefficient of resistance" and is the change of resistance per 1° C.

In Table I is given a list of the values of the specific resistance ρ in Ω m. mm.², the conductivity λ in $\frac{1}{\Omega}$, and the temperature coefficient γ for the more commonly used solid conducting materials, whilst Table II gives the corresponding quantities for some representative liquid substances.

Table III gives the specific resistances in Ω cm. cm.² and the temperature coefficients for a number of *pure metals* as determined by Fleming and Dewar, and Table IV gives the specific resistances for a number of important *resistance alloys*, also as measured by Fleming and Dewar. In Table V will be found the values of the specific resistances for some of the more important *insulating materials*.

It is convenient to classify electrical resistance materials according to the type of work for which they are intended. Resistance may be required for the purpose of precision measurements such as are used in the various type of bridge systems: as series or shunt resistances for current regulation, and for electric radiators and industrial furnaces for the generation of heat. In the following schedule such a classification is given.

ALLOYS FOR PRECISION RESISTANCE UNITS

	Copper	*Silver*	*Manganese*	*Tin*	*Nickel*	*Iron*	*Aluminium*	
Manganin	86	--	12	-	2	-	—	Per cent.
Isabellin	84	--	13			-	3	
Novoconstant	82 5	—	12	—	—	1 5	4	
NBW 108	-	82	10	8	—			
,, 173	—	80	17	3		-	—	
Gold Chrome	2·1 to 2·4 per cent. Chromium : Rest Gold							

RESISTANCE ALLOYS FOR INDUSTRIAL FURNACES AND FOR RADIATORS

Iron-Chromium : Iron-Chromium-Aluminium : Iron-Chromium-Silicon : Iron-Nickel-Chromium

RESISTANCE ALLOYS FOR SERIES AND SHUNT REGULATORS

Iron-Chromium : Iron-Chromium-Silicon : Iron-Chromium-Aluminium : Iron-Manganese-Chromium : Iron-Silicon

TABLE I

Conductor Material	*Specific Resistance at 15° C. in ρ in Ω/m./mm.²*	*Conductivity at 15° C. λ in Reciprocal Ohms or Siemens*	*Temperature Coefficient α at 15° C.*	*Density*
Aluminium	0·03	33·5	+ 0·0037	2·7
Aluminium-Bronze	0·13 to 0·29	3·45 to 7·7	+ 0·001	—
Bismuth	1·2	0·833	+ 0·0037	9·8
Copper	0·017 to 0·0175	57 to 58·8	+ 0·004	8·93
Copper Wire for Transmission Lines	0·0175	57	0·004	8·93
Amorphous carbon	40 to 60	0·025 to 0·017	—	1·85 to 2·05
Graphite	7 to 12	0·14 to 0·08	—	2·21 to 2·23
Gas Coke	100 to 600	0·00166 to 0·01	− 0·0003 to − 0·0008	—
Gold	0·022	45·5	+ 0·0035	19·32
Iron (Annealed)	0·016	—	+ 0·0045	7·85
Cast Steel	0·20	—	+ 0·0045 to + 0·005	7·8
Electric Steel (Stalloy)	0·125		—	7·2
Lead	0·21	4·76	+ 0·0041	11·37
Mercury	0·95	10·5	+ 0·0009	13·567
Nickel	0·1 to 0·12	8·33 to 10	+ 0·0041	8·9
Platinum	0·094 to 0·11	9·09 to 10·64	+ 0·0024	21·5
Tantalum	0·15 to 0·165	6 to 6·66	+ 0·003	16·6
Tin	0·12	8·33	+ 0·0045	7·29
Tungsten	0·07	14·275	+ 0·0051	18·8
Zinc	0·06	16·6	+ 0·0039	7·1

TABLE II.—LIQUID CONDUCTORS

Substance	*Specific Resistance ρ at 15° C. Ω m./mm.²*	*Conductivity λ at 15° C. $= \frac{1}{\rho}$*	*Temperature Coefficient α*
Copper sulphate:			
5 per cent.	5×10^5	2×10^{-6}	− 0·022
10 ,,	$3·1 \times 10^5$	$3·225 \times 10^{-6}$	
25 ,,	$2·3 \times 10^5$	$4·165 \times 10^{-6}$	
Zinc Sulphate:			
5 per cent.	$5·2 \times 10^5$	$1·92 \times 10^{-6}$	
10 ,,	$3·1 \times 10^5$	$3·225 \times 10^{-6}$	− 0·24
20 ,,	$2·3 \times 10^5$	$4·35 \times 10^{-6}$	
30 ,,	$2·1 \times 10^5$	$4·76 \times 10^{-6}$	
Sulphuric Acid:			
5 per cent.	$4·8 \times 10^4$	$2·08 \times 10^{-5}$	
10 ,,	$2·6 \times 10^4$	$3·85 \times 10^{-5}$	
20 ,,	$1·54 \times 10^4$	$7·14 \times 10^{-5}$	− 0·02
30 ,,	$1·35 \times 10^4$	$7·4 \times 10^{-5}$	

TABLE III.—PURE METALS

(*Fleming and Dewar*)

Metal	*Specific Resistance at 0° C. in Ohms per Centimetre Cube*	*Mean Temperature Coefficient α between 0° C. and 100° C.*
Silver (electrolytic and well annealed) .	$1{\cdot}468 \times 10^{-6}$	0·00400
Copper (electrolytic and well annealed) .	$1{\cdot}561 \times 10^{-6}$	0·00428
Gold (annealed)	$2{\cdot}197 \times 10^{-6}$	0·00377
Aluminium (annealed)	$2{\cdot}665 \times 10^{-6}$	0·00435
Zinc	$5{\cdot}751 \times 10^{-6}$	0·00406
Nickel (electrolytic)	$6{\cdot}935 \times 10^{-6}$	0·00618
Iron (annealed)	$9{\cdot}065 \times 10^{-6}$	0·00625
Platinum (annealed)	$10{\cdot}917 \times 10^{-6}$	0·003669
Tin (pressed)	$13{\cdot}048 \times 10^{-6}$	0·00440
Lead (pressed)	$20{\cdot}38 \times 10^{-6}$	0·00411
Bismuth (electrolytic)	110×10^{-6}	0·00433
Mercury (pure)	$94{\cdot}07 \times 10^{-6}$	0·00098

TABLE IV.—RESISTANCE ALLOYS

(*Fleming and Dewar*)

Alloy	*Specific Resistance ρ in Ohms per Centimetre Cube at 0° C.*	*Mean Temperature Coefficient α at 15° C.*	*Percentage Composition of Alloy*
Platinum-Silver .	$31{\cdot}582 \times 10^{-6}$	0·000243	Pt 33 : Ag 66
Platinum-Iridium .	$30{\cdot}896 \times 10^{-6}$	0·000822	Pt 80 : Ir 20
Gold-Silver . . .	$6{\cdot}28 \times 10^{-6}$	0·00214	Au 90 : Ag 10
Manganese-Steel .	$67{\cdot}148 \times 10^{-6}$	0·00127	Mn 12 : Fe 78
Nickel-Steel . . .	$29{\cdot}452 \times 10^{-6}$	0·00201	Ni 4, 35
German-Silver . .	$29{\cdot}982 \times 10^{-6}$	0·000273	Cu 50 : Zn 30 : Ni 20
Platinoid. . . .	$41{\cdot}731 \times 10^{-6}$	0·00031	Similar to German Silver but contains a trace of tungsten
Manganin . . .	$46{\cdot}678 \times 10^{-6}$	0·0000	Cu 84 : Mn 12 : Ni 4

General Notes on Resistance

When a pure metal is alloyed with a very small percentage of another metal, the specific resistance of the alloy is widely different according to the nature of the metals used. Thus, an alloy of pure copper with 3 per cent. aluminium has a specific resistance about 5½ times that of pure copper. The presence of a very small proportion of a *non-metallic* element in an otherwise pure metal has a very great effect in increasing the specific resistance of the metal: for example, pure iron with 0·01 per cent. of carbon has a specific resistance of about 10 microhms per centimetre cube, whereas an addition of 3·4 per cent. of silicon increases

TABLE V.—SPECIFIC RESISTANCE OF INSULATORS

Substance	*Specific Resistance ρ in Ohms per Centimetre Cube*	*Temperature in ° C.*
Bohemian Glass	61×10^{14}	60
Mica	84×10^{12}	20
Gutta-percha	$7{,}000 \times 10^{12}$	0
	450×10^{12}	24
Flint Glass	$1{,}020 \times 10^{12}$	60
Vulcanised India-rubber	$1{,}500 \times 10^{12}$	15
Shellac	$9{,}000 \times 10^{12}$	28
Pure India-rubber	$10{,}900 \times 10^{12}$	24
Ebonite	$28{,}000 \times 10^{12}$	46
Paraffin	$34{,}000 \times 10^{12}$	46
Linseed Oil	5×10^{10}	18
Olive Oil	1×10^{12}	18

the specific resistance about $4\frac{1}{2}$ times. Certain metallic elements also show the same effect: thus, platinoid has a specific resistance which is 30 per cent. greater than that of German silver, although it differs from it merely in containing a small trace of tungsten.

Special alloys are manufactured, of which the specific resistances are exceptionally high; such alloys are used when it is required to consume as much power as possible for a minimum amount of resistance material, such as in the case of a resistance furnace. A well-known alloy of this type is nickel-chrome, and in the following Table VI are given some data of this material.

TABLE VI

Size of Wire Diameter		*Resistance in ohms per 1,000 yards*	
S.W.G.	*Inch*	200° C.	400° C.
16	0·064	451	494
18	0·048	800	879
20	0·036	1,423	1,590
22	0·028	2,354	2,583

The current-carrying capacity of nickel-chrome for well-ventilated open spirals is given in Table VII.

Another well-known high-resistance material is "Eureka", that is, the copper-nickel alloy (60 Cu : 40 Ni), otherwise known as "constantan", having a specific resistance of $49 \times 10^{-6} \Omega$/cm. cm.^{2}, a characteristic property of this alloy being that its temperature coefficient is practically zero.

TABLE VII

Size	Rise in Temperature 100° C. Amperes	200° C. Amperes	Size	Rise in Temperature 100° C. Amperes	200° C. Amperes
22	1·6	2·1	14	6	10
20	2·0	3·2	12	10	15
18	3·0	4·2	10	16	24
16	4·2	7·0	8	22	30

The alloys of nickel and chromium were patented by A. L. Marsh in 1906 for high-temperature service. These alloys contain from 65 to 80 per cent. of nickel and from 15 to 20 per cent. of chromium, and in addition to their excellent electrical properties of high specific resistance they have proved to be immune from the destructive effects of oxidation even when continually exposed to the atmosphere at temperatures up to 1,100° C. For household electric radiators and heating appliances, as well as for industrial furnaces, these alloys are exclusively used. The specific resistance of these alloys is about 1·2Ω m. mm.^{2} at 1,000° C. For temperatures greater than about 1,200° C. silicon carbide heating elements are used. Another alloy which is now available and suitable for these purposes is a nickel-free alloy of iron, chromium, and aluminium.

It is to be observed that the majority of metals when in a finely powdered state are practically non-conductors, and a mass of metal powder or filings may be made to pass suddenly into the conducting state by being exposed to the influence of an electromagnetic wave. The same is true of the loose contact of two metallic conductors: thus, if a steel point presses lightly against a metallic plate of, say, aluminium, it is found that this contact, if carefully adjusted, is non-conducting, but that if an electromagnetic wave is generated in the neighbourhood, the contact suddenly becomes conducting. These characteristic features were utilised formerly as detectors for receiving radio signals.

Certain dry chemical compounds can be changed with extreme rapidity from very good conductors of electricity into almost perfect insulators by the application of a slight degree of heat. For example, lead-peroxide (PbO_2) has a specific resistance of about 2·5Ω/cm./cm.^{2}, the resistance varying with the pressure with which it has been compressed. At a temperature of about 150° C. lead peroxide is reduced to red-lead (Pb_3O_4), which has a specific resistance of about 60×10^6Ω/cm./cm.^{2} At slightly higher temperatures red-lead reduces to litharge (PbO), which is practically an insulator. This property of lead oxide is made use of in one form of lightning arrester.

The specific resistance of mercury has been investigated with exceptional care, as on its value is based the definition of the international ohm, as has been given already in Chapter I. The resistance of the metal selenium is affected by light, and it is a better conductor in the light than in the dark. This property is utilized for various practical purposes. The resistance of bismuth is increased when placed in a magnetic field, and if the field is transverse to a rod of bismuth the increase of resistance is far greater than when the direction of the magnetic field is in the direction of the axis of the rod. The following Table VIII gives the resistance of a bismuth wire when kept at constant temperature (19° C.) and placed in a transverse magnetic field (Henderson).

TABLE VIII

Intensity H of Transverse Magnetic Field in Oersted	*Resistance of Bismuth Wire*
0	1
12,500	1·63
27,450	2·54
38,900	3·34

At a temperature of 185° C. a transverse magnetic field of intensity 21,800 oersted increases the resistance of a bismuth wire 150 times. This increase of resistance of bismuth in the presence of a magnetic field is used for the purpose of exploring the distribution of intensity of such a field.

The specific resistance of liquid is, generally speaking, much higher than that of any solid conductor. One of the best conducting liquids is fused lead chloride, which has a specific resistance of about 370,000 microhms per centimetre cube, whereas the specific resistance of any alloy is seldom greater than about 100 microhms per centimetre cube.

A conductor material frequently used for overhead transmission lines is "aldrey", which is an alloy containing 0·3 to 0·5 per cent. Mg, 0·4 to 0·7 per cent. Si, 0·3 per cent. Fe, and the rest aluminium. At 20° C. the conductivity of aldrey is $\lambda = 30$, whereas for aluminium $\lambda = 34$ and for copper $\lambda = 57$ (see Table I). The resistance temperature coefficient is $\alpha = 0{\cdot}0036$ for aldrey, 0·004 for aluminium, and 0·0038 for copper. The safe stress in kilograms per square millimetre is 13 for aldrey, 8 for aluminium and 19 for copper.

Pioneer work on the measurement of the specific resistance of pure metals at very low temperatures was carried out by H. Kammerlingh Onnes at Leyden. It had already been deduced from theoretical considerations that the specific resistance of mercury at a temperature of 4·25° absolute (i.e. $-268{\cdot}75°$ C.) would still be measurable, but at

2 absolute (– 27° C.) would become almost indefinitely small. Onnes, however, found that at a temperature of 4·19 absolute the value of the specific resistance falls suddenly to an almost immeasurably small value. This critical temperature marks the transition from the conditions of normal conductivity to "supra-conductivity". When the temperature had fallen below the critical temperature it was found to be possible to pass a current at a density of 1,000 amperes per square millimetre through a conductor 1 metre long without any appreciable difference of potential between the ends.

By using 1,000 turns of lead wire for which the critical temperature is about 7·3 absolute, see Fig. 16, 0·1 mm. diameter, it was possible to obtain a magnetic induction density in air of 100,000 oersted.

By the use of a lead wire spiral with the two ends welded together by the oxy-hydrogen blow-pipe, the spiral was found to have a resistance of one twenty-billionth of an ohm when the temperature was reduced below the critical value, whereas at the normal temperature of the room the resistance of the spiral was 736 ohms. In this way a current of about 0·6 amps. excited in the spiral lasted for several days.

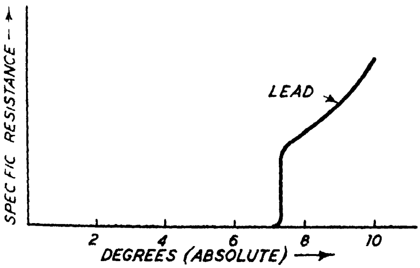


Fig. 16.

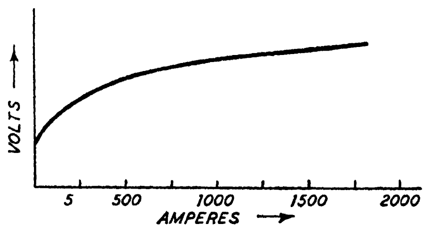


Fig. 17.

Resistance Units

The field of application for resistance units is extraordinarily extensive, and such units are constructed in accordance with an immense range of constructional types. In most cases resistance wire is used, mounted on an insulating support. For very heavy current loading the conductor may be formed as a tape or band. Instead of winding the wire on insulating supports it may be woven with asbestos filaments to form a tape, and in such a condition the conductor shows good cooling properties; but, most important of all, it possesses very small inductance. For small currents the resistance unit may be formed by winding the wire on an asbestos core, and such flexible units are used for radio receivers since they occupy very little space and can be used as connecting leads in suitable parts of the circuit.

Resistance units are also constructed of carbon and other materials,

but since in such cases the resistance is largely dependent upon the p.d., have large self-capacitance and, when used in amplifiers, develop " resistance noises ", they are in consequence little used. As coupling resistance, however, resistances of metal or carbon dust deposited on insulating bodies are very widely used. For example, resistances of the order of several megohms can be formed by dusting a layer of such powder on an insulating surface and then scraping away the dust in such a manner that a conducting zig-zag path is formed.

A synthetic resistance material is now manufactured having a ceramic base and of which the characteristic is such that its resistance decreases as the applied pressure increases and is consequently specially adapted for use with lightning arresters on overhead transmission lines. In this way, excess pressure rises on the line are avoided which would otherwise be developed when the lightning current discharges to earth through the resistance. In Fig. 17 is shown a typical shape of the characteristic for such a resistance material.

Liquid Conductors (Electrolytes)

The characteristic feature of a liquid conductor of electricity is that the conduction is accompanied by chemical action. Such a liquid conductor is termed an *electrolyte*, whilst the accompanying chemical action is termed *electrolysis*. Pure distilled water is practically an insulator, but by the addition of a small quantity of sulphuric acid it becomes an electrolytic conductor. The specific resistance of such liquid conductors ranges from about 1·3 ohms per centimetre cube for a 30 per cent. nitric acid solution, and still lower values for fused salts, to about 10,000 ohms per centimetre cube for pure river water, whilst for distilled water, alcohol, oils, etc., the specific resistance is practically infinity. All such liquid conductors become insulators when frozen.

A further characteristic of liquid conductors is the negative temperature coefficient of resistance, that is to say, the resistance decreases as the temperature rises, as is shown by Table II.

If a current of electricity is passed through acidulated water the consequent chemical action will release hydrogen at one pole and oxygen at the other pole, and this action is, in fact, the principle of electrolysis which was discovered in the year 1800 and which has achieved such immense industrial importance for the large-scale production of these gases. The principal features of the apparatus for performing the electrolysis of water are shown in Fig. 18. Two tubes *A* and *B* are filled with acidulated water and inverted in a vessel which is also filled with water. Near the end of each tube is a platinum electrode, and these electrodes are respectively connected to the terminals of the source of d.c. When the current passes in the direction shown in Fig. 18 the tubes *A* and *B* become gradually filled with gas, viz. hydrogen in the *B* tube, which contains the negative electrode, or *cathode*, and oxygen in the tube

A which contains the positive electrode, or *anode*. The volume of the hydrogen gas which collects in the tube *B* will be twice the volume of the oxygen gas which collects in the tube *A*. Although these gases are liberated at the respective electrodes, there is no trace of the free gases to be observed in the body of the liquid between the electrodes the gases are first seen in the immediate neighbourhood of the electrodes.

The action can be explained somewhat as follows. Suppose the water is acidulated with sulphuric acid (H_2SO_4). In Fig. 19 a chain of molecules of H_2SO_4 is shown extending between the electrodes. Considering, then, one of these molecules, say, *d*, it is assumed that the electric force which acts between the two electrodes exerts a directive action on the constituents of the molecules, viz. the hydrogen *ion* and the *sulphion* (SO_4), so that the hydrogen moves with the current and

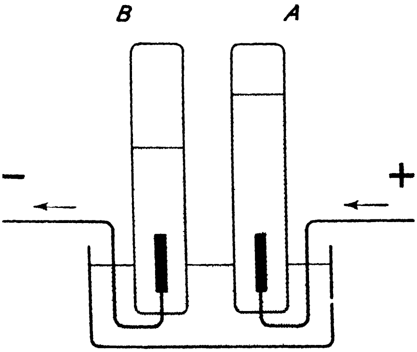


Fig 18.

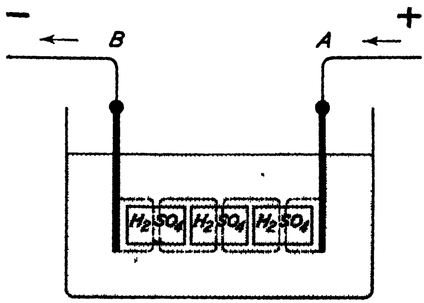


Fig 19

the sulphion moves in the opposite direction. The hydrogen of the molecule *d* will then combine with the sulphion of the molecule *c* and the hydrogen of the molecule *c* with the sulphion of molecule *b*, so that, actually, *hydrogen becomes released at the cathode B.*

Similarly, the sulphion of molecule *d* combines with the hydrogen of molecule *e*, and so on, until eventually the sulphion is released at the anode *A*. The sulphion SO_4, however, cannot exist in a free state and consequently attacks the water at the anode, combining with the hydrogen and releasing oxygen. The oxygen and hydrogen thus appear at the respective electrodes as though the water alone were decomposed by the action of the electric current. It is to be observed in this connection that the total amount of sulphuric acid in the water will remain constant

It is now known that chemical forces are of an electric nature and chemical affinity is an aspect of the mutual action of the electric fields of the atoms. Thus the force between the two oppositely charged ions

H_2 and SO_4, may be assumed to be roughly that given by Coulomb's Law of inverse squares, viz. (see expression (1), Chapter I),

$$E = \frac{1}{\varepsilon} \frac{q \times q^1}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where q and q^1 are the respective charges of the ions, d the distance between them, and ϵ the dielectric constant. Since the value of ϵ is about 80 times greater for water than for air, the value of the electric force between the ions H_2 and SO_4 in water will be only $\frac{1}{80}$th of the value when in air and under these circumstances the thermal motion may become sufficiently strong to *dissociate* these ions as soon as the acid is dissolved in the water. That is to say, *this dissociation takes place before any p.d. is connected across the electrodes A and B.* When the p.d. is applied, the individual ions having become already dissociated, are free to move in the manner described in the foregoing (see Fig. 19), or, in other words, an electric current will flow.

If the relationship between the p.d. and the current for such acidulated water be represented by a graph, it will be seen that, for constant temperature, a diagram such as that shown in Fig. 20 will be obtained. That is to say, for values of the p.d. less than the value e, which is a characteristic quantity for the liquid concerned, only a very small " leakage current " or " diffusion current " will flow, whilst for higher values of the applied p.d. a straight-line relationship will be obtained, as shown in Fig. 20, and which is in accordance with Ohm's Law. The relationship between current and applied p.d. as defined by this straight line will then be,

$$I = \frac{V - e}{R} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where R is the resistance of the electrolyte. The e.m.f. e is the " counter e.m.f. of polarisation " and corresponds to the chemical energy of dissociation of the water, viz., about 1·5 to 1·7 volts. It is seen, therefore, that electrolysis cannot begin unless the applied p.d. is equal to about 1·7 volts.

The Laws of Electrolysis

Suppose in Fig. 21 A, B and C represent three voltameters, and suppose A and B are connected in parallel and the parallel combination in series with C. If a current is passed through the circuit, it is found that the amount of chemical action in C is equal to the sum of the actions in A and B. That is to say, the amount of oxygen liberated in C is equal to the sum of the amounts of oxygen liberated in A and B and, similarly, the amount of hydrogen liberated in C is equal to the sum of the amounts of hydrogen liberated in A and B. Further, the amount of chemical action is found to be independent of the size of the electrodes of the

cell and is directly proportional to the time for which the current flows. These results are summarised in Faraday's first law of electrolysis, viz.

The chemical action, or mass of a substance liberated from an electrolyte, is proportional to the quantity of electricity which passes through the electrolyte.

Next, suppose a current is passed through dilute sulphuric acid between two platinum electrodes and through a solution of copper sulphate between two copper electrodes, these two electrolytic cells being connected in series. It will be found that the mass of hydrogen liberated at the cathode of the first cell is to the mass of copper deposited in the second cell as 1·008 : 31·8. In other words, when 1·008 mgm. of hydrogen have been liberated at the cathode of the first cell, 31·8 mgm. of copper will have been deposited at the cathode of the second cell. These numbers, however, are in the same ratio as the chemical equivalents of

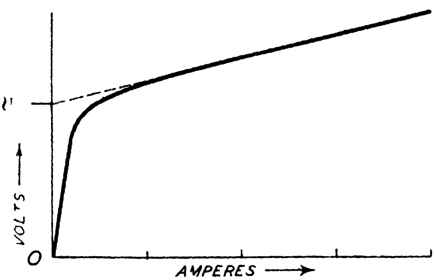


Fig. 20.

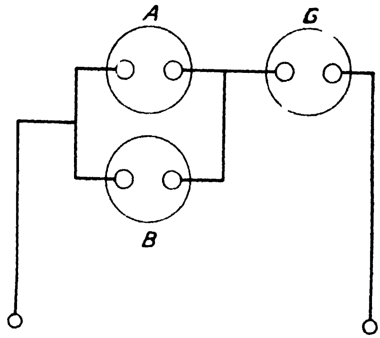


Fig. 21.

hydrogen and copper, respectively. Further, when 1·008 mgm. of hydrogen have been liberated at the cathode of the first cell, 8 mgm. of oxygen will have been released at the anode, and these numbers are also in the same ratio as the chemical equivalents of hydrogen and oxygen, respectively. These results are summarised in Faraday's second law of electrolysis, viz.

The mass of a substance liberated from any electrolyte when a given quantity of electricity passes through the electrolyte is proportional to the chemical equivalent of that substance.

Electrochemical Equivalents

In order to liberate 1·008 mgm. of hydrogen and 8 mgm. of oxygen by the electrolysis of acidulated water, the passage of 96,540 coulombs is necessary, that is to say, 1 coulomb liberates $1{\cdot}044 \times 10^{-5}$ gm. of hydrogen.

The mass of a substance in grams liberated by the electrolysis of a liquid

when 1 coulomb passes, is termed the electrochemical equivalent of the substance.

From the results given below, it is seen that the electrochemical equivalent is proportional to the chemical equivalent. For hydrogen, the chemical equivalent is 1, and hence the constant by which the chemical equivalent of a substance must be multiplied in order to obtain the electrochemical equivalent of that substance is $1{\cdot}044 \times 10^{-5}$.

A list of chemical equivalents and electrochemical equivalents is given in Table IX.

TABLE IX

Element	*Symbol*	*Atomic Weight*	*Valency*	*Chemical Equivalent*	*Electro-Chemical Equivalents (i.e. Gram per Coulomb)*
Aluminium	Al	27·1	3	8·96	0·000093
Bromine	Br	79 96	1	79·32	0·000828
Chlorine	Cl	35·45	1	35·17	0·000367
Copper (from cuprous salts)	Cu	63·6	1	63·09	0·000659
Copper (from cupric salts)	Cu	63 6	2	31·54	0·000329
Gold	Au	197·2	3	65·21	0·000681
Hydrogen	H	1·008	1	1	0·00001044
Iron (from ferrous salts)	Fe	55·9	2	27·73	0·000289
Iron (from ferric salts)	Fe	55·9	3	18·485	0·000193
Lead	Pb	206·9	2	102·63	0·0010714
Mercury (from mercurous salts)	Hg	200	1	198·4	0·002071
Mercury (from mercuric salts)	Hg	200	2	99 2	0·001036
Nickel	Ni	58·7	2	29·12	0·000304
Nitrogen	N	14·04	3	4·64	0·0000484
Oxygen	O	16 00	2	7·94	0·0000829
Potassium	K	39·15	1	38·84	0·000405
Silver	Ag	107 93	1	107·07	0·001118
Sodium	Na	23·05	1	22·86	0 0002386
Tin (from stannous salts)	Sn	119	2	59·02	0·000616
Tin (from stannic salts)	Sn	119	4	29·51	0·000308
Zinc	Zn	65 4	2	32·44	0·000338

The amount of a substance liberated by electrolysis in a given time is therefore a measure of the current which passes through the electrolyte. So many causes, however, tend to affect the perfect accumulation of the ions at the electrodes that an accurate determination of the electrochemical equivalent or, conversely, of the magnitude of the current in this way is not easy. Large copper electrodes in copper sulphate give good results, but the most satisfactory arrangement is that used by Lord Rayleigh to find the electrochemical equivalent of silver. A platinum bowl is filled with a neutral solution of silver nitrate containing about 15 parts of salt to 100 parts of water. A silver plate wrapped in filter paper dips into the solution and the current passed from silver

plate as anode to platinum bowl as cathode, the current density being about 0·01 amp. per square centimetre. The bowl is weighed before and after the passage of the current and the weight of silver deposited is thus obtained. In this way it has been found that a current of 1 amp. passing for 1 sec., that is to say, the passage of 1 coulomb of electricity produces a deposit of 0·001118 gm. of silver. This determination has been used as the definition of the ampere (see also Chapter I), viz.

A current of 1 amp. is such that, when flowing through a solution of silver nitrate, it deposits silver at the rate of 1·118 mgm. per second.

The chemical equivalent of silver is 107·07, and hence the ratio .

$$\frac{\text{electrochemical equivalent}}{\text{chemical equivalent}} = \frac{0{\cdot}001118}{107{\cdot}07} = 1{\cdot}044 \times 10^{-5},$$

and this number is the same for all ions. In Table IX will be found useful data relative to the more common elements. In each case the electrochemical equivalent is obtained by multiplying the chemical equivalent by $1{\cdot}044 \times 10^{-5}$. The atomic weight of oxygen is taken to be 16, which gives the atomic weight of hydrogen as 1·008

Energy Involved in Electrolytic Action

Consider the electrolysis of acidulated water, as described on page 51. As stated on page 54 it is known that 1 coulomb of electricity releases $1{\cdot}044 \times 10^{-5}$ gm. of hydrogen. If a current of i amp. flows for t sec. the amount of hydrogen released will be,

$$i.t(1{\cdot}044 \times 10^{-5}) \text{ gm.}$$

Now experiment shows that when 1 gm. of hydrogen combines with oxygen to form water, 34,000 calories are released, that is, $14{\cdot}2 \times 10^{11}$ ergs Hence to release a mass of $i.t(1{\cdot}044 \times 10^{-5})$ gm. from water requires an expenditure of $i \times t \times 1{\cdot}044 \times 10^{-5} \times 14{\cdot}2 \times 10^{11}$ ergs., that is, $1{\cdot}485\ i \times t$ joules. If v is the p.d. at the electrodes during the electrolysis, the energy supplied by the current in sending $i \times t$ coulombs through the electrolyte is $v \times i \times t$ joules. Hence the applied p.d. must be at least about 1·5 volts, and for this reason one Daniell primary cell, for example, is not sufficient to decompose water, since its e.m.f. is only 1·07 volts.

Electrolysis and the Electron

The facts of electrolysis have led to the conclusion that all monovalent atoms carry the same charge of electricity which is regarded as an ultimate unit of electric quantity and is termed the *electron*. For each atom of a substance released with its charge at the electrode, the corresponding quantity of electricity must flow through the electrolyte. For example, if N be the number of atoms in a mass of 1 gm. of hydrogen and e the electric charge associated with each atom, then

$$N \times e = 96{,}540 \text{ coulombs,}$$

since 96,540 coulombs are required to liberate 1 gm. of hydrogen as already stated on page 53. Now N is known to be about $60{\cdot}6 \times 10^{22}$ at 0° C. and 760 mm. barometric pressure, so that

$$e = \frac{96{,}540}{60{\cdot}6 \times 10^{22}} = 1{\cdot}59 \times 10^{-19} \text{ coulomb}$$

$$= 4{\cdot}77 \times 10^{-10} \text{ electrostatic c.g.s. units} \quad . \quad . \quad (15)$$

and this is the magnitude of the charge of an electron [see also expression (1)]. The quantity N is known as Avogadro's Number or, sometimes, Loschmidt's number (see also Chapter IV, page 115). If the substance has a valency greater than unity, the atom of the substance is charged with the same number of electrons as the number which denotes the valency. Thus, an atom of a divalent element carries a charge of two electrons, and so on.

Electrolysis in Practice

Electrolytic action has most important industrial applications such as the recovery of metals from the ores by electrolysis, notably aluminium : the refining of copper for use as electrical conductors : electro-plating : electro-typing : the isolation of hydrogen and oxygen from water. Electrolytic meters are used for the measurement of the quantity of electricity supplied to a consumer.

Electrolysis is a serious source of corrosion in gas and water pipes which are laid underground. For example, when such pipes are laid parallel and near to tramway lines which form the return circuit for the tramway supply, the current leaks from the tram lines and flow into and along the pipes, to return to the negative main of the supply. At the place where the current leaves the pipes, corrosive action is developed during electrolysis of the liquid in the neighbouring earth material, and unless special precautions are taken, the pipes may in time become completely destroyed.

When iron or steel is immersed in water (especially salt water) it quickly corrodes and the corrosion can be attributed to electrolysis taking place between different parts of the iron which are not in the same physical condition, e.g. in boilers in which certain parts have become hardened by hammering, caulking, etc. Corrosion is specially serious when two of the immersed parts of the same structure are of different metals. In order to prevent such corrosion it is frequently the practice to provide zinc plates immersed in the water and in good electrical connection with the iron or steel plate which it is desired to protect. The result of this arrangement is that the corrosion is transferred from the iron to the zinc, and in order to maintain effective protection, the zinc plates must be frequently renewed.

If two metals of different electrolytic voltages are continuously in contact in a moist atmosphere, "contact corrosion" occurs. The metal

of low electrolytic potential is destroyed by the one at the higher electrolytic potential. Practical experience shows that real danger of corrosion only begins when a voltage difference of more than 0·3 volt has been developed. The following Table shows the electrolytic potential of a number of pure metals referred to hydrogen as the datum.

TABLE X

	Volts		*Volts*
Magnesium	– 1·8	Nickel	0·20
Aluminium	– 1·45	Tin	– 0·146
Manganese	– 1·10	Lead	– 0·132
Zinc	– 0·77	Hydrogen	0
Chromium	– 0·56	Copper	+ 0·35
Iron	– 0·43	Silver	+ 0·80
Cadmium	– 0·42	Mercury	+ 0·86
Cobalt	– 0·23	Gold	+ 1·5

Electric Accumulators or Secondary Cells

If a current is sent through acidulated water between platinum electrodes a counter e.m.f. is developed at the electrodes (see also page 52). If the source of electric supply is then disconnected and the electrodes of the cell are connected by a wire, a current will flow through this electrolytic cell for a short time, the direction of the current being the reverse of that which flowed when the electric source of supply was connected to the cell. Such an arrangement is a very simple form of electric accumulator or secondary cell, although of no practical value in such a simple form, since the storage capacity is extremely small.

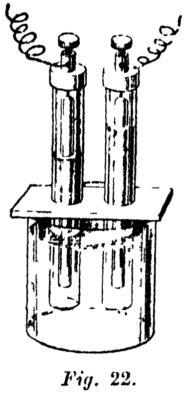


Fig. 22.

The earliest form of practical accumulator was Grove's "Gas Battery" invented in 1842. One cell of such a battery is shown in Fig. 22. Two glass tubes, each closed at one end, are filled with, and dipped into, acidulated water. Each tube is sealed at the upper end and contains a platinum plate joined to a platinum wire which is connected to a terminal outside the tube. Upon passing a current through the cell, hydrogen gas collects in the tube from which the current leaves, and oxygen gas in the tube at which the current enters, the volume of hydrogen being twice that of oxygen. If the cell is now removed and the two terminals joined together, a current will flow from the tube containing the oxygen through the outer connecting wire to the tube containing the hydrogen and so through the cell. That is to say, the direction of the current through the cell will now be the reverse of that supplied to the cell when it was being "charged". The current will continue to flow until the hydrogen and oxygen in the respective tubes have disappeared.

Lead Accumulators

In 1860 Planté invented the first lead cell, which consisted of two lead plates immersed in dilute sulphuric acid. When the cell was charged by connecting the plates to the poles of a d.c. supply of sufficient high e.m.f. to send a current through the cell, lead peroxide (PbO_2) formed at the anode, i.e. the lead plate which was connected to the positive terminal of the supply, but the lead plate which formed the cathode remained unaltered. When the cell was discharged, each of the lead plates became coated with lead sulphate ($PbSO_4$). When again charged, the $PbSO_4$ on the positive plate became changed to lead peroxide (PbO_2) and the $PbSO_4$ on the negative plate became reduced to spongy lead.

Planté discovered that by charging and discharging, and then charging in the reverse direction, the capacity of the cell could be greatly increased. By repeating this procedure many times the cell attained its maximum capacity, and this process is technically known as "forming" the plates. The surfaces of the plates were thus rendered more porous and the active or effective surface area enormously increased, although the mechanical strength was greatly reduced in this way. In modern cells of the Planté type the plates are formed chemically, e.g. by boiling in dilute nitric acid, or electrolytically, by adding sodium nitrate to the electrolyte.

In 1880 Faure introduced the form of lead cell in which the active material was applied to the plates as a paste instead of being produced by "forming" the plates. Faure used a paste of red lead (Pb_3O_4) and sulphuric acid for both plates. The mixture formed lead sulphate ($PbSO_4$), and this, on the positive plate, was converted to PbO_2 by the liberated oxygen during the first charge, whilst the lead sulphate on the negative plate was reduced to spongy lead. In modern practice the Faure type of plate is formed as a grid to support the paste. For the negative plates a paste of litharge (PbO) and sulphuric acid is used, and for the positive plate a paste of red lead and sulphuric acid. During the first charge the red lead of the positive plate is oxidised to lead peroxide and the litharge of the negative plate is reduced to spongy lead.

The Alkaline Accumulator

The type of secondary cell was invented by Edison, and has now become widely applied to a great variety of purposes. The purpose of the invention was to provide a more robust construction and one which would be lighter than the lead cell for a given capacity, and to avoid the use of lead, sulphuric acid and celluloid. The positive plates have as active material, nickel hydroxide mixed with fine metallic nickel flake to render it more conducting. Helical tubes made from perforated nickel steel strips are filled with alternately arranged thin layers of nickel hydroxide and fine metallic nickel flakes. The tubes are surrounded by solid steel rings for the purpose of consolidating the active material.

A complete positive plate is an assemblage of such units, a nickel steel frame being used as a mounting for the rows of tubes. The negative plates are an assemblage of small units of oblong pocket form containing iron oxide incorporated with a trace of mercury to improve the conductivity. The assembled plates are contained in a nickelled, cold rolled, sheet-steel box welded at the seams. The electrolyte is a solution of potash (KHO) in distilled water with a small percentage of lithia. The average working voltage of the Edison cell is about 1·2 volts per cell as compared with the value of about 2 volts for a lead accumulator cell.

Heat Energy of an Electric Current

In the year 1841 J. P. Joule had shown that the heat produced by an electric current flowing in a conductor is proportional to the resistance of the conductor, the square of the current strength, and the time for which the current flows. That is

$$H = 0{\cdot}239(I^2.R.t) \text{ calories} \qquad . \quad . \quad . \quad (16)$$

where I is the current in amperes,
R is the resistance of the conductor in ohms,
t is the time in seconds for which the current flows.

The "calorie" is the heat necessary to raise the temperature of 1 gm. of water through 1° C. in the neighbourhood of the temperature of 4° C. This definition refers to the "gram-calorie" in distinction to the "kilogram-calorie", which is the amount of heat required to raise the temperature of 1 kg. of water through 1° C.

In 1843 Joule found that in order to raise the temperature of 1 lb. of water through 1° C. it was necessary to supply 1,399·5 ft.-lb. of energy. That is to say, in order to raise the temperature of 1 gm. of water through 1° C. an amount of energy equal to 3·08 ft.-lb. must be supplied, or, in other words,

$$1 \text{ calorie} = 3{\cdot}08 \text{ ft.-lb.} \qquad . \quad . \quad . \quad . \quad (17)$$

From the expressions (16) and (17) the heat generated by an electric current of I amps. flowing through a resistance of R ohms for a time of t sec. is

$$0{\cdot}239(I^2.R.t) \times 3{\cdot}08 = 0{\cdot}737(I^2.R.t) \text{ ft.-lb.};$$

but

$$\begin{aligned} I^2Rt &= (\text{Volts} \times \text{Amperes} \times \text{Seconds}) \\ &= \text{Watt-seconds (i.e. Joules).} \end{aligned}$$

from which it follows that

$$1 \text{ joule} = 0{\cdot}737 \text{ ft.-lb.},$$

or

$$746 \text{ watts} = 550 \text{ ft.-lb. per second} = 1 \text{ horse-power} \quad . \quad (18)$$

For example,

$$\begin{aligned} 1 \text{ B.o.T. unit of electric energy is} &= 1 \text{ kWh.} \\ &= 1{,}000 \times 3{,}600 = 3{\cdot}6 \times 10^5 \text{ joules} \quad . \quad . \quad (19) \end{aligned}$$

that is,

$$1 \text{ B.o.T. unit} = 0{\cdot}239 \times 3{\cdot}6 \times 10^5 = 860{,}000 \text{ calories} \quad . \quad (20)$$

Also, one gallon of water = 8 pints = 4·546 litres = 4,546 gm., so that it is easy to show that 1 B.o.T. unit will raise the temperature of nearly 2 gallons of water from 0° to 100° C.

Heating and Cooling Curves

Suppose heat is being supplied to a homogeneous conductor at a constant rate of Q calories per second. If, for example, an electrical conductor of resistance R ohms is heated by carrying a current of I amps., then, as shown in the previous paragraph

$$Q = 0{\cdot}239 I^2 Rt \text{ calories per second.}$$

Let c be the specific heat of the material of the conductor, that is, the number of calories necessary to raise the temperature of 1 gm. of the material through 1° C.

Let M gm. be the mass of the conductor: S sq. cm. the superficial area: and λ the cooling constant—that is, the number of calories lost by the conductor to the surrounding medium in one second per *square centimetre* of surface and per 1° C. temperature difference between the conductor and the surrounding medium. (*Note.*—The constancy of λ is "Newton's Law of Cooling".)

In the time δt sec. the heat supplied to the conductor is $Q\delta t$ calories. Let θ be the temperature difference between the conductor and surrounding medium and $\delta\theta$ the rise of temperature in the time δt. Then the heat supplied $Q\delta t$ is partly expended in raising the temperature of the conductor by an amount $\delta\theta$ and the remainder is transferred to the surrounding medium, so that

$$Q\delta t = c\,.\,M\,.\,\delta\theta + S\,.\,\lambda\,.\,\theta\,.\,\delta t.$$

Eventually, a condition of equilibrium will be reached in which the heat supplied to the conductor just balances the heat which is lost to the surrounding medium, and the temperature will then remain constant.

Let Θ be this steady temperature difference between the conductor and the surrounding medium, then

$$Q = S\,.\,\lambda\,.\,\Theta$$

so that,
$$Q\,.\,\delta t = c\,.\,M\,.\,\delta\theta + \frac{\theta}{\Theta}Q\delta t$$

that is,
$$\frac{\delta\theta}{\Theta - \theta} = \frac{Q\,.\,\delta t}{\Theta c M} = \frac{\delta t}{\tau}$$

where $\tau = \dfrac{\Theta c M}{Q}$ and is the time which would be required to produce

the final temperature rise Θ if no heat were transferred to the surrounding medium. Then, since $\tau = \dfrac{\Theta c M}{Q}$ and $Q = S\lambda\Theta$, it follows that

$$\tau = \frac{cM}{S\lambda}$$

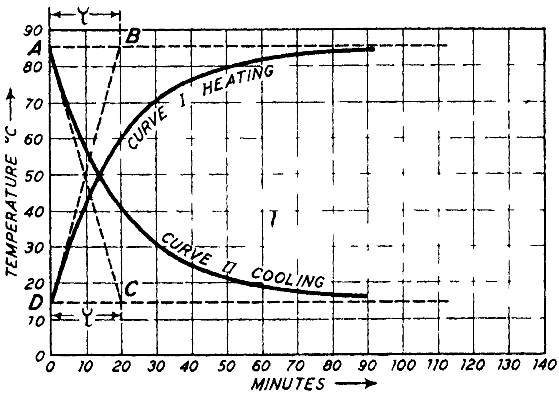
and is a constant quantity which depends upon the size, shape, and material of the conductor but is independent of the rate at which heat is supplied. The factor τ is termed the " heating time constant " of the conductor. Since

$$\frac{\delta\theta}{\Theta - \theta} = \frac{\delta t}{\tau}: \int_{\theta=0}^{\theta=\theta} \frac{d\theta}{\Theta - \theta} = \int_{t=0}^{t=t} \frac{dt}{\tau}$$

then taking the initial conditions to be $t = 0$ and $\theta = 0$, it follows that

$$\theta = \Theta(1 - e^{-t/\tau}) \quad . \quad . \quad . \quad (21)$$

where $e = 2.73$ and is the base of natural logarithms.



This equation gives the temperature rise θ above the surrounding medium at any time t in terms of the final temperature rise Θ and the time constant τ. In Fig. 23 a temperature curve I is shown as calculated from this equation and for the conditions that the time constant $\tau = 1{,}200$ sec. and the final temperature rise is $\Theta = 70°$ C., that is, an initial temperature of 15 C. is assumed and a final temperature of 85 C. The equation for the temperature rise is thus

$$\theta = 70(1 - e^{-t/1{,}200}).$$

Differentiating θ with respect to t for the equation (21), gives

$$\frac{d\theta}{dt} = \frac{\Theta}{\tau} e^{-t/\tau}.$$

At the time $t = 0$: $\tau = \dfrac{\Theta}{\left.\dfrac{d\theta}{dt}\right]_{t=0}}$

so that the time constant is the time which would be required to reach the final temperature if the initial rate of rise were to be maintained constant. In Fig. 23 the time constant τ is given by the intercept AB on the horizontal line through A of the tangent through D of the heating curve DB.

It is easily shown that if a homogeneous body is brought to a temperature Θ C. above the surrounding medium and is then allowed under steady conditions, the cooling curve will be given by

$$\theta = \Theta . e^{-t/\tau} \qquad . \quad . \quad . \quad . \quad (22)$$

This curve is marked II in Fig. 23 and the Curves I and II shown in this Fig. 23 are respectively the heating and cooling curves of the same body.

The time which elapses before the temperature of the heated body rises to within x per cent. of the calculated final temperature is found as follows:

$$\frac{\Theta - \theta}{\Theta} = \frac{x}{100} : \text{ so that } e^{-t/\tau} = \frac{x}{100} : \text{ or } t = \tau \log_e\left(\frac{100}{x}\right) \qquad . \quad . \quad (23)$$

In the following table are shown some corresponding values of x and t:

x	10%	5%	2·5%	1%	0·5%
t	2·3τ	3τ	3·7τ	4·6τ	5·3τ

It is seen, for example, that after a time equal to 4·6x (time constant) the temperature has reached a value which is only 1 per cent. less than the final calculated value. Similar considerations apply to the cooling curve II. From the equation (23) it follows that the temperature rises to within x per cent. of its final value in a time which is independent of Q, the constant rate at which heat is being supplied to the body.

The value of the time constant τ for a given body may be determined from its cooling curve II by drawing the tangent to the curve at origin as shown in the diagram. Further, if any two points on the heating curve be taken, and if these points are respectively defined by the quantities $\theta_1 : t_1$ and $\theta_2 : t_2$, then

$$\Theta = \frac{\theta_2 - \theta_1}{e^{-t_1/\tau} - e^{-t_2/\tau}}$$

so that $S.\lambda = \dfrac{Q}{\Theta}$ is found and, knowing τ and $S.\lambda$, all the necessary data are known for constructing the heating curve of the body for any given constant rate of heat supply Q.

Fuses

When a current passes through a wire, the temperature commences to rise and would continue to rise indefinitely if no heat were lost to the surrounding medium. The temperature which is actually attained is that at which the rate of loss of heat to the surrounding medium is equal to the rate at which the heat is being developed in the wire. By Newton's law of cooling, the loss of heat by radiation per unit of area surface at each instant is proportional to the excess of temperature of the wire above the surrounding medium.

Let l cm. be the length of the wire,
d ,, ,, ,, diameter of the wire,
ρ be the specific resistance in ohms per centimetre cube,
R ohms be the resistance,
I amperes be the current,
θ be the temperature rise above the surrounding medium,
λ be the radiation constant.

Then, the heat generated $= 0{\cdot}24I^2.R$ calories per second
and the heat radiated is $= \pi d.l.\lambda.\theta$ calories per second,
consequently, $\pi d\,l.\lambda.\theta = 0{\cdot}24I^2R,$
from which it is found that $I = kd^{3/2}\sqrt{\theta}$ amps. . . . (24)

where k is a constant. Hence for a wire of given material, the current which is necessary to fuse the wire (the fusing temperature being θ) is proportional to the diameter of the wire raised to the power 1·5. Actually, however, the practical fuse problem is not quite so simple, since the fixture of the ends of the wire contribute to the dissipation of the heat, and in cases where two or more wires are adjacent, the mutual heating effects of the wires influence the result. The formula deduced in the foregoing, therefore, is only to be taken as a general indication of the way in which the fusing current varies with the diameter of the wire.

In recent years great progress has been made in the development of fuses, particularly for use with high-tension systems, and some account of these developments will be found in *Engineering*, page 264, October 3, 1941

Insulators

An electric insulator is a substance which has an extremely high specific resistance, viz. of the order of magnitude given in Table V, page 46. Such substances are also termed "dielectrics", and this term is more particularly associated with the insulation layers of electric condensers.

Another important characteristic of a dielectric is its "angle of loss", which may be defined as follows with reference to Figs. 24 and 25. In a perfect condenser in which no dielectric losses occur, the current

vector corresponding to an alternating p.d. applied to the terminals of the condenser will be at right angles to the pressure vector, as shown in Fig. 24 (see also Chapter XI). In general, however, there will be power losses due to both dielectric hysteresis and dielectric insulation resistance, in consequence of which the current vector will be displaced by less than 90 ahead of the pressure vector. In other words, the current vector will have a component in phase with the pressure vector as shown in Fig. 25. The angle δ by which the current vector is displaced from the ideal position of 90° ahead of the pressure vector is

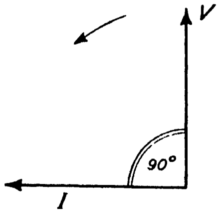


Fig. 24.

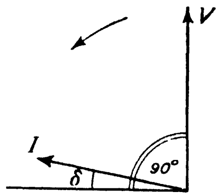


Fig. 25.

the "angle of loss", and for all good insulating materials this angle is very small. The angle δ is usually defined by its tangent, and consequently, in practice, by the term "angle of loss" is generally understood the quantity $\tan \delta$. In Table II, Chapter IV, are given a list of the more important insulating materials with the corresponding values of the dielectric constant ϵ, the angle of loss, $\tan \delta$, and the dielectric breakdown strength in kilovolts per millimetre. It is to be observed that (for convenience) the quantity given in the second column of this table is $10^4 \times (\tan \delta)$, whilst the numbers in brackets give the frequency for which the individual values of the dielectric constants have been measured.

Insulation Materials

So long as electrical engineers were only concerned with low-tension work, no serious difficulties were encountered as regards insulation. When, however, considerations of efficiency and economy required the adoption of high pressures both for generators and for transmission systems, a consequent requirement arose for an increase of the temperature at which the insulation could be worked, and it was soon found that the problems of insulation under such conditions became of serious significance. Cases frequently occurred of inexplicable breakdowns of the insulation and gave rise to serious disturbances of the continuity of the electric supply as well as endangering the safety of the whole plant. It became obvious that until a thorough knowledge was obtained of the physical phenomena associated with dielectric stress, it was hopeless to attempt to solve some of the more important practical problems

encountered in changing over to the use of machines of very large outputs and transmission systems of very high pressures.

During the last two decades, fundamental research on the electrical strength of insulating materials has been undertaken by investigators in many parts of the world and tolerably clear ideas have now been formulated as regards the electric strength of gaseous and liquid insulating materials, but the problems relating to solid insulators have been found to be much more complex. For this type of insulation it was generally agreed that the cause of breakdown was of a thermal nature and the phenomenon was accordingly termed the "heat rupture" of the material. Starting from the fact that no substance is a perfect insulator but that all have a definite (though small) conductivity, it was realised that the conductivity was not homogeneous throughout the body of the material, and at those placed where the conductivity was greatest the heat generated was also greatest, and this in turn further increased the conductivity. If the heat developed in this way does not become dissipated sufficiently, the heat balance is disturbed with the eventual destruction of the insulation material. Although this view can be said to give, on the whole, a correct picture of the sequence of events which culminate in breakdown, it nevertheless requires a great deal of co-operative research on the part of chemists, manufacturers and designers, before the practical problems of insulation technique can be fully solved.

The primary drawback of the commonly used insulating materials for machines, transformers, and cables, is the relatively low temperature rise which can be safely allowed. For temperatures which are appreciably greater than 100 C. such insulating materials lose to a great extent their mechanical and electrical strength. The greater the proportion of organic material used, the lower is the permissible temperature rise, and, consequently, upon this limitation will depend the output of electrical machines. If it were possible to raise the upper limit of permissible temperature, then the weight of electrical machines, transformers, and other appliances could be correspondingly reduced. For example, this would be an immense advantage as regards electrical vehicles and would correspondingly increase the carrying capacity, that is the useful load, of such vehicles. From such considerations it became clear that indefatigable research was urgently needed with a view to discovering insulating materials which would stand with safety, higher temperature rises than those at present in use. A further serious drawback of the existing insulating materials is the fact that the coefficients of expansion are different from those of the electrical conductors which they are required to insulate. This causes periodic changes in the stresses to which the insulating material is subjected as the electrical load changes, and this leads to a more or less gradual deterioration of the electric strength, eventually leading to a breakdown. Changes in

the mechanical stress of insulating materials caused by changes in the electric load and consequent changes in the heating conditions, form the most prolific sources of deterioration of electric and mechanical strength. A still further defect of many of the modern insulating materials is their sensitivity to moisture.

Solid Insulators

These are, in general, synthetic materials and a distinction must be drawn between the insulating base itself and the binding material with which it is moulded and which must also have suitable electrical properties. The oldest and even to-day very widely used basic insulating materials are of organic origin, that is, textile fibres, threads, and filaments, such as silk, cotton, and flax, which can be spun or woven. In the untreated condition, however, the electric strength of such materials is little better than air, and in fact, owing to the amount of moisture due to their porous nature, which they will absorb, the electric strength will generally be less than that of air. In order to improve their electric strength they must be impregnated with a suitable shellac or varnish. When it is necessary to reduce the insulating material to an absolute minimum, as, for example, in those cases in which the available space is relatively small, as in measuring instruments and for fractional horse power motors, silk or enamel insulated wires have hitherto been used whereas for the normal insulation of the conductors of motors, cotton insulation is used. Instead of silk, however, it is now becoming common practice to use spun cellulose, and cellulose thread or paper instead of cotton.

One of the most important basic insulating materials is paper. It is used for insulating the iron stampings for electrical machines and transformers, for insulating cables, and for the manufacture of tough paper insulating accessories such as tubes and sheets in combination with synthetic resins, and it was long thought that the only textile material which could satisfactorily withstand the temperature limit requirements was paper. Recent developments, however, have shown that cellulose paper can be manufactured which shows satisfactory performance and, with the suitable preliminary treatment, cellulose paper has proved to be equally effective as the best linen-made paper, and, in fact, linen paper is less able to withstand attack from mineral oil than cellulose paper. There are now a great variety of cellulose papers available for insulation purposes, the only insulation purpose for which it is not yet able to supplant linen paper being for condensers, for which the requirements are exceedingly thin paper (7 to 8 microns*), and so far nothing has yet been found to equal linen paper for this purpose.

Very important insulating materials for machines and transformers are presspahn and allied substances which can be manufactured in rolls

* 1 micron = 10^{-6} metre.

or sheets and the basic materials of which are the same as for paper. Since this material is mainly subjected to bending stresses, it must possess great toughness and great resistance to temperature deterioration.

Synthetic Resin Insulating Materials

Of natural resins used for insulating purposes the oldest and still the most important is shellac. One of the drawbacks of shellac in its rivalry with the synthetic resins has been its comparatively low softening point unless previously heat-cured for long periods. Chemists have, however, succeeded in improving the thermal characteristics and have also successfully overcome the "greening" of copper in contact with shellac. At the same time, this improved shellac is able to show an increased resistance to water and petroleum hydrocarbons.*

The most important synthetic resins are the thermo-setting phenol-formaldehyde group. During the past few years they have formed the raw material for an immense range of finished products (plastics), of which "bakelite" is one of the most familiar examples. They are heat resisting and impervious to water, and their initial dielectric values are well maintained over long periods of hard service These resins have been brought to such a high stage of development as regards both their electrical and mechanical properties, and their cost so reduced, that they can now be considered to be on a competitive basis with the older materials such as rubber, shellac and porcelain. Although originally developed for electrical purposes, they are now used in a great variety of different industries. Their main drawback for electrical purposes is the tendency to "track" under discharge conditions, and in order to minimise this defect of surface leakage, moulded materials are often coated with an alkyl resin solution such as glyptal varnish.

The B.S.S. No. 771 refers to five grades of moulding of this type of material, viz. two which are wood-filled for general purposes, two which are fabric-filled with shock-resisting properties, and one, which possesses the highest heat-resisting quality, is reinforced by an inorganic filling material such as mica or asbestos.

Tubes and cylinders of synthetic resin-bonded paper are available, having the requisite mechanical strength, and are used for the manufacture of insulating tanks to hold oil-immersed apparatus and for testing purposes. Synthetic resin-bonded paper is used in high-tension work for "post" insulators, leading tubes, and for current transformers, their outstanding advantage being shock-resisting properties to short-circuits and their high oil-resistant quality. For pressures up to 3,000 volts this material is used for insulating the slots in electrical generators and motors, and when immersed in oil can be used for the highest voltages. For oil switches it is used as leading-in tubes and for explosion chambers: for transformers it is used for terminal insulation and as supports

* See also G. E. Haefely: *Journal, I.E.E.*, May 1941, Part I, p. 179.

between coils as well as for cylindrical insulating tubes. It is also very widely used for light-current work and for radio engineering.

As a substitute for paper insulation for use with very high frequencies, materials which have been developed from "Trolitul" (a synthetic form of paraffin) are now available. Their outstanding electrical characteristic is their very low loss angle (see Table II Chapter IV) at very high frequencies. These materials can be manufactured in the form of flexible fibres and tapes without impairing its electrical properties. The angle of loss is $\tan \delta = 2 \times 10^{-4}$ as compared with 150×10^{-4} for paper at high frequencies. They also have the advantage of being completely non-hygroscopic. Long fibres can be spun into threads, and these are used for insulating cables for ultra high-frequency work such as for co-axial cables for television and telecommunication.

Some Ceramic and Other Insulation Materials

PORCELAIN has been for a long time and still is an indispensable material for the insulation of power transmission lines and for many other outdoor purposes of insulation. Although no synthetic substance has yet been found to take its place in its ability to stand up to all weather and atmospheric conditions, yet synthetic substances, due to their greater toughness, accuracy in dimensions and machineability, have replaced porcelain for insulating parts which are not exposed to the weather or working under oil or compound.

Soft felspathic porcelain fired in an oxidising atmosphere is now used and gives improved physical properties and greater consistency than that which was manufactured some years ago. If still better physical properties are required, and if expense is of secondary consideration, steatite (soapstone) is now used. The loss angle of this material is lower than that of porcelain, and recent developments in the use of bonding materials for soapstone have resulted in an appreciable reduction in the loss angle, so that suitable ceramic materials are now available for high-frequency dielectric purposes. By varying the ratio of the magnesia (MgO) to the silica (SiO_2) content of soapstone, still better results have been obtained. *Frequentite* (see Table II, Chapter IV) is representative of this class. By adding titanium dioxide (rutile) the dielectric constant of the material is much increased, and by varying the ratio of TiO_2 to the other constituents a range of ceramic materials is obtained with dielectric constants varying from 40 to 100, as, for example, *condensa C* (see Table II, Chapter IV).

GLASS.—When silica (SiO_2) is fired with, for example, potash, soda, lime and lead oxide, glass is formed. Although glass has always been recognised as a good insulating material it is only recently that its full possibilities are being exploited for practical purposes. Recent experience shows that glass insulation of the line-and-suspension type are becoming

formidable rivals to porcelain for many purposes. Apart from its good electrical properties, its resistance to high temperatures and moisture are valuable qualities, and for some purposes its transparency is also advantageous. Now that the production of alkali-free glass in the form of a fabric spun and woven from very fine thread has been established, a wide field of application has been opened up. As an inorganic textile material fibre-glass is more suitable than asbestos, due to its greater uniformity, its chemical purity and its excellent heat-conducting properties. Like all woven material, glass-fibre insulation has to be varnish treated for almost all its uses as an insulator. Owing to the great mechanical strength of glass fibre, the process of varnishing the cloth can be carried on the same plant as is used for treating cotton cloth and silk.

Whereas varnished cotton becomes brittle at high temperatures, resulting in a decrease of electrical strength, mechanical glass fibre is practically unaffected and retains its flexibility The difference in the resistivity of the two materials is particularly marked owing to the non-hygroscopic characteristic of fibre glass. The tensile strength of this material is considerably higher than that of cotton or asbestos. Motors with fibre-glass insulated windings can be run at much higher temperatures, giving better utilisation of the active materials and therefore a decrease in the weight per horse-power. From synthetic resin-bonded glass fibre boards, slot wedges can be rough-cut and, owing to their great rigidity, can be forced into the slot, thereby adjusting themselves to the exact profile, and in this way an expensive machining operation can be eliminated.

MICA has been used as an insulating material ever since the electrical industry began, and it is still the most important because of its high degree of reliability in service under the most severe conditions—atmospheric, physical, and thermal. It is chiefly used for electrical purposes in the form of *micanite*, consisting of mica splittings bonded together and further treated, and this material in various forms still constitutes the main type of insulation for rotating machinery, particularly generating plant and heavy-duty traction motors. Built-up mica is still the most suitable material for insulating commutator segments and modern heating appliances.

A combination of mica and glass is now manufactured as an insulating material, and mouldings of this substance can be obtained with very fine limits of precision—much more so than is possible with steatite or ceramic mouldings, and it can be satisfactorily used under humid conditions.

ASBESTOS is still used as asbestos papers, fabrics, boards, cements and wire coverings. Its great advantages are resistance to heat, its chemical immunity and its freedom from deterioration.

SLATE and MARBLE are used for switchboards but are gradually being

substituted to a greater extent by impregnated cement-asbestos boards and synthetic resin panels.

Oil Substitutes

On account of the growing importance which is being attached to the elimination of fire risks in electrical installations, non-inflammable liquid substitutes for oil have been developed for immersed transformers switches, condensers, and other appliances.

Rubber

If it were not on account of the fact that natural rubber has a tendency to age prematurely, the search for synthetic substitutes would not have come to be regarded as so supremely important as it has been in recent years. Ebonite and similar vulcanised rubber compounds have so far maintained their positions as insulating materials for scientific instruments and for telephone and radio applications where the temperatures do not exceed 70° C. Like natural rubber, the synthetic compounds are highly polymerised hydrocarbons. Poly iso-butylene is one of the basic compounds of synthetic rubber and is claimed to be able to resist boiling water without detriment to its electrical properties. At 800 Hz its loss angle is about 4×10^{-4} at 20° C., increasing by only 1 per cent. at 85° C., whilst the electrical strength and resistivity are found to be very satisfactory.

Dielectric Strength

The modern view of the mechanism of insulation breakdown is based on the following experimental evidence (Boning's Theory) ;

(*i*) The conduction of electricity in solid insulating materials is accounted for by the existence of a multitude of minute channels in the body of the material which are filled with moisture and thus provide electrolytic conduction paths through which an insulating current actually passes. The insulation material itself is to be regarded as non-conducting, and experiments have shown that the conductivity can be influenced by certain treatments, for example, by drying out the colloidal types and by repeatedly re-crystallising the crystalline types.

(*ii*) The boundary walls of the conducting channels are lined with adsorbed " boundary ions " which may be either positive or negative and, in general, are held in position by intensely powerful forces and require a correspondingly intense applied electric field in order to tear them away.

(*iii*) In addition to the boundary ions of item (*ii*), there are " supplementary " or " creeping " ions which are also attached to the boundary walls and electrically neutralise the boundary ions. The adhesive force, however, by which the supplementary ions are attached to the boundary walls is, in general, considerably less than that which holds the boundary

ions, and consequently the supplementary ions can be driven away by a much less intense applied electric field.

(*iv*) In the conducting channels themselves are the normal dissociated ions of the electrolyte by means of which the insulation current is carried.

In Fig. 26 is shown diagrammatically, the arrangement of the different groups of ions in a conducting channel.* When the insulating material is subjected to an applied p.d , a current will flow, the magnitude of which will correspond to the conductivity of the electrolyte. If the applied p.d. is large enough, the supplementary ions of item (*ii*) will be swept away and the boundary ions of item (*i*) will remain and so give rise to a corresponding *space charge* (see also page 30).

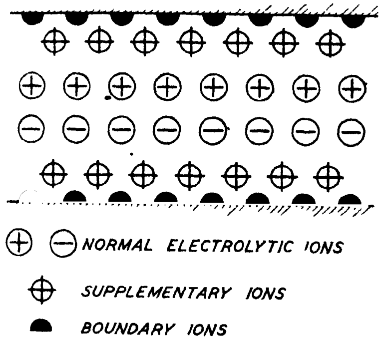


Fig 26

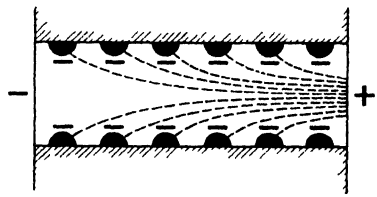


Fig 27

Suppose now that a disc of insulating material is taken and an electrode of the same diameter as the disc is applied to each face. If a p d. V be applied to these electrodes of a magnitude which is sufficient to drive off the supplementary ions from the conducting channel walls, and assuming that the boundary ions are negatively charged, the conditions will be as shown in Fig. 27. At the moment of applying the p d to the electrodes a current I will flow such that

$$I = \frac{V}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where R is the electrolytic resistance of the moisture-filled channels. That is to say,

$$R = \rho . \frac{d}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where ρ is the specific resistance of the electrolyte,
A is the cross-sectional area of the channel,
d is the length of the channel.

The space charge, that is, the negatively charged boundary ions, will partially neutralise the electric field due to the applied p.d. and will operate to produce a *back e.m.f.* as follows :

* Figs. 26–32 are reproduced from *Engineering*, April 2, 1943, pp. 261, 262.

If σ is the surface density of the space charge, the electric force due to this space charge will be $\frac{4\pi\sigma}{\epsilon}$, which, for convenience, may be written $2b$. At any distance x from the negatively charged electrode the intensity of this force will be

$$E_S = 2bx = \frac{dE_S}{dx} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and the applied p.d. must have a component V_S — E_S to compensate this opposing force. In addition, the applied p.d. must have a component V_I to provide the electric force which is necessary to drive the current through the electrolyte, that is,

$$E_I = \frac{dV_I}{dx} = \frac{d}{dx} I\rho\frac{x}{A} = I\frac{\rho}{A} \quad . \quad . \quad . \quad . \quad (28)$$

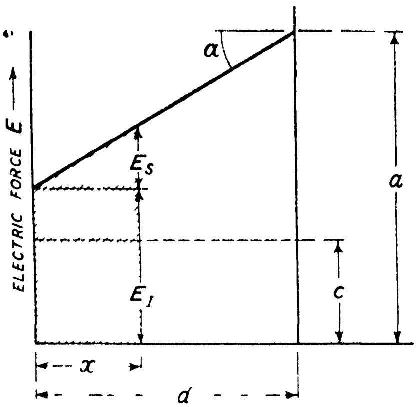


Fig. 28

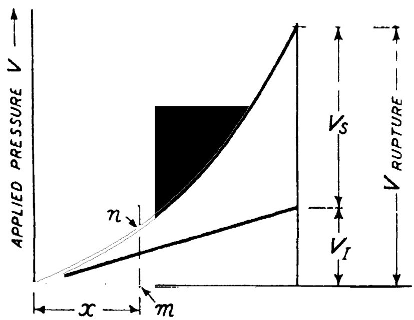


Fig. 29.

The total electric force, therefore, which has to be provided by the applied p.d. will be

$$E = E_I + E_S = I.\frac{\rho}{A} + 2bx \quad . \quad . \quad . \quad . \quad (29)$$

and this is shown by the diagram of Fig. 28 where $\tan\alpha = \frac{E_S}{x} = 2b$. If a is the intensity of the force necessary to break down the insulation, that is, the maximum applied intensity of force which the insulation can stand, then, putting $E = a$, for $x = d$ in equation (29) gives,

$$a = I\frac{\rho}{A} + 2bd$$

or

$$I\frac{\rho}{A} = a - 2bd,$$

so that (29) may now be written

$$E = a - 2bd + 2bx.$$

The breakdown applied p.d. will consequently be given by

$$V_{max} = V_I + V_S = \int_0^a \{(a - 2bd)dx + 2bx\,dx\}$$

i.e.

$$V]_{RUPTURE} = (a - 2bd)d + bd^2 = \mathbf{ad - bd^2} \quad . \quad . \quad (30)$$

This result is shown by the diagram of Fig. 29, where the ordinate at any distance x from the negative electrode represents the area of the corresponding shaded portion of the diagram of Fig. 28. When the applied pressure reaches the rupture value, the boundary ions will be torn away from the boundary walls and this will be effected suddenly —a kind of relay action being set in operation—in consequence of which, intense heat will be developed, gases will be formed, and the insulation material will be destroyed.

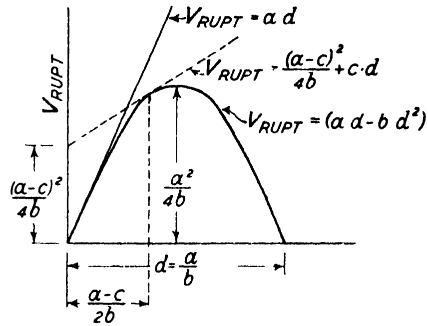


Fig. 30.

Reference to expression (30) for the rupture pressure shows that the magnitude of this pressure is a parabolic function of the distance d, as shown in Fig. 30, and further, the pressure will have a maximum value when

$$\frac{\partial}{\partial d}(V_{RUPTURE}) = 0 = a - 2b.d$$

and consequently,

$$(V_{RUPTURE})_{max} = \frac{a^2}{4b} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It is also to be observed that, if there is no space charge, that is, if $b = 0$, which implies that the supplementary ions have not been displaced so that the space charge of the boundary ions is neutralised, then

$$V_{RUPTURE} = a.d \quad . \quad . \quad . \quad . \quad . \quad (32)$$

that is, a straight-line function of the thickness d of the insulation as shown in Fig. 30.

If the applied pressure V becomes reduced, or, alternatively, if the thickness of the insulation d is increased, the field intensity at the negative electrode of the channel will eventually fall to some value c. If, now, c is also the field intensity which is necessary to dislodge the supplementary ions from the channel walls, then any further reduction of the

applied p.d. will result in the supplementary ions reassembling on the channel walls as shown in Fig. 31. For this condition,

$$V = c.d + b.d'^2 ;$$
$$a = c + 2b.d'$$

and eliminating d' gives $(V - c.d) = b\left(\dfrac{a-c}{2b}\right)^2,$

so that, $$V_{RUPTURE} = \frac{(a-c)^2}{4b} + c.d = k + c.d \quad . \quad . \quad (33)$$

which is a straight line, as shown in Fig. 30.

At a given constant temperature, insulation material is characterised by the three quantities, a, b, c, as defined in the foregoing, and experiments have shown that the effects of temperature can be classified into three groups as follows:

(*i*) A range of low temperatures for which experimental data show that the rupturing pressure is independent of the temperature, and in this range the breakdown pressure rises proportionally with the insulation thickness

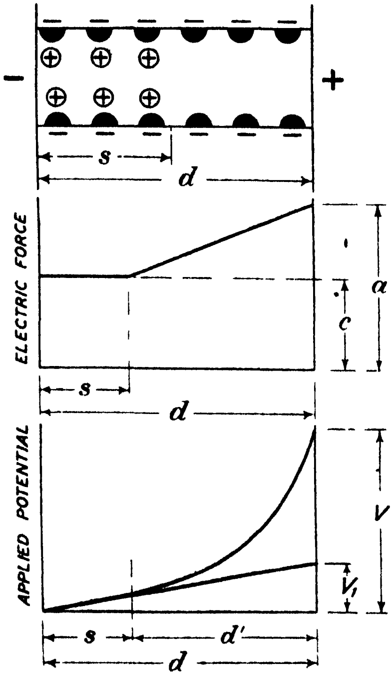


Fig 31.

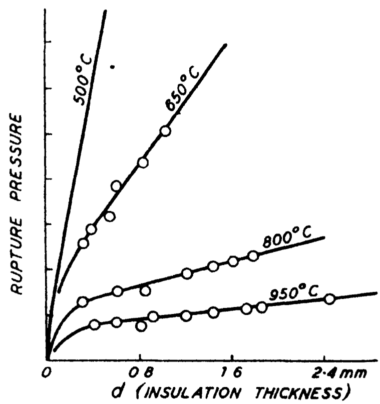


Fig 32

(*ii*) Experimental data show that the rupturing pressure falls slowly as the temperature rises, and this range is termed the "thermal electric range" of pressure breakdown.

(*iii*) The rupturing pressure falls rapidly as the temperature increases, and this range is termed the "thermal breakdown" range.

Böning has shown that by defining the quantities a, b and c, as respective functions of the temperature, the expressions (32) and (33) can be made to give results which are in accordance with the actual experimental data. In Fig. 32 are shown the calculated and experimental results for the rupturing pressure of " Frequenta ".

Whilst there can be no doubt as to the validity of Böning's theory as applied to many insulation materials, it can hardly be applied without modification to substances such as glass.

Chapter III

COULOMB'S LAW : FIELDS OF ELECTRIC FORCE : POTENTIAL

Coulomb's Law

FOR purposes of calculation, electricity may be treated as though it were a material substance, each particle of which acts on every other particle in accordance with Coulomb's Law, viz.

If two quantities of electricity, q, q', are respectively concentrated at points r cm. apart, and situated in a medium of which the dielectric constant is ε, then the force between them will be

$$E = \frac{q \cdot q'}{\varepsilon \cdot r^2} \text{ dynes} \quad . \quad . \quad . \quad . \quad (1)$$

If this force is one dyne when $q = q'$; $r = 1$ cm. and $\varepsilon = 1$ (see also Chapter I, page 2), then the magnitude of each of the concentrated quantities of electricity will be *one electrostatic c.g.s. unit.* Some idea of the magnitude of this unit of quantity is obtained by noting that if two pith balls, each weighing 1 gram and hung by silk fibres 1 metre long without appreciable weight, and if the pith balls are equally electrified to the extent that they separate 10 cm. from each other, then the charge on each pith ball will be about 70 electrostatic c.g.s. units of quantity

This electrostatic unit is very small in comparison with the quantities which come into account in practice, and, consequently, a multiple of this unit, viz. the *Coulomb*, is taken as the practical or technical unit. The coulomb is equal to 3×10^9 electrostatic c.g.s. units, and the reason why this particular multiple has been chosen will be found in Chapter I.

Representation of the Electric Field by Lines of Force

Lines of electric force are lines drawn in the field such that the direction of the line through any point gives the direction of the force at that point, that is to say, the direction in which a unit of positive electricity would tend to move if placed at that point.

Suppose that one c.g.s. unit of positive electricity is placed at the centre of a spherical surface of 1 cm. radius. The force experienced by another unit quantity placed anywhere on the surface of the sphere will, by definition, be unity, and *one unit line of force is then said to cross each square centimetre of the spherical surface.* The strength of an electric field may thus be represented by the number of unit lines of force which cross one square centimetre of area, the area being such that its plane is at right angles to the direction of the lines of force at the place considered.

Since one unit line of force crosses each square centimetre of area of the surface of the sphere of unit radius when a unit charge is placed at the centre, it follows that 4π unit lines will cross the whole surface of the sphere or, otherwise stated, each unit charge of electricity gives rise to 4π unit lines of force.

The *flux of force* across any area in an electric field is the number of unit lines of force which cross that area. Otherwise, the flux of force across an area may be represented as follows. Let E be the intensity of the electric force that is, the "electric intensity" at a point in the area δS, and let E_n be the component at right angles to this area, the flux of force across the area will be $E_n . \delta S$. Now suppose in Fig. 1, that A and B are two charged surfaces forming the boundaries of an electric field. On the surface A draw an element of area δS and at every point of the contour of this element let lines of force be drawn. These lines will then enclose a tube which will reach the surface B in a direction at right angles to the area δS_1. This tube is termed a *tube of force*. If E is the intensity of the force at the point P in the element of area δS, then the product $E \delta S$ is the *flux of force* within the tube, assuming that the intensity of the force is uniform over the whole area δS.

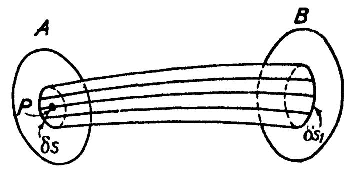


Fig. 1.

Electric Surface Density

If a quantity of electricity of q electrostatic c.g.s. units exists on a surface of area S sq. cm., the average electric surface density will be,

$$\sigma = \frac{q}{S}$$

and the density at any given point of an electrified surface is,

$$\sigma = \frac{dq}{dS} \text{ electrostatic c.g.s. units per square centimetre.}$$

From unit surface area on which the density is $+\sigma$, $4\pi\sigma$ unit lines of force will start, and at unit surface on which the density is $-\sigma$, $4\pi\sigma$ unit lines of force will terminate. Experiments conclusively prove that when a charge of electricity is given to an insulated conductor the electricity resides on the outside surface so that there is neither electric charge or electric force within the conductor. When a charge of electricity is given to an insulated conductor, therefore, the electricity will spread over the surface so that the distribution will give no resultant force within the conductor. It then becomes a mathematical problem to determine the surface density at every point of the conductor which is necessary to fulfil this requirement. It is only in a few special cases, however, that the mathematical problem can be solved, but, generally

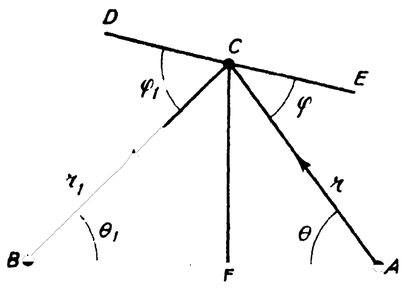


Fig 2

speaking, the distribution will be such that the density will be the greatest on sharp projections of the surface—such as points and edges.

Methods of Drawing Lines of Force in Some Typical Cases

EXAMPLE 1. *Two Equal and Opposite Charges concentrated respectively at the Points A and B (Figs 2 and 3).*—Suppose at the points A and B are respectively placed one positive and one negative electrostatic unit of electricity and consider the force at any point C. Let the distance AC be r cm. and the distance BC be r_1 cm. The force at C due to the charge $+1$ at A will then be $\frac{1}{\epsilon r^2}$ dynes, whilst the force at C due to the charge -1 at B will be $\frac{1}{\epsilon r_1^2}$ dynes, and these forces will act in the respective directions shown in Fig 2. Let DCE be the tangent to the

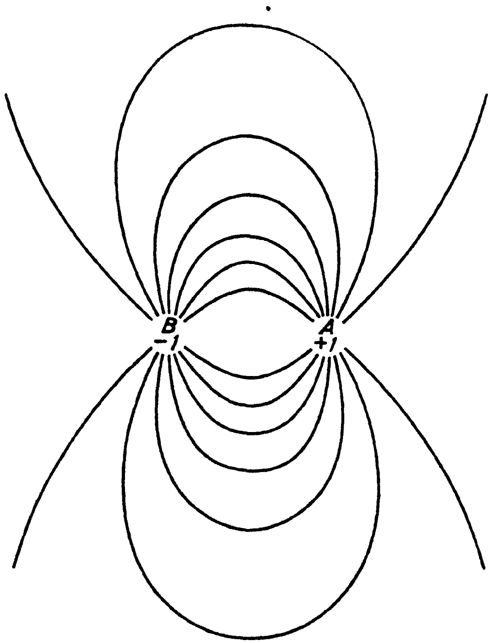


Fig 3.

line of force which passes through C. Then by definition of a line of force, it follows that the resultant force in a direction at right angles to DCE will be zero, that is

$$\frac{1}{\varepsilon r^2}\sin\phi = \frac{1}{\varepsilon r_1{}^2}\sin\phi_1 \quad . \quad . \quad . \quad . \quad (2)$$

But,

$$CF = r_1 \sin\theta_1 = r\sin\theta : \text{and, } r_1\frac{d\theta_1}{ds} = \sin\phi_1 : -r\frac{d\theta}{ds} = \sin\phi \quad . \quad (3)$$

where ds is a small element of length of the line of force which passes through C, the positive direction being from A towards B. Hence from (2) and (3),

$$\sin\theta\frac{d\theta}{ds} + \sin\theta_1\frac{d\theta_1}{ds} = 0$$

that is,

$$\cos\theta + \cos\theta_1 = \text{a constant} \quad . \quad . \quad . \quad (4)$$

The lines of force shown in Fig. 3 have been drawn by means of equation (4), each line of force corresponding to a different value of the constant on the right-hand side of this equation. The electric charge at A is assumed to be $+1$ unit and at B -1 unit.

If the charges at A and B are not equal, but if, for example, the

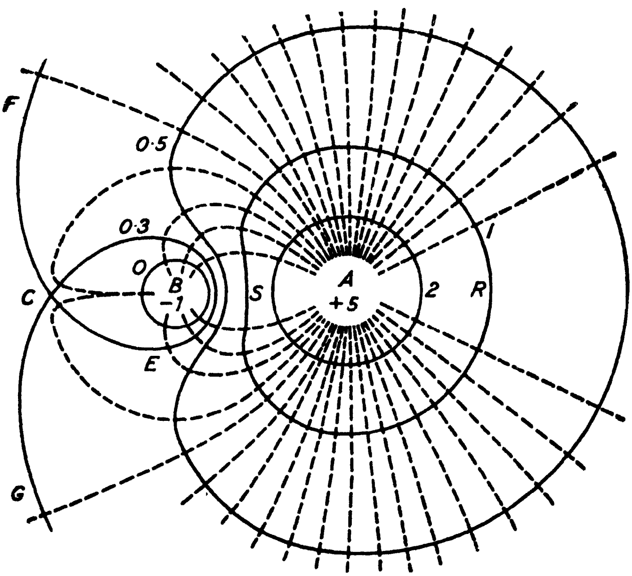


Fig. 4.

charge at A is $+q$ units and the charge at B is -1 unit, then the equation for the lines of force becomes

$$q \cos \theta + \cos \theta_1 = \text{a constant} \quad . \quad . \quad . \quad (5)$$

In Fig. 4 the *broken-line curves* represent the lines of force as derived by means of equation (5) for the condition that a charge of $+5$ units is at A and a charge of -1 unit is at B. The full-line curves defined the equi-potential surfaces the equation for which is given on page 79, (see also Fig. 16, page 89) At the point marked C in this figure the force is zero.

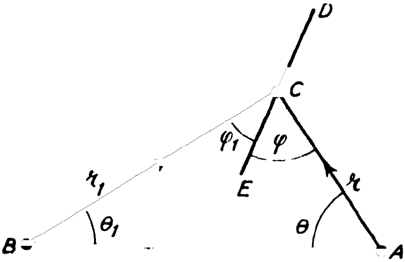


Fig. 5.

EXAMPLE 2. *Like Charges are concentrated at Points A and B (Fig.* 5).—Suppose in Fig. 5 one unit of positive electricity is concentrated at each of the points A and B and consider the force at C due to these respective charges, viz $\frac{1}{\epsilon r^2}$ due to the charge at A and acting in the direction AC and $\frac{1}{\epsilon r_1{}^2}$ due

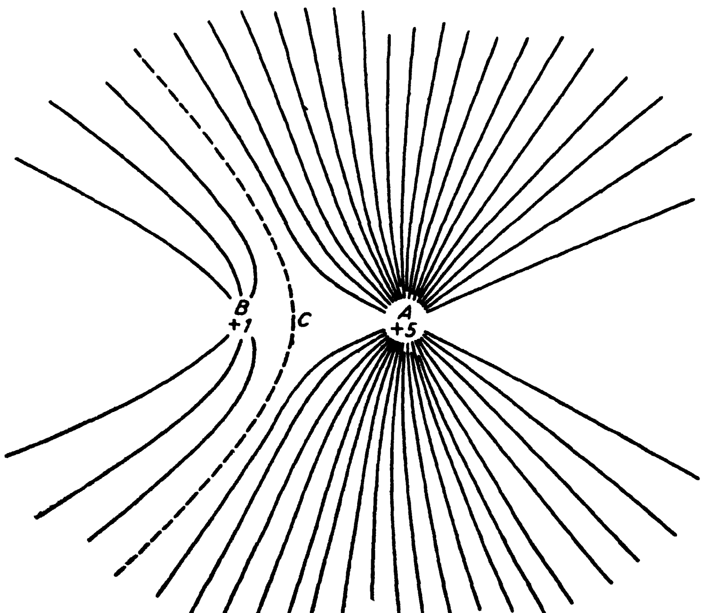


Fig. 6.

to the charge at B and acting in the direction BC. The resultant force will then be as shown by ECD so that the force at C in a direction at right angles to ECD will be zero, hence,

$$\frac{1}{\varepsilon r^2}\sin\phi = \frac{1}{\varepsilon r_1{}^2}\sin\phi_1$$

but
$$r_1\frac{d\theta_1}{ds} - \sin\phi_1 : r\frac{d\theta}{ds} = \sin\phi : r_1\sin\theta_1 - r\sin\theta,$$

from which it follows that the equation which defines the lines of force in this case is

$$\cos\theta - \cos\theta_1 = \text{a constant} \quad . \quad . \quad . \quad (6)$$

If there are + q units of electricity at A and + 1 unit at B, then the general equation for the lines of force becomes

$$q\cos\theta - \cos\theta_1 = \text{a constant} \quad . \quad . \quad . \quad (7)$$

In Fig. 6 are shown some of the lines of force due to a charge of + 5 units at A and a charge of + 1 unit at B. The force at the point C will be zero and the broken line curve through C forms the boundary which separate the lines of force emanating from the charge and those which emanate from the charge B.

The Electric Force at any Point Outside a Charged Sphere is the same as if the Charge were Concentrated at the Centre of the Sphere

It has been shown on page 77 that one electrostatic c.g.s. unit of quantity gives rise to 4π unit lines of force, so that a quantity q will give rise to $4\pi q$ unit lines. It has also been stated on page 77 that there is neither electric charge nor electric force inside a charged conductor.

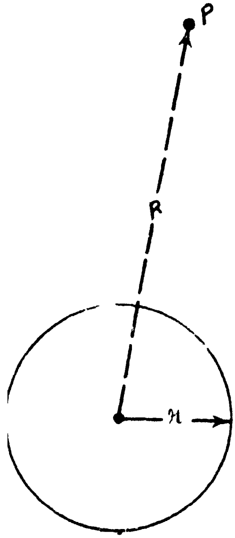


Fig. 7.

When a charged spherical conductor is isolated in space, it is clear that the charge will distribute itself uniformly over the surface so that the density σ will be the same at every point on the surface, the total charge on the surface being $\frac{4\pi r^2\sigma}{\varepsilon}$, and consequently the total flux which issues from the surface of the conductor will be $\frac{4\pi(4\pi r^2\sigma)}{\varepsilon}$, where r cm. is the radius of the sphere. In order to find the intensity of the electric force at any point P outside the sphere and distant R cm. from the centre, consider a spherical surface drawn concentrically with the charged sphere and passing through P (see Fig. 7). The total flux passing through this spherical surface will be the same as that which issues from the surface of the charged sphere, viz.

$16\pi^2 r^2\sigma$, so that the electric intensity over the imaginary spherical surface through P will be

$$\frac{16\pi^2 r^2\sigma}{\varepsilon 4\pi R^2} = \frac{Q}{\varepsilon R^2},$$

where Q is the total charge on the spherical conductor, and ε is the dielectric constant of the medium in which the sphere is placed. That is to say, the electric intensity at the point P will be the same as if the whole charge Q were concentrated at the centre of the sphere.

The Electric Force at a Point Indefinitely near the Surface of a Charged Conductor in Equilibrium is Equal to $4\pi\sigma$ where σ is the Surface Density near the Point

In Fig 8 an element δS of an electrified surface is shown. Consider a tube of force of which the element δS forms one end, then if E is the force just outside the element and σ is the surface density on δS in electrostatic units per square centimetre, the flux of force in the tube will be $E.\delta S$. But the flux of force due to the charge $\sigma.\delta S$ is $\frac{4\pi\sigma.\delta S}{\varepsilon}$, so that $E.\delta S = \frac{4\pi\sigma.\delta S}{\varepsilon}$, that is,

$$E = \frac{4\pi\sigma}{\varepsilon} \text{ dynes} \quad . \quad . \quad . \quad . \quad (8)$$

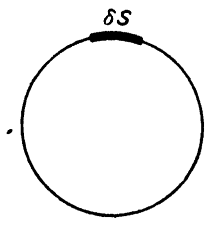


Fig 8.

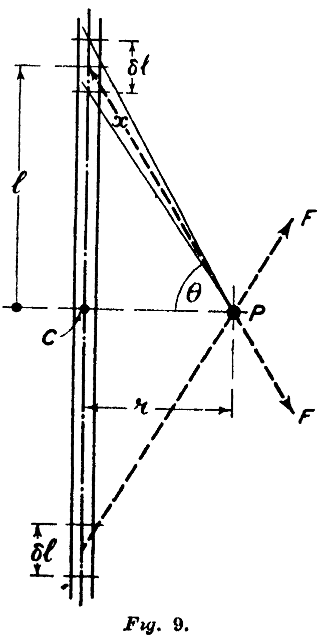


Fig. 9.

The Electric Force at a Point Distant r cm. from the Axis of a Charged Long Straight Wire

Let q electrostatic units per centimetre length be the charge on the wire and consider an element of length δl cm. (Fig. 9). The force at P due to the charge on the element δl will then be

$$e = \frac{q\,\delta l}{\varepsilon x^2} \text{ dynes,}$$

acting in the direction PF. But, $x \cos\theta = CP = r$: $x\frac{d\theta}{dl} = \cos\theta$, so that

$$e = \frac{q d\theta}{\varepsilon r} \text{ dynes.}$$

The component of e in the direction CP will be

$$e \cos\theta = q\frac{\cos\theta \, d\theta}{\varepsilon r} \text{ dynes,}$$

whilst the component of E in the direction at right angles to CP will be neutralised by the component at right angles to CP and due to the symmetrically placed charged element δl_1. Hence the total force at P due to the charged wire will be in the direction CP and of magnitude

$$E = \int_{-\pi/2}^{+\pi/2} e \cos\theta \, d\theta = \int_{-\pi/2}^{+\pi/2} \frac{q \cos\theta}{\varepsilon r} d\theta = \frac{2q}{\varepsilon r} \text{ dynes} \quad . \quad (9)$$

The Electric Force between Two Oppositely Charged Discs arranged Parallel to one another and at a Small Distance Apart

In Fig. 10 (a) let AB and CD be the two charged discs arranged parallel to each other and suppose that the distance apart is d cm. where d is small in comparison with the diameter of each disc. Let the inner face

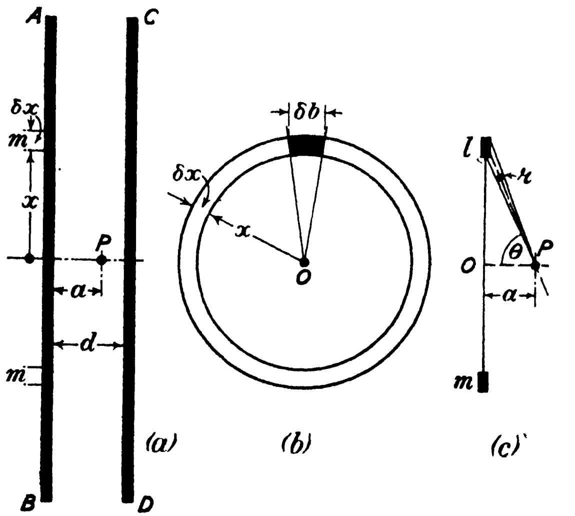


Fig. 10.

of the disc AB be uniformly charged with a surface density $+\sigma$ and the inner face of CD with a density $-\sigma$ electrostat c.g.s. units per square centimetre. It is to be observed here that when a charge is given to either

of the discs AB or CD arranged as shown in Fig. 10 (a), it can be deduced from Coulomb's Law that, except at places near the edges of the discs the charge will so distribute itself that there will be uniform density over the inner surface of each disc.

Consider now the force at the point P which lies on the common axis of the discs and at a distance a cm. from AB. In Fig. 10 (b) is shown a circular strip nm on the face of the disc AB of radius x cm. and breadth δx. An element of the surface of the width δb will have the charge $\sigma\delta b.\delta x$, and the force at P due to the total charge on the circular strip will be in the direction CP (Fig. 10 (c) and of the magnitude

$$\delta E = \frac{\sigma 2\pi x\,\delta x}{\kappa r^2}\cos\theta \text{ dynes}$$

but

$$\sin\theta = \frac{x}{r}\,;\quad \cos\theta = \frac{r\,d\theta}{dx}$$

so that

$$dE = \frac{\sigma 2\pi}{\kappa}\sin\theta\,d\theta \text{ dynes},$$

and consequently the resultant force at P due to the charge on the whole inner surface of the disc AB will be

$$E_1 = \int_0^{\pi/2} dE = 2\pi\frac{\sigma}{\kappa} \text{ dynes}.$$

The force at P due to the charge on the disc CD will have the same value, so that the total force at any point P in between the discs will be

$$E = 2E_1 = \frac{4\pi\sigma}{\kappa} \text{ dynes} \quad . \quad . \quad . \quad . \quad (10)$$

It will be seen from the expression (10) that the magnitude of this force is independent of the distance a of the point P from the disc, and consequently the lines of force between the two charged surfaces will be parallel and equi spaced straight lines except at places near the rims of the plates.

The Electrostatic Pressure, or the Electric Force on each Element of a Charged Conductor Due to the Remainder of the Charge.

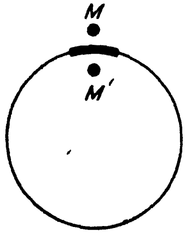


Fig 11.

Consider two points, M and M' in Fig. 11 such that M is close to the outside surface of a closed and charged conductor whilst M' is close to the inside surface and opposite to M. The total intensity F of the force at the point M is made up of the force f due to the charge on the elementary surface δS near M and the force f' due to the charge on the rest of the surface.

Let σ electrostatic c.g.s. units per square centimetre be the surface density on the element δS, then from expression (8) the force at M will be as on page 82,

$$E = f + f_1 = \frac{4\pi\sigma}{\varepsilon} \text{ dynes.}$$

But since the total force at M' just inside the surface is zero, whilst the force f_1 due to the whole of the charged surface except δS, is of the same magnitude at M' as at M, that is, the force f at M becomes $-f$ at M' then,

$$f - f_1 = 0 : \text{ and } f = f_1$$

so that
$$E = 2f = \frac{4\pi\sigma}{\varepsilon}$$

or
$$f_1 = \frac{2\pi\sigma}{\varepsilon} \text{ dynes} \quad . \quad . \quad . \quad (11)$$

The force on the charged element δS due to all the surface other than the element itself, is

$$f_1\sigma\,\delta S = \frac{2\pi\sigma^2}{\varepsilon}\delta S \text{ dynes,}$$

and, for an element of surface $\delta S = 1$ sq. cm., the force on this charged element due to the charge on the rest of the surface is

$$\frac{2\pi\sigma^2}{\varepsilon} = \frac{\varepsilon . E^2}{8\pi} \text{ dynes per square centimetre} \quad . \quad . \quad (12)$$

This quantity is termed the "electrostatic pressure" at the part of the charged conductor considered. Since this quantity is proportional to E^2 it is always directed outwards from the conductor, whether the sign of the charge itself is positive or negative.

For the atmosphere, under normal conditions of temperature and barometric pressure, the limiting value for the electrostatic pressure is about 540 dynes per square centimetre, this value corresponding to a potential gradient of 32,000 volts per centimetre. If the electrostatic pressure at the surface of a charged conductor in air at normal temperature and barometric pressure, exceeds this value of about 640 dynes per square centimetre, the air will cease to insulate and the charge will begin to leak away from the conductor (e.g. "corona" effect).

The result given by expression (12) may be applied to the electrified surface of the earth. Suppose, for example, as commonly occurs, the voltage gradient near the earth's surface is 1 volt per centimetre and that the lines of force are at right angles to the earth's surface, that is to say, the force near the surface is,

$$E = \frac{1}{300} \text{ dynes,}$$

the electrostatic pressure on 1 sq. cm. of the earth surface due to the charge on the rest of the surface will be

$$2\pi\sigma^2 \frac{1}{(300)^2}\,\frac{1}{8\pi}\ \text{dynes} = 4{\cdot}5 \times 10^{-7}\ \text{dynes},$$

and this force is much too small to lift even the lightest body.

Electrical Potential

The potential of a conductor depends not only on the quantity with which it is charged, but also on the shape and on the other charges in the neighbourhood. The potential difference between two charged conductors is measured by the amount of energy which would be required to transfer a unit of positive electricity from the conductor at the lower potential to that at the higher potential.

If one erg of work is expended in transferring 1 electrostatic c.g.s unit of positive electricity from conductor B to conductor A, the potential difference between the two conductors is said to be 1 electrostatic c.g.s. unit. For most technical purposes, the electrostatic unit is too large and, consequently the practical unit is taken as 1 volt, that is to say 300 volts = 1 electrostatic c.g.s. unit. It is easily proved that the work done in transferring electricity between two points which are at different potentials is independent of the path which is followed in the transfer Suppose, in Fig. 12, the electric force at the point A is E dynes, and consider a point B which is distant from A by the small amount δx. In moving 1 electrostatic unit of positive electricity from A to B the amount of work done by the electric field will be

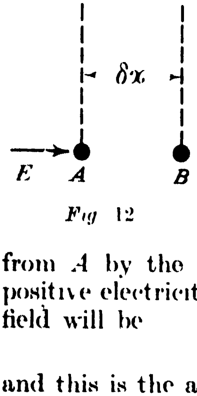


Fig 12

$$E.\delta x \text{ ergs},$$

and this is the amount δV by which the potential A exceeds that of B, that is

$$\delta V = E.\delta x: \quad \text{or } E = -\frac{dV}{dx} \qquad . \quad . \quad . \quad (13)$$

Hence, the intensity of the electric force at A is equal to the rate of change of potential at that point and is in that direction which corresponds to the maximum rate of fall of potential. For practical purposes, the earth is taken to be at zero potential.

Potentials of Various Arrangements of Conductors

EXAMPLE 1. *An insulated Charged Sphere isolated in Space.*—Let $+Q$ electrostatic units be the charge on the sphere the radius of which is r cm. (Fig. 13). It has already been shown on page 82 that the electric force at points outside such a sphere is the same as if the whole charge

were concentrated at the centre. The electric force at a point P distant x cm. from the centre, therefore, is

$$F = \frac{Q}{\varepsilon x^2} \text{ dynes.}$$

The work done in moving 1 electrostatic c.g.s. unit of positive electricity from the point P through a distance δx towards the sphere, that is to say, a distance $-\delta x$ cm. is

$$\delta V = -\frac{Q}{\varepsilon x^2}\delta x \text{ ergs,}$$

so that the work done in bringing unit positive charge from an infinite distance to the surface that is, the potential of the sphere will be

$$V = \int_{\infty}^{r} \frac{Q}{\varepsilon x^2} dx = \frac{Q}{\varepsilon r} \text{ electrostatic c.g.s units.} \quad . \quad (14)$$

This is also the potential at a point P distant r cm. from a concentrated charge of $+Q$ electrostatic units.

EXAMPLE 2. *A Conducting Sphere totally enclosed in a Concentric Spherical Shell* (Fig. 14).—Let $+Q$ electrostatic c.g.s. units be the charge on the inner sphere of radius r_1 cm., so that the charge on the [redacted] inner surface of the spherical shell of radius r_2 cm. will be $-Q$. The force at any point P in the space between the sphere and shell and distant x cm. from the common centre will be $\frac{Q}{\varepsilon x^2}$ dynes, due to the charge on the sphere,

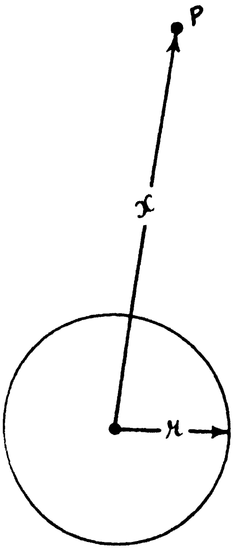


Fig. 13.

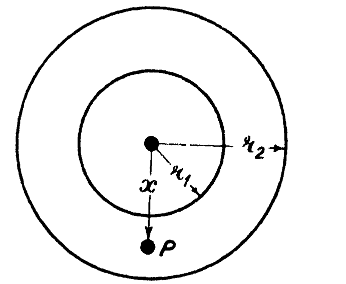


Fig. 14.

whilst the force due to the charge on the shell will be zero (see p. 77). Hence the potential difference between the sphere and spherical shell will be

$$V = \int_{r_2}^{r_1} -\frac{Q}{\varepsilon x^2} dx = \frac{Q}{\varepsilon}\left(\frac{r_2 - r_1}{r_2 . r_1}\right) \text{ electrostatic c.g.s. units} \quad . \quad (15)$$

Example 3. *The Potential Difference between Two Equally and Oppositely Charged Plates at a Small Distance Apart.*—On page 84 it was shown that the electric force at a point in between the two plates is constant and equal to $\frac{4\pi\sigma}{\epsilon}$ dynes for points which are not near the edges of the plates. The potential difference between the plates will therefore be

$$\frac{4\pi\sigma}{\epsilon}d = \frac{4\pi Q}{\epsilon S}d \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where S sq. cm. is the area of the charged face of each of the plates and Q electrostatic c.g.s. units is the total charge on each plate.

Equipotential Surfaces

The lines of force at the surface of a charged conductor in equilibrium are perpendicular to the surface because otherwise there would be a force tangential to the surface of the conductor tending to move the charge over the surface, and the distribution of electricity would accordingly be altered until a steady state has been reached such that the lines of force became directed at right angles to the charged surface. A surface drawn in an electric field such that the potential at every point of the surface is the same, is termed an equipotential surface. This may be expressed by the equation (13) (see page 86)

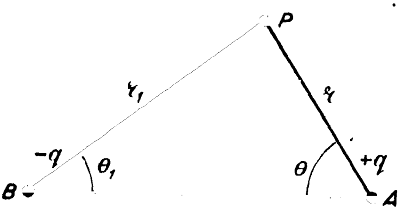


Fig. 15.

$$E = -\frac{dV}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where x is measured in a direction tangential to the surface at the point considered.

Example 1. Suppose in Fig. 15 a charge of $+q$ units is concentrated at the point A and a charge of $-q'$ units at the point B. The potential at any point P distant r and r_1 cm. respectively from the points A and B is then

$$\frac{q}{\epsilon r} - \frac{q'}{\epsilon r_1} \text{ electrostatic c.g.s. units}$$

and the surface which satisfies the equation

$$\frac{q}{r} - \frac{q'}{r_1} = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

is an equipotential surface. For the particular case that the constant is zero the equation becomes

$$\frac{q}{r} - \frac{q'}{r_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and this is the bi-polar equation for a sphere. In Fig. 16 are shown, by the full-line curves, the traces in the plane of the paper of a number of equipotential surfaces for the case in which the charge concentrated at A is $+5$ units and the charge concentrated at B is -1 unit, the equation for the equipotential surfaces thus being

$$\frac{5}{r} - \frac{1}{r_1} = \text{a constant} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

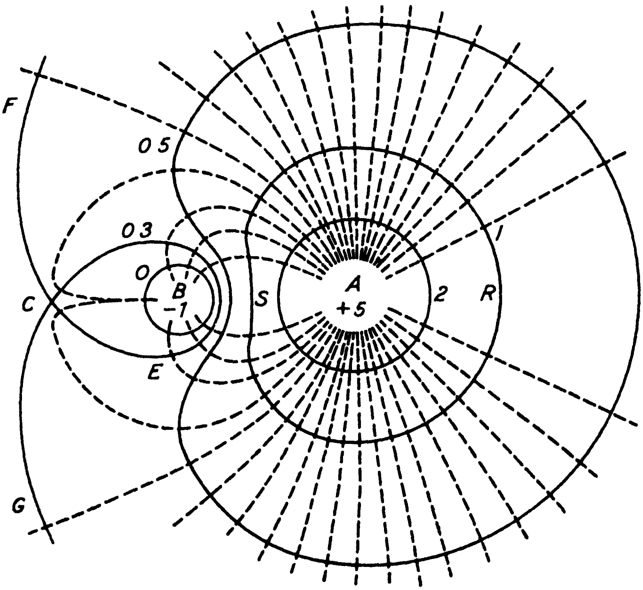


Fig. 16.

The spherical surface for which the potential is zero is then

$$\frac{5}{r} - \frac{1}{r_1} = 0\,; \quad \text{that is } 5r_1 = r$$

and the trace of this surface is shown in Fig. 16 by the circle O, the centre of which is to the left of the charge which is concentrated at B. The equipotential surfaces are given by the revolution of the system shown in Fig. 16 about the axis AB, each equipotential curve in this figure will then generate an equipotential surface. All points inside the circle O are at negative potential.

In Fig. 16 the lines of force (broken-line curves) which pass between the charges at A and B respectively are also drawn, the equation for the lines of force being (see also Fig. 4 and equation (5), page 80),

$$5 \cos \theta - \cos \theta_1 = \text{a constant}$$

as has been seen already on page 80. At the point C in Fig. 16, that is for which

$$\frac{5}{(CA)^2} = \frac{1}{(CB)^2},$$

the force is zero and the equipotential surface through this point C forms a loop. For the particular condition chosen, viz. $AB = 5$ cm. and $+5$ units at A, -1 unit at B, the potential of this looped surface is $+0 \cdot 3$ electrostatic c.g.s. units. For positive potentials below this value, the equipotential surfaces will be two distinct surfaces, one inside the loop CE and the other completely outside the surface CFG. For points at a higher potential than $+0 \cdot 3$ the equipotential surface will pass between the loop CE and the point A, as, for example, the surface corresponding to the curve SR.

EXAMPLE 2. *A Single Overhead Transmission Line.*—In Fig. 17 is shown a single overhead transmission line A of diameter d cm. and at a height h cm. above the earth's surface. If, for example, the transmission is d.c. at a pressure $+E$, there will be as electric charge $+q$ per centimetre length of the line. The electrostatic field due to this charge will be the same as that produced by the system shown in Fig. 18, in which A_r denotes the actual line charged with $+q$ units per centimetre length, and A_i is its image in the plane CD, this image being charged with $-q$ units per centimetre length. If the lines of force

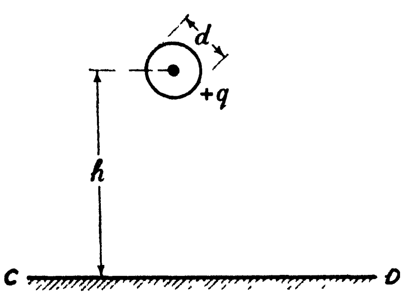


Fig. 17.

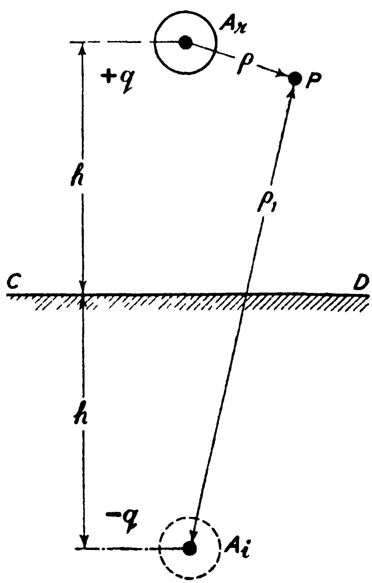


Fig. 18.

due to these two equally and oppositely charged conductors be drawn, then the field so obtained between the line A_r and the plane CD (Fig. 18) will be identical with the field between A and the earth's surface (Fig. 17).

The potential at any point P in Fig. 18 and due to the charged line A_r will be (see expression (9), page 83)

$$v_r = \int_{d/2}^{\rho}\left(-\frac{2q}{\varepsilon\rho}\right)d\rho = \frac{2q}{\varepsilon}\log_e\left(\frac{d}{2\rho}\right)$$

and the potential at P due to the charged line A_i will be

$$v_i = \int_{d/2}^{\rho_1}\frac{2q}{\varepsilon\rho_1}d\rho_1 = \frac{2q}{\varepsilon}\log_e\frac{2\rho_1}{d}.$$

Hence, the resultant potential at P due to the equal and oppositely charged lines A_r and A_i will be

$$v = v_r + v_i = \frac{2q}{\varepsilon}\log_e\frac{\rho_1}{\rho}\left\{\begin{matrix}\text{in electrostatic units when}\\ q \text{ is in electrostatic units}\end{matrix}\right\} \quad . \quad . \quad (21)$$

It will be seen from Chapters I and XVI that

$$\varepsilon . \mu = \frac{1}{c^2},$$

where ε is the dielectric constant in any system of units, μ is the magnetic permeability in the same system of units, and $c = 3 \times 10^{10}$ cm. per second and is the velocity of light in open space. The potential at P may then be written

$$v = 2q\mu c^2 \log_e\frac{\rho_1}{\rho}\left\{\begin{matrix}\text{in electromagnetic units for}\\ q \text{ in electromagnetic units}\end{matrix}\right\} \quad . \quad . \quad (22)$$

and $\mu = 1$ for open space.

The potential of the line A_r is then obtained by putting $\rho = \frac{d}{2}$ and $\rho_1 = 2h$ in expression (22), that is to say, by assuming the point P of Fig 18 to be on the line conductor A_r. Hence, the potential of the line A (Fig. 17) is

$$v = \frac{2q}{\varepsilon}\log_e\frac{4h}{d} \text{ electrostatic units} \quad . \quad . \quad . \quad (23)$$

and this leads to the expression for the *capacitance* of the line A to earth, viz (see also page 104 and Chapter XII, page 372)

$$C = \frac{q}{v} = \frac{\varepsilon}{2\log_e\left(\frac{4h}{d}\right)}\left\{\begin{matrix}\text{electrostatic c.g.s. units per}\\ \text{centimetre length of the line}\end{matrix}\right\} \quad . \quad (24)$$

Suppose the charge on the line A, Fig. 19, is $+1$ electrostatic unit per cm. length, then, since for air $\varepsilon = 1$,

$$v = 2\log_e\frac{\rho_1}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the surface over which the potential is v will be found by plotting the simultaneous values of ρ_1 and ρ (Fig. 18) which will satisfy the equation (25). This equation therefore denotes that the equipotential surfaces will be given by the equation

$$\frac{\rho_1}{\rho} = \text{a constant} \quad . \quad . \quad . \quad . \quad (26)$$

that is to say, the equipotential surfaces will be cylindrical surfaces. Thus, if

$h = 5m = 500$ cm., then for $v = 1$ electrostatic unit of potential

$$\log_e \frac{\rho_1}{\rho} = 0{\cdot}5 : \text{ that is } \frac{\rho_1}{\rho} = 1{\cdot}64,$$

and for the point a in Fig. 19, $\rho_1 + \rho = 1{,}000$ cm. : $\dfrac{\rho_1}{\rho} = 1{\cdot}64$

so that $\rho = 380$ cm.

For the point b in Fig 19, $\rho_1 - \rho = 1{,}000$ cm. : $\dfrac{\rho_1}{\rho} = 1{\cdot}64$,

so that $\rho = 560$ cm. and the diameter of the equipotential circle therefore, for which $v = +1$ is 940 cm. Similarly, the equipotential circle for $v = +2$ can be found and is drawn in Fig 19.

The equation for the lines of force between the two charged cylinders as shown by the broken line curves of Fig. 19, may be found by reference to Fig. 20. Suppose at any point C the tangent to the line of force through this point is DCE. Since the resultant force at C in the direction at right angle to DCE will be zero, then for $q = 1$,

$$\frac{2}{\epsilon r_1} \sin \phi_1 - \frac{2}{\epsilon r} \sin \phi = 0 \quad . \quad . \quad . \quad . \quad (27)$$

but

$$r_1 \frac{d\theta_1}{ds} = \sin \phi_1 : \qquad r \frac{d\theta}{ds} = -\sin \phi \quad . \quad . \quad . \quad . \quad (28)$$

where ds is a small element of length along the line of force the positive direction of which is from A_r to A_l.

From the expressions (27) and (28) it follows that

$$\frac{d\theta_1}{ds} + \frac{d\theta}{ds} = 0$$

so that

$$\theta_1 + \theta = \text{a constant} \quad . \quad . \quad . \quad . \quad (29)$$

The lines of force, therefore, are circles which pass through the axes of the conductors A_r and A_l, and the centres of which lie on the line FG

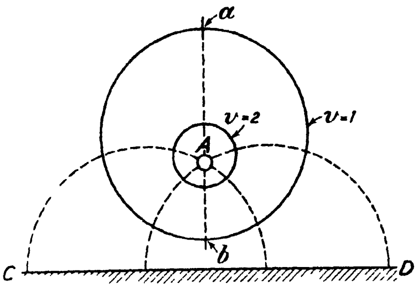


Fig. 19.

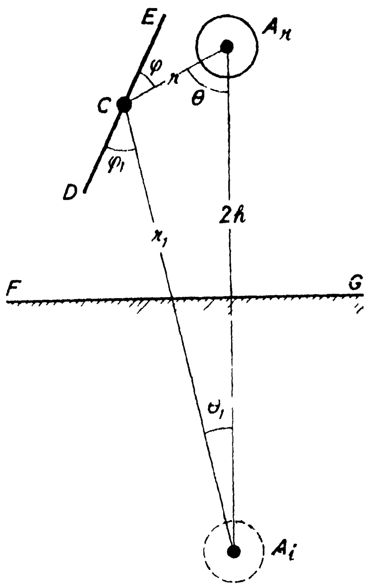


Fig. 20.

(Fig. 20), which intersects at right angles the line joining the axes of the conductors A_r and A_i. These circles (broken-line curves) will, of course, cut the equipotential surfaces of Fig. 19 at right angles.

EXAMPLE 3. *The Earth Wire on a Transmission Line Mast.*—In order to minimise the dangerous possibilities of surges on overhead transmission lines which are generated by atmospheric electrical disturbances, an earthed wire is arranged above the transmission lines and directly connected to each mast so that it is maintained at earth potential.

In general there exists over the surface of the earth an electrostatic field due to atmospheric electricity, and this field is assumed to be directed vertically, the static charges to which the field is due being located at high altitudes in space. The equipotential surfaces of the field will therefore be horizontal planes and, the potential of the earth being zero, the potential v_a at any height y will be positive and may be taken to be directly proportional to that height, that is

$$v_a = v_0 \cdot y \qquad (30)$$

In any given locality the atmospheric field strength may reach a value from 10 volts to 100 volts per centimetre and may very rapidly increase to values of the order of 1,000 volts per centimetre even when there is no obvious disturbance such as a thunderstorm.

In Fig. 21 are shown diagrammatically the lines of force and the trace of one equipotential surface due to atmospheric electricity. In Fig. 22 is shown the earth wire of diameter d cm. which is supported at a height h above the earth's surface and is metallically connected to each mast. Since this wire is placed in the electric field of the atmosphere which is at positive potential, it will become charged with

negative electricity, and the magnitude of this charge may be found as follows.

If the charge is $+Q$ electromagnetic c.g.s. units per centimetre length of the wire and if, in Fig. 23, A_1 is the electrical image of the wire in the earth's surface, then this image will have a charge of $-Q$ units per

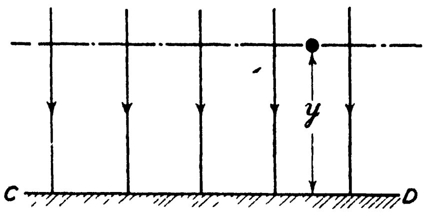


Fig. 21.

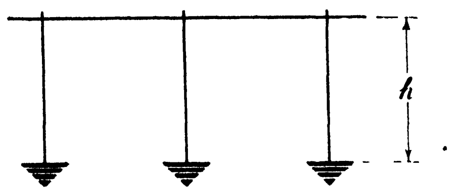


Fig 22

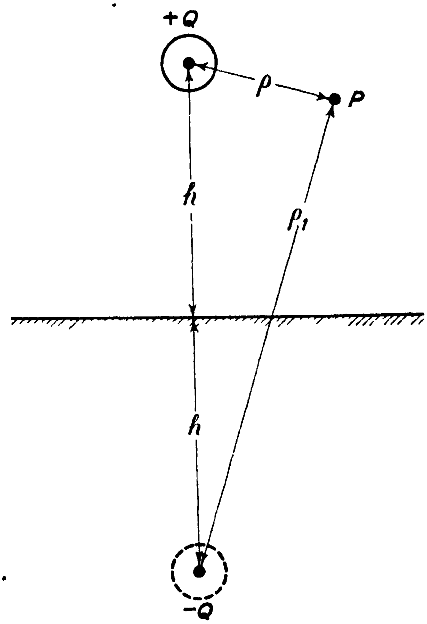


Fig. 23.

centimetre length. At any point P in Fig. 23, the position of which is defined by the distances ρ and ρ_1, the potential due to the charges $+Q$ and $-Q$ respectively, will be (see expression (22), Example 2)

$$v_e = 2c^2Q \log_e \frac{\rho_1}{\rho} \text{ electromagnetic units} \quad . \quad . \quad (31)$$

where $c = 3 \times 10^{10}$ cm. per second and is the velocity of light in open space. For a point on the surface of the wire itself, $\rho_1 = 2h$: $\rho = \frac{d}{2}$, so that

$$v_e = 2c^2Q \log_e \frac{4h}{d} \text{ electromagnetic units} \quad . \quad . \quad (32)$$

The total potential v_y at the surface of the earth wire, however, must be zero, so that

$$v_y = v_{a\,y\,h} + v_e = v_0 . h - 2c^2Q \log_e \frac{4h}{d} = 0 \quad . \quad . \quad (33)$$

and the magnitude of the charge on the wire will consequently be

$$Q = \frac{v_0 h}{2c^2 \log_e \frac{4h}{d}} \text{ electromagnetic units per centimetre length} \quad . \quad (34)$$

and substituting this value for Q in equation (31), the potential at any point P which is at a height y cm. above the earth's surface will then be given by the expression

$$v_y = v_a + v_e = v_0 y - v_0 h \frac{\log_e \frac{\rho_1}{\rho}}{\log_e \frac{4h}{d}} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The two components of which the resultant potential is the algebraic sum are therefore :

(i) $v_0 y$, which defines the equipotential surfaces represented by a horizontal plane at the height y.

(ii) $v_0 h \dfrac{\log_e \frac{\rho_1}{\rho}}{\log_e \frac{4h}{d}}$ (which defines the cylindrical equipotential surface which passes through the point P of height y cm. above the earth as already found in the foregoing Example 2.)

Suppose the earth wire is supported at a height of 10 m. = 1,000 cm. above the earth's surface, and assume that the potential gradient in the earth's field is $v_0 = 100$ volts per centimetre = 10^{10} electromagnetic units per centimetre. The diameter of the earth wire is $d = 1$ cm., so that $\log_e \frac{4h}{d} = 8{\cdot}3$. Then from expression (34)

$$Q = -\frac{10^{10} \times 10^3}{2 \times 9 \times 10^{20} \times 8{\cdot}3} = -0{\cdot}67 \times 10^{-9} \text{ electromagnetic units per centimetre,}$$

that is

$$-Q = -20{\cdot}1 \text{ electrostatic units per centimetre} \quad . \quad (36)$$

In order, however, to find the potential at any point P (Fig. 23) due to the charge $-Q$ on the earth wire, it is not necessary to find the numerical value of Q since the component (ii) of expression (35) is a general one and is directly applicable to any point P the position of which is defined by the quantities ρ and ρ_1. That is to say, the potential at any point P (Fig. 23) due to the charge $-Q$ on the earth wire is

$$v_e = -\left(\frac{v_0 h}{\log_e \frac{4h}{d}}\right) \log_e \frac{\rho_1}{\rho} \quad . \quad . \quad . \quad . \quad (37)$$

where the quantity in brackets is a constant for any point P. Thus,

for the numerical data given in this example, the quantity in brackets is

$$\frac{v_0 h}{\log_e \dfrac{4h}{d}} = \frac{100 \times 10^8 \times 10^3}{8\cdot3} = \frac{10^{13}}{8\cdot3}$$

since 1 volt = 10^8 electromagnetic units, and consequently, for any point P, the expression (37) gives,

$$\log_e \frac{\rho_1}{\rho} = -v_e \frac{8\cdot3}{10^{13}}$$

or, more conveniently,

$$\log_{10} \frac{\rho_1}{\rho} = -\frac{8\cdot3 \times v_e}{2\cdot3 \times 10^{13}} = -v_e \frac{3\cdot61}{10^{13}}.$$

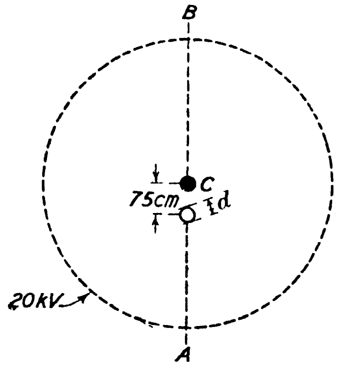


Fig. 24.

If, for example, it is desired to find the cylindrical equipotential surface defined by

$$v_e = -20{,}000 \text{ volts}$$

that is,

$$v_e = -2 \times 10^{12} \text{ electromagnetic units}$$

then

$$\log_{10} \frac{\rho_1}{\rho} = \frac{2 \times 10^{12} \times 8\cdot3}{2\cdot3 \times 10^{13}} = 0\cdot72 \,;$$

and

$$\frac{\rho_1}{\rho} = 5\cdot25 \,;$$

and this is the equation of a circle (see also Example 2). The diameter and centre of this circle are easily found as follows.* For the point A, Fig. 24, then in Fig. 23,

$$\rho + \rho_1 = 2h = 2{,}000 \text{ cm.} : \frac{\rho_1}{\rho} = 5\cdot25 : \rho = 320 \text{ cm.}$$

* *Engineering*, June 21, 1940, pages 608, 609.

For the point B (Fig. 24), then in Fig. 23,

$$\rho_1 - \rho = 2h = 2{,}000 \text{ cm.} : \frac{\rho_1}{\rho} = 5{\cdot}25 : \rho = 470 \text{ cm.}$$

so that the diameter AB of this circle, Fig. 24, $320 + 470 = 790$ cm. and the centre lies $\frac{790}{2} \quad 320 = 75$ cm. above the axis of the earth wire.

In this way, the equipotential circles can be drawn for a series of values of v_e. To obtain the resultant equipotential surfaces it is only then necessary to find the algebraic sum of the negative potential due

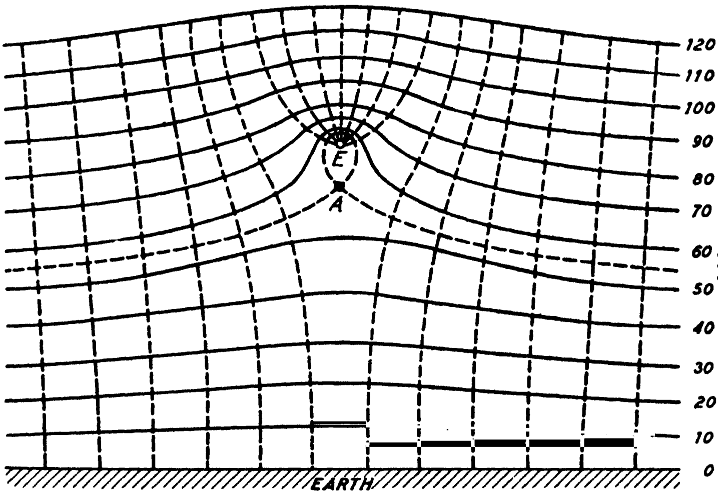


Fig 25.

to the induced charge on the earth wire and the positive potential due to the atmospheric field as represented by the horizontal lines shown in Fig. 21. The resultant equipotential surface so obtained are shown in Fig. 25.

Now since the potential of the earthed wire E (Fig. 25) is zero and the potential of the earth surface is also zero, it follows that there must be some point in the vertical through the earth wire E at which the potential has a maximum positive value. The position of this point A is easily found by differentiating the expression (35) with respect to ρ and equating the result to zero, that is

$$\frac{dv_y}{d\rho} = 0 = \frac{d}{d\rho}\left\{v_0(1{,}000 - \rho) - \frac{v_0 1{,}000}{8{\cdot}3}\log_e\frac{(2{,}000 - \rho)}{\rho}\right\}$$

so that $$- v_0 \quad v_0 \cdot \frac{1{,}000}{8{\cdot}3} \cdot \frac{\rho}{2{,}000 \quad \rho} \quad \frac{2{,}000}{\rho^2} = 0$$

from which it is found that $\rho = 128$ cm. (or 1,872 cm.). The value 128 cm. refers to the actual earth wire and is marked by E in Fig. 25. The alternative value of 1,872 refers to the image.

Since the electric force is defined by the equation

$$f = \frac{dv_y}{d\rho}$$

it follows that the force at the conductor A, Fig. 25, must be zero since $\frac{dv_y}{d\rho} = 0$ at A.

The force f_a at A due to the atmospheric field is

$$f_a = 100 \text{ volts per centimetre,}$$

and the force f_e at A (see Fig. 2, page 78) due to the charge of $Q = -20{\cdot}1$ electrostatic units per centimetre on the earth wire and $Q = +20{\cdot}1$ units on its image, is given by

$$f_e = 2Q\left[\frac{1}{\rho} + \frac{1}{2{,}000 - \rho}\right] 300 \text{ volts per centimetre}$$
$$= -100 \text{ volts per centimetre,}$$

so that the resultant force at A is

$$f = f_a + f_e = 0.$$

It is seen, therefore, that by providing an earth wire above the transmission line it is possible to obtain a space in which the electric force due to the atmospheric field is largely neutralized, and by arranging the transmission lines in this space they become screened to a great extent from the fluctuations of the atmospheric field strength.

Chapter IV

CAPACITANCE: DIELECTRIC CONSTANT: ENERGY OF THE ELECTRIC FIELD

The Capacitance of an Electric Field

In practice, the electric fields which are most frequently met with are those in which the boundaries on which the respective ends of the lines of force terminate are conducting surfaces. The potential difference between the boundaries of such a field is given by $\int E\, ds$, where E is the electric intensity at any point in the field and the integration is performed along a line of force from one boundary to the other. The value of E is, by Coulomb's Law, directly proportional to the quantity of electricity on each boundary and, consequently, the value of the potential difference between the boundaries is directly proportional to the electric charge on each. Hence

$$\int E\, ds = V = \frac{Q}{C}$$

or

$$C = \frac{Q}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where C is a constant for any given configuration of the boundaries and any given insulating medium in which the boundaries are placed. This constant is termed the *capacitance* of the field formed by the two boundary surfaces and the insulating medium.

The combination of two conducting surfaces separated by a non-conducting medium is termed an *electric condenser*, and the capacitance of such a condenser is defined as follows:

A condenser has a capacitance of 1 electrostatic c.g.s. unit if one electrostatic c.g.s. unit of quantity of electricity on each surface (i.e. one positive unit on one surface and one negative unit on the other surface) raises the potential difference between the two conducting surfaces by one electrostatic c.g.s unit.

The practical unit of capacitance is the *farad* and its subdivisions the microfarad (i.e. $1\ \mu F = 10^{-6}$ F), and the *millimicrofarad* (i.e. $1\ m\mu F = 10^{-9}$ F) and the *micromicrofarad* (i.e. $1\ \mu\mu F = 10^{-12}$ F), this last subdivision being also termed the *pico-farad* (pF).

An electric field has a capacitance of one farad if one coulomb of quantity on each conducting boundary surface produces a potential difference between the surfaces of 1 volt.

Since 1 coulomb $= 3 \times 10^9$ electrostatic c.g.s. units of quantity and 300 volts $=$ 1 electrostatic unit of potential, it follows that :

$$\left.\begin{array}{lcl} 1 \text{ farad} & = & 9 \times 10^{11} \\ 1\ \mu\text{F} & = & 9 \times 10^{5} \\ 1\ \mu\mu\text{F} & = & 0{\cdot}9 \end{array}\right\} \text{electrostatic units (i.e. in centimetres ; see also Chapter I).}$$

It will be seen by reference to expression (5) that the capacitance has the dimensions of a length when electrostatic units are used, that is, when it is assumed that the dielectric constant ε is a pure number Consequently, it is common practice to speak of the magnitude of a capacitance as so many centimetres as an alternative to the statement that the magnitude is so many electrostatic c.g.s. units, that is 1 farad $= 9 \times 10^{11}$ cms.

The Capacitance of a Tube of Force

Consider a tube of force which extends from one boundary conducting surface element δS_1 to the other boundary surface conducting element δS_2 as shown in Fig. 1. The quantity of electricity at each end of this tube of force will then be

$$q = \sigma_1 . \delta S_1 = \sigma_2 . \delta S_2$$

where σ_1 and σ_2 are the respective values of the surface densities. If E_1 is the electric intensity at δS_1 and E_2 the intensity at δS_2, then (see page 82, expression (8))

$$E_1 = \frac{4\pi\sigma_1}{\varepsilon} ; \quad E_2 = \frac{4\pi\sigma_2}{\varepsilon},$$

so that
$$q = \frac{E_1\varepsilon}{4\pi}\delta S_1 = \frac{E_2\varepsilon}{4\pi}\delta S_2 .$$

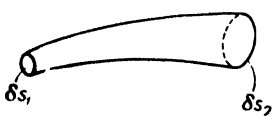


Fig 1.

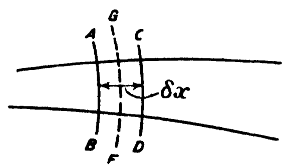


Fig. 2.

The potential difference between the ends of the tube of force, that is between the surface elements δS_1 and δS_2 will be $\int E\,ds$, the integration being performed from one surface element to the other. The quantity of electricity on each of the surface elements is then

$$q = \frac{\varepsilon . E_1 . \delta S_1}{4\pi} = \frac{\varepsilon . E_2 . \delta S_2}{4\pi} = \frac{\varepsilon . E . \delta S}{4\pi} \quad . \quad . \quad (2)$$

where E is the electric intensity at any point P in the tube and δS is the cross sectional area of the tube at that point (see Fig. 2). Since the areas $\delta S_1 : \delta S : \delta S_2$ are each at right angles to the lines of force of the tube, each of these areas must be a portion of the respective equipotential surfaces which cut across the tube at these positions. Hence the capacitance of the tube is

$$C = \frac{q}{\int E\,dx} = \frac{\varepsilon . E_1 . \delta S_1}{4\pi \int E\,dx} = \frac{\varepsilon . E_1 . \delta S_1}{4\pi . E_1\,\delta S_1 \int \frac{dx}{\delta S}}$$

that is,

$$C = \frac{\varepsilon}{4\pi} \frac{1}{\int \frac{dx}{\delta S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The capacitance of an elementary volume $ABCD$ (Fig. 2) of the tube is therefore

$$\delta C = \frac{\varepsilon}{4\pi} \frac{1}{\frac{\delta x}{\delta S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where δx is the length of the elementary volume measured along the mean line of force and δS is the mean cross sectional area of the element, that is δS is the element of the equipotential surface GF which is intercepted by the tube.

EXAMPLE 1. *Capacitance of an Isolated Spherical Conductor.*—On page 87 it has been shown that the potential of an isolated sphere of radius r cm. charged with a quantity of Q units of electricity is

$$V = \frac{Q}{\varepsilon r},$$

from which it follows that the capacitance of the sphere is

$$C = \frac{Q}{V} = \varepsilon . r \text{ electrostatic e.g.s. units}$$

or

$$C = \frac{\varepsilon . r}{9 \times 10^5} \mu\text{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLE 2. *Capacitance of a Sphere which is totally enclosed in a Conducting Spherical Shell.*—Let r_1 cm. be the radius of the sphere and r_2 cm. the radius of the inner surface of the conducting shell as shown in Fig. 3. From expression (15) on page 87 it is seen that if a quantity of electricity Q is given to the sphere, the p.d. between the surfaces of the sphere and spherical shell will be

$$V = \frac{Q}{\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

and consequently the capacitance of such a condenser is

$$C = \frac{Q}{V} = \frac{\varepsilon . r_1 . r_2}{r_2 - r_1} \text{ electrostatic c.g.s. units (i.e. centimetres),}$$

or

$$C = \frac{\varepsilon . r_1 . r_2}{9 \times 10^5 (r_2 - r_1)} \text{ cm.} \qquad . \quad . \quad . \quad . \quad (6)$$

EXAMPLE 3. *The Capacitance of the Field Between two Parallel Conducting Plates which are Close Together.*—From expression (16) on page 88 it is seen that the potential difference between two such plates which are respectively charged with $\pm Q$ units as shown in Fig. 4 will be

$$V = \frac{4\pi Q}{\varepsilon . S} d \text{ electrostatic c.g.s. units}$$

where d cm. is the (small) distance between the plates and S sq. cm is the area of each of the charged surfaces. Hence the capacitance of such a plate condenser will be

$$C = \frac{\varepsilon . S}{4\pi d} \text{ electrostatic c.g.s. units,}$$

or

$$C = \frac{\varepsilon . S}{113 \times 10^5 d} \mu F \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

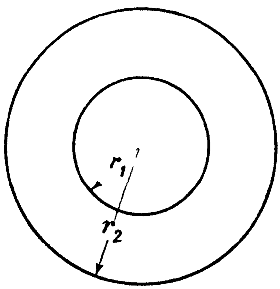


Fig. 3.

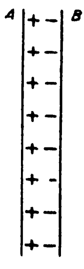


Fig. 4.

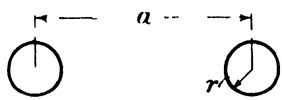


Fig. 5.

EXAMPLE 4. *Capacitance of the Field between Two Long Cylindrical Wires arranged with their Axes Parallel and in the Same Plane* (*Fig.* 5).—This is essentially the same problem as has been considered already in Chapter III (expression (24) on page 91), from which it will be seen that the capacitance is

$$C = \frac{\varepsilon}{4 \log_e \frac{2a}{d}} \text{ electrostatic c.g.s. units per centimetre length} \quad . \quad (8)$$

where d is the diameter of each wire and a is the distance between the axes of the two wires (see also Chapter XII, page 372),
that is,

$$C' = \frac{0{\cdot}0122\varepsilon}{\log_{10}\dfrac{2a}{d}}\ \mu\text{F per kilometre of double line} \quad . \quad . \quad . \quad (9)$$

EXAMPLE 5. *A Disc Isolated in Space. Radius r cm. and Thickness d cm.* (*Fig.* 6).

$$C = \varepsilon.\frac{2r}{\pi}\left(1 + \frac{d}{\pi r}\right)\ \text{electrostatic c.g.s. units} \quad . \quad . \quad (10)$$

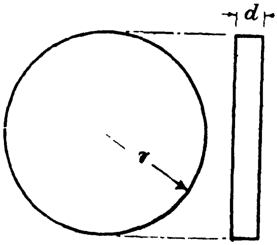


Fig. 6.

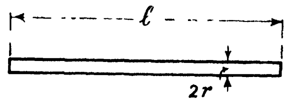


Fig. 7.

EXAMPLE 6. *An Isolated Straight Wire of Length l cm. and Radius r cm.* (*Fig.* 7).

$$C = \frac{\varepsilon.l}{2\log_e\dfrac{l}{r}}\ \text{electrostatic units} \quad . \quad . \quad . \quad (11)$$

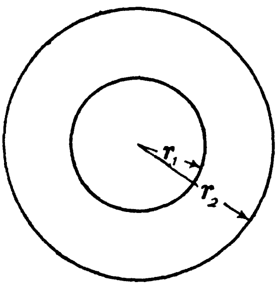


Fig. 8.

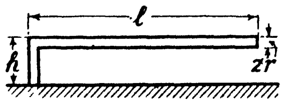


Fig. 9.

EXAMPLE 7. *A Concentric Cable of Length l cm., Core Radius r_2 cm. and Sheath Radius r_1 cm.* (*Fig.* 8).

$$C = \frac{\varepsilon.l}{2\log_e\dfrac{r_2}{r_1}}\ \text{electrostatic units} \quad . \quad . \quad . \quad (12)$$

EXAMPLE 8. *A Straight Horizontal Wire h cm. above the Earth's Surface and of Radius r cm., the Wire being Earthed at One End* (*Fig.* 9)

$$C = \frac{\epsilon . l}{2 \log_e \frac{2h}{r}} \text{ electrostatic units} \quad . \quad . \quad . \quad (13)$$

EXAMPLE 9. *A Straight Vertical Wire Earthed at the Lower End, e g a Simple Vertical Antenna Wire of Length l cm. and Radius r cm.* (*Fig.* 10)

$$C_{STATIC} = \frac{\epsilon . l}{2 \log_e \frac{2l}{r}} \text{ electrostatic units} \quad . \quad . \quad . \quad (14)$$

See also Chapter XVI, page 508.

It is important to observe that this formula is only strictly applicable as applied to the *static* condition. That is to say, it is only valid if at any given moment, the current has the same value at every point in

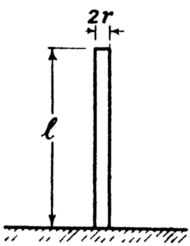


Fig 10

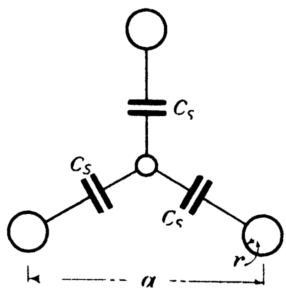


Fig 11.

the wire, and this is only true when the wavelength of the applied pressure is large as compared with the length of the wire. For high frequency currents, however, the current will be sinusoidally distributed throughout the length of the wire, and in this case the so-called *dynamic capacitance* of the antenna will be given by the expression

$$C_{dyn} = \frac{c . C_{STATIC} \sin \frac{2\pi f l}{c}}{2\pi f l} \quad . \quad . \quad . \quad . \quad (15)$$

where c is the wave velocity of propagation along the wire in centimetres per second and $\omega = 2\pi f$, where f is the frequency of the current in the line. When the frequency of the current is equal to the natural frequency of the antenna, that is, when $f = f_0 = \frac{c}{4l}$, then

$$C_{dyn.} = \frac{2}{\pi} C_{STATIC} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Reference should also be made to Chapter XVI, page 508, for further information about antennae.

EXAMPLE 10. *A Symmetrically Arranged Three-Phase Overhead Transmission Line* (*Fig.* 11).

Star capacitance per phase,

$$C = \frac{0{\cdot}024}{\log_{10}\frac{a}{r}}\mu\text{F per km.} \qquad . \quad . \quad . \quad . \quad (17)$$

Three-Phase Cables

For modern electric power stations and distribution systems three-phase cables are very extensively used, and although the transmission of power by means of cables is much more costly than by means of overhead lines, cable transmission has the advantage that it is much less subject to damage, and consequently the maintenance costs are reduced to a minimum. In city areas, for example, where high voltage overhead lines would not be allowed, cable transmission is indispensable.

Rubber-insulated cables are used for all kinds of installation work such as dwelling-house wiring, and since such cables cannot withstand mechanical shocks or stresses, they must be laid in protecting tubes when necessary. Rubber insulated cables may be used for pressures up to 25 kV., but for purposes of important transmission and distribution it may be said that all modern power cables are paper insulated with impregnated paper and are provided with a lead sheath. Not only is such impregnated paper insulation more reliable than rubber, but it is also practically unaffected by age. Fig. 12*a* shows the constructional features of a three-phase cable * for 6 kV. with circular cores, each core having a cross-sectional area of 70 sq. mm. Each of the three conductor cores is provided with its own paper insulating sheath and the three cores are then assembled with a filling of paper, jute, or other suitable material. The whole is enclosed in a paper insulating sheath over which is drawn a seamless lead sheath. Such cables are distinguished by the overall paper insulating sheath and are termed " belted " cables. When such a paper-insulated cable is buried in the ground there is a possibility of acidulated water attacking the lead sheath, and for this reason an outer coating of bituminised paper is provided, and over this a layer of impregnated jute. In order to provide against mechanical damage, two layers of steel tape are wrapped over the jute covering and the steel tape is protected from rust by an outer layer of bituminised jute.

In Fig. 12*b* is shown the cross-section of a belted cable,* also for 6 kV., each core having a section of 70 sq. mm. In this case, however, the cores are built up in sector form and consequently the overall diameter

* See T. Buchhold, *Elektrische Kraftwerke und Netze.*

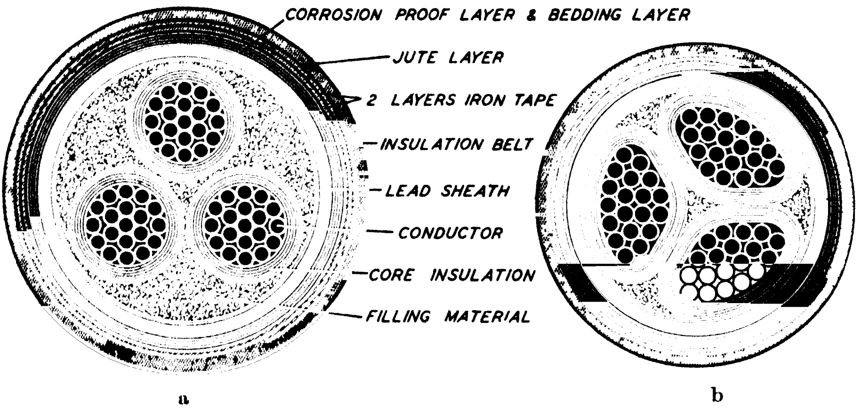


Fig 12

of the cable is greatly diminished as compared with the type shown in Fig 12*a*, the two diagrams of Figs 12*a* and 12*b* being drawn to the same scale Cables with sector shaped cores cannot be safely used for pressures greater than about 10 kV, whereas the belted type of cable shown in Fig 12*a* can be used for pressure up to about 20 kV Owing to the fact that the direction of the electric stress is not perpendicular

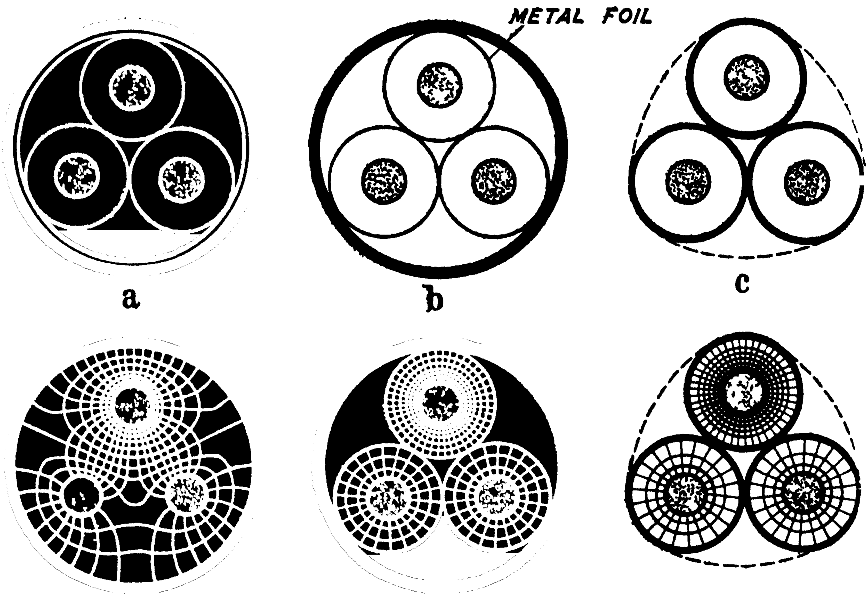


Fig 13

to the surface of the paper insulation * (see Fig. 13*a*), the belted type of three core cable is not suitable for pressures greater than about 20 kV.

Experimental test data have shown that paper insulation will not stand such high electric stress when in a direction parallel to the surface of the paper as when the stress is at right angles to the surface. The first practical realisation of the importance of maintaining the electric stress at right angles to the surface of the paper is due to Höchstädter, and for this purpose he provided each core with a screen of metal foil over the surface of its paper insulating sheath, as shown in Figs. 13*b* and 14*a*, this ensuring that the lines of electric force shall act in a radial direction. An alternative construction to the metal foil coating applied

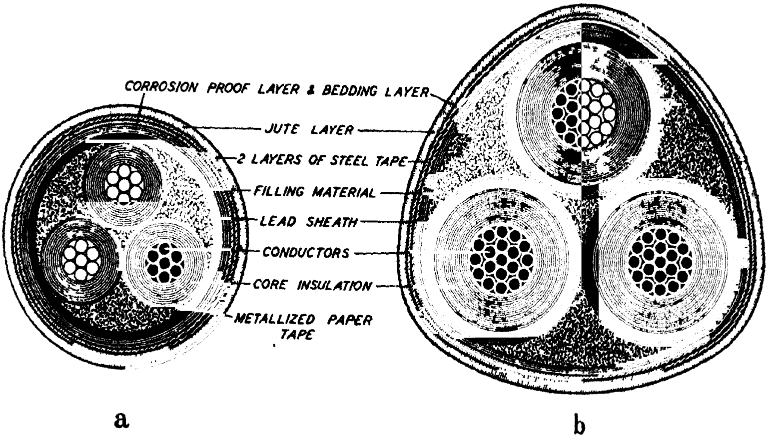


Fig. 14.

over the paper insulating sheath of each of the three cores is to use a lead sheath over each core and to assemble the three cores within a common overall corrosion-proof sheath, as shown in Fig. 14*b*. This arrangement has certain practical advantages over the Hochstadter screened type, or Fig. 13*b*.

The permissible temperature rise of paper-insulated cables is relatively very small, and the reason for this is as follows. When the cables become warm due to its loading, the impregnated compound expands more than the paper, so that the compound presses against the lead sheath, which consequently stretches. When the cable cools down again, the compound contracts, but will not return to its original position, and in consequence there is considerable risk of cavities being formed which will give rise to glow discharges (see Chapter II, page 33 and Chapter III, expression (12)) and gradually destroy the insulation. The glow discharge also

* See T. Buchhold, *Elektrische Kraftwerke und Netze.*

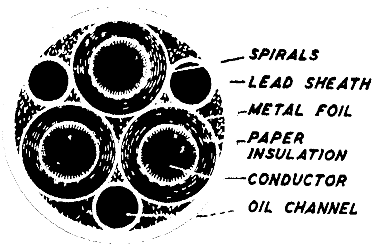


Fig. 15.

involves power dissipation, the amount of which can be measured by determining the so-called "loss angle" (see Chapters II, page 64, and XI, page 363).

For the highest voltages which are now used in practice, it is essential that all possibility of cavitation in the insulation should be eliminated and for this purpose there are two distinct types of construction now available, viz. oil filled cables, and those in which gas under pressure is maintained in the cable by means of a gas-tight sheath. In Fig. 15 is shown an example of a three-phase oil-filled cable having channels which are filled with a thinly fluid oil, thus keeping the paper insulation saturated. When the temperature of the cable rises, the oil expands and, spaced at intervals along the route of the cable, are expansion chambers into which the surplus oil can flow. In the following Table I are given characteristic data for a range of power cables from low-pressure distribution cables to those for the highest pressures which are now used in practice.

TABLE I

	Cross section sq mm	*Pressure kV. Line Phase*	*Operating Load kVA.*	*Capacitance µF per km. per Phase*	*Max. Pressure Gradient at Core Surface kV/mm.*	*tan δ*	*Dielectric Loss. Watts per km.*	*Temp. Rise at Sheath °C.*
SOLID PAPER INSULATION	*Single Sheath Cable*							
	3 × 150	0·5 0·29	265	0·96	0·23	0·01	0·77	0·0007
	3 × 150	6 3·5	3,300	0·60	2·2	0·01	70	0·06
	3 × 150	10 6	4,000	0·50	2·3	0·01	170	0·15
	Screened Cable							
	3 × 150	16 10	7,000	0·40	2·8	0·01	380	0·2
	3 × 150	30 17·6	13,000	0·30	3·3	0·01	860	0·4
	3 × 150	60 35	22,500	0·21	4·5	0·01	2,420	0·8
OIL-FILLED	1 × 240	60 35	47,000	0·38	5·7	0·004	685	0·4
	1 × 240	100 68	73,500	0·29	6·7	0·004	1,230	0·75
	1 × 240	220 127	115,000	0·20	9·3	0·004	4,060	1·60

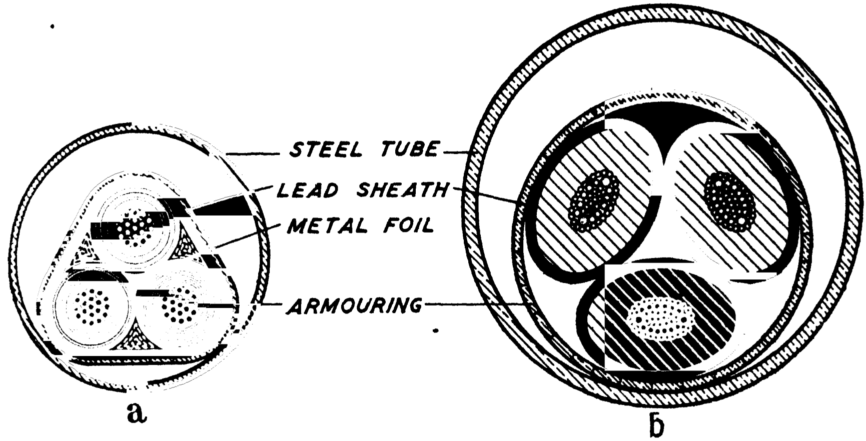


Fig. 16.

In Fig. 16*a* is shown a section of a gas-filled three phase cable having a common lead sheath for the three cores, whilst Fig. 16*b* shows the structural features of a gas-filled cable in which each core has a separate lead sheath.*

For further information on cables, see Chapter XV, page 467.

Some Practical Constructional Forms of Condensers

For practical purposes it is necessary to construct condensers for given values of the capacitance and having as small dimensions as possible. For these purposes, two plates separated by a dielectric is the basic type. If the surface area of each plate is s sq. cm. and the distance apart of the plates is d cm., then the capacitance is given by the expression

$$C = \frac{\varepsilon . s}{4\pi d} \text{ cm.} \tag{18}$$

When a p.d. V is applied to the plates there will be established a uniform electric field in the space between them. Apart from the "fringing" near the edges of the plates, the intensity of this electric field will be given by the expression

$$E = \frac{V}{d} \text{ volts per cm.}$$

If however, the insulating medium between the plates consists of layers of thicknesses, $d_1 : d_2 : d_3$ . . . . and dielectric constants $\varepsilon_1 : \varepsilon_2 : \varepsilon_3$ . . . respectively, then the field strength between the plates will not be uni-

* See also T. Buchhold, *Elektrische Kraftwerke und Netze*.

form. The capacitance of such a condenser will then be given by the expression

$$C = \frac{s}{4\pi\left\{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} + \ldots \frac{d_n}{\varepsilon_n}\right\}} \text{ cm.} \quad . \quad . \quad . \quad (20)$$

Inspection of the expression (7) on page 102 shows that the capacitance of a condenser can be increased by:

(*i*) Increasing the surface area of the plates.

(*ii*) Decreasing the distance between the plates.

(*iii*) Choosing an insulating medium having a high value for the dielectric constant.

All three of these methods are used in practice, either singly or in combination.

The increase of the surface area of the plates is obtained by using two aluminium foil tapes, or bands, wound together with a similar tape-

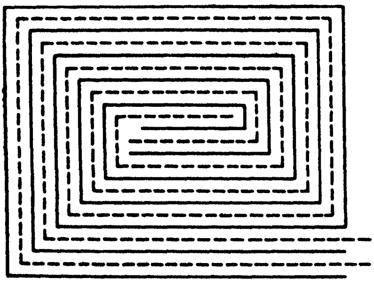


Fig. 17.

Fig. 18.

form of insulation material as shown in Fig. 17, in which case it is to be observed that the condenser will also have a certain amount of inductance. Another method is to arrange the plates as shown in Fig. 18. If there are a total of n such plates the capacitance of the condenser will be

$$C = \frac{\varepsilon(n-1)s}{4\pi d} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If the plates are arranged so that the two sets can be moved relatively to each other, the capacitance can be varied. For condensers with semicircular plates the capacitance is proportional to the angle α, by which the two sets are displaced relatively to each other, that is

$$C_\alpha = \frac{C_{180^\circ} - C_0}{180^\circ}\alpha + C_0 \quad . \quad . \quad . \quad . \quad (22)$$

where C_{180} is the capacitance when the two sets of plates are interleaved to the fullest extent and C_0 the capacitance when the plates are

turned so that they are separated to the fullest extent. For such condensers the minimum amount of insulating material is required and an important consequent feature is that the "loss angle" will be very small.

For radio tuning circuits it is frequently undesirable to have the capacitance proportional to the angle α since the adjustment for certain angular ranges is required to be finer than in others, and consequently, in most such condensers as used for receiver circuits the plates are made with a kidney-shaped profile which gives a finer frequency relationship. The maximum capacitance of receiving condensers is from about 250 to 1 000 cm., but for certain special purposes, such as short-wave reception, and neutrodyne reception, they may be of very much smaller capacitance.

A special form of condenser is the "differential" type in which the rotating member turns between two separated sets of fixed plates. When the rotor is turned, its capacitance to one set of the fixed plates is increased by the same amount as its capacitance to the other set of fixed plates is reduced.

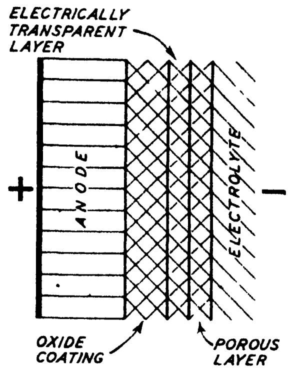


Fig. 19.

When oil or other insulating material such as mica is used for the dielectric, the capacitance increases proportionally with the dielectric constant ε. For condenser units of fixed capacitance, the use of the newer insulating materials such as "Condensa" makes it possible to obtain relatively large capacitance values in units of small size. For example, a unit comprising a tube 4·2 cm. long and 0·8 cm. diameter has a capacitance 700 cm. when Condensa is used as the insulating material, and only 100 cm. when "Calit" is used (see Table II, page 114).*

The capacitance of a condenser may be varied by varying the distance between the plates. For example, if insulating material of large dielectric constant is used a relatively large capacitance is obtained in small units, and by varying the force with which the plates are pressed together the capacitance can be varied. Such so-called "squeeze" condensers are used, for example, for the tuning of band filters instead of rotary condensers.

Especially large capacitances are obtained by the use of electrolytic units in which the plates are separated by an electrolyte. When direct current is passed through such a system, the plate which forms the anode becomes coated with an extremely thin oxide film which functions as a dielectric and prevents the further passage of the direct current. The arrangement thus forms a condenser system in which the plates are separated by an extremely thin dielectric (see Fig. 19). In this way it is possible to obtain a capacitance of 8 μF with plates of

* F. Vilbig: *Hochfrequenztechnik.*

only 500 sq. cm. surface area. Such condensers are chiefly used for smoothing d.c. pressures of relatively low magnitude.

It is to be observed that whilst in the case of condensers, definite values of the capacitance are required, yet in the case of sockets, coil supports, switches, and leading-in tubes for the plate connections, the capacitance should be reduced to a minimum. For this reason small metal parts with small surface areas must be used and relatively large distances between any such metal surfaces, whilst the insulating material should be of low dielectric constant and very small loss angle (see Chapter II, page 64).

Apart from the magnitude of the capacitance, the most important characteristics of a condenser are the magnitude of the losses and the permissible loading. Losses arise due to inferior insulating material glow and brush discharge, and dielectric hysteresis. These losses depend upon the composition of the dielectric, the size of the condenser and the frequency on which it operates.

Systems of Condensers

(*i*) Condensers in Series. If a number of condensers A, B, C, are connected in series as shown in Fig. 20 and the system is charged by connecting the terminals S, S, to a source of e.m.f., then if the plate a of condenser A becomes charged with a quantity $+Q$ the other plate a_1 will receive a charge $-Q$. An equal charge $+Q$ will then pass to the plate b of condenser B and consequently a charge $-Q$ will be induced on plate b_1, and so on. In other words, each condenser will become charged with the same quantity of electricity Q.

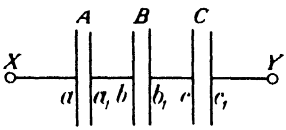


Fig. 20.

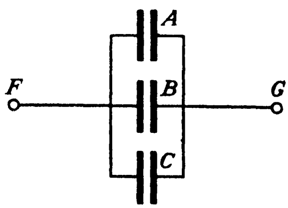


Fig. 21.

If the capacitances of the individual condensers are respectively $C_A : C_B : C_C$ . . . and the corresponding values of the terminal p.d.'s of the individual condensers are $V_A : V_B : V_C$ . . . . then

$$V_A = \frac{Q}{C_A} : V_B = \frac{Q}{C_B} : V_C = \frac{Q}{C_C} \dots$$

If V is the p.d. across the whole series, then

$$V = V_A + V_B + V_C + \dots = Q\left[\frac{1}{C_A} + \frac{1}{C_B} + \frac{1}{C_C} + \dots\right] = \frac{Q}{C}$$

where C is the capacitance of the single condenser which is equivalent to the series combination of Fig. 20, that is to say,

$$\frac{1}{C} = \frac{1}{C_A} + \frac{1}{C_B} + \frac{1}{C_C} + \ldots \quad . \quad . \quad . \quad (23)$$

(ii) CONDENSERS IN PARALLEL.—When the condensers are connected in parallel, as shown in Fig. 21, each will receive the same value of the applied p.d., and if $C_A : C_B : C_C \ldots$ are the respective values of the capacitances of the individual condensers, then

$$V = \frac{Q_A}{C_A} = \frac{Q_B}{C_B} = \frac{Q_C}{C_C} = \ldots$$

The total charge given to the parallel system will then be

$$Q = Q_A + Q_B + Q_C + \ldots$$

or $$Q = V(C_A + C_B + C_C + \ldots) = V.C,$$

where C is the capacitance of the single condenser which is equivalent to the parallel arrangement, that is to say,

$$C = C_A + C_B + C_C + \ldots \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The Dielectric Constant

Faraday found that the capacitance of a condenser depends not only on the size, shape, and distance apart of the conducting surfaces, but also on the nature of the insulating medium between the surfaces, and he gave the name *specific inductive capacity* to that property of an insulating medium on which the capacitance of a condenser depends. The term *dielectric constant* is now commonly used in practice for this quantity, so that

$$\text{dielectric constant } \epsilon = \frac{\text{Capacitance of condenser with the given dielectric}}{\text{Capacitance of same condenser with a vacuum (or air) as the dielectric}}.$$

For a vacuum the value of the dielectric constant is $\epsilon = 1$, and in Table II, page 114 will be found the values of ϵ for a number of insulating materials of practical importance (see also Chapter II, page 52).

The reason why the capacitance is affected when a block of matter is placed between the plates of a condenser was explained by Faraday as follows. Suppose that, between the opposite charged plates of a condenser as shown in Fig. 22, an uncharged metal sphere is introduced and consider what happens. The sphere becomes "polarised" so that on the left-hand side a positive charge assembles and on the right-hand side a positive charge assembles. Due to these induced charges on the sphere, the condenser plate on the left-hand side is able to take an increased negative charge and the right-hand plate an increased positive charge, that is to say, the capacitance of the condenser will have increased whilst the p.d. across the terminals has been maintained constant.

TABLE II

Insulating Material	*Dielectric Constant* ϵ	*Angle of Loss* $\times 10^{4}$, *i.e.* $10^{4} \times \tan\delta$ *for* (. . . *Hz.*)	*Breakdown Strength in kV. per mm*
Alcohol	26·3		
Bakelite	4·8–5·3	126–300 (800) : 100 (10^{3}) : 220 (10^{7})	23
Amber	2·8	50 (6×10^{6})	
Calit	6·5	3·7 (3×10^{6}) : 3·4 (10^{7}) : 2·5 (5×10^{7})	
Cellon	7·0	700 (800)	
Celluloid	4·0–4·1	500 (2×10^{6})	
Ceresin	2·1–2·3	0·3 (800)	
Condensa	40–50	8·5 ($7·5 \times 10^{6}$) : 7·2 (3×10^{6}) : 6·4 (6×10^{6})	
Condensa C	80–100	20–40 (3×10^{6})	
Ebonite	2·0–3·5	24–230 (800) : 60–80 (6×10^{6})	Up to 34
Frequenta	6·1	3·0 (3×10^{6}) : 2·6 (5×10^{6})	
Frequentit	6·1	8 (3×10^{6}) : 6·8 (10^{7}) : 6 (5×10^{7})	
Gas	1	0	
Glass	5–12	130–240 (800) : 35–75 (10^{6})	12–20
Mica	4–8	2–10 (800) : 1·7 (3×10^{6}–5×10^{7})	20–60
Rubber	2·5		
Synthetic Porcelain	5·4–6·4		
Wood	2·5–6·8		
Vulcanised Rubber	2–3·5		
Air	1·0006	0	3·2
Marble	8·5	1,000 (800)	
Micanite	4·5–6		25–35
Mycalex	8	18 (3×10^{5}–5×10^{7})	
Oiled Paper	2		
Paper	1·8–2·6	40 (800)	
Paraffin	1·7–2·3	0·8 (800) : 3·9 (6×10^{6})	30
Paraffin Oil	2–2·5		13
Pertinax	4·8–5·4	250 (800) : 230–390 (6×10^{6}) : 1,000 (6×10^{7})	10–20
Petroleum	2–2·2		10
Porcelain	5–6·7	110–140 (800) : 135 (6×10^{5})	15
Presspahn	3·4	265 (5×10^{6}) : 580 ($1·2 \times 10^{7}$)	11–22
Quartz	4·5–4·7	1 (3×10^{5}–10^{7}) : 1·1 (5×10^{7}–10^{8})	
Quartz-glass	3·7–4·2	1·8 (3×10^{5}–3×10^{7}) : 1·7 (10^{7})	
Shellac	2·7–3·8		
Slate	6·6–7·4	3,400 (800) : 2,500 (6×10^{6})	
Sulphur	3·6–4·1		
Sealing wax	4		
Steatite	6·4	20 (10^{6}) : 15 (5×10^{7})	
Turpentine	2·3		
Trolitul	2·2	3·9 (10^{6}) : 4·5 (10^{7}) : 5·4 (5×10^{7})	
Tourmaline	6	5·2 (2×10^{6})	
Vulcanised Fibre	4·1	670 (2×10^{6}) : 1,000 ($1·2 \times 10^{7}$)	5
Water	80		

In a precisely similar way Faraday viewed the effect of the insulating medium between the plates of the condenser. He considered that the individual molecules behave like minute spherical conductors which, however, were not able to pass on their electrical charges from one to the other, and it was on this hypothesis of Faraday that the idea of the "displacement current" was introduced by Maxwell, this term imply-

ing that only "displacement" of the bound electric charges is possible, in direct contrast to the case of a metal conductor in which at least one type of electric particle, viz. the electron (see Chapter II, page 25) can be considered as being almost unrestrictedly mobile. In actual fact, the modern view regarding these phenomena is in strikingly close agreement with the rough outline given in the foregoing. An atom consists, for example, of a positive nucleus and a number of revolving electrons travelling in orbits encircling the nucleus. If such a structure is placed in an electric field the electrons and the positive nucleus will be displaced in opposite directions, so that the atom may be said to have received an "induced electric moment", see page 116, just as in the case of the Faraday metal sphere when introduced between the charged plates of the condenser as shown in Fig. 22. The sum of such effects of the individual atoms is therefore one of the causes of the characteristic property of the material which is defined as the "dielectric constant".

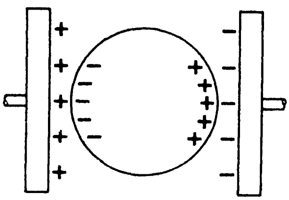


Fig. 22.

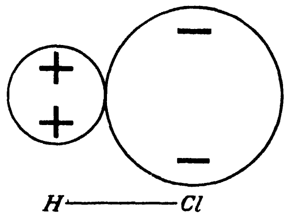


Fig. 23.

Clausius and Mosotti, assuming that each molecule could be regarded as a perfectly conducting sphere of radius a and that the polarisation of the aggregate of the molecules is the sum of the polarisation of the component molecules, showed that the dielectric constant ε could be expressed by the following equation, viz.

$$\left(\frac{\varepsilon - 1}{\varepsilon + 2}\right)\frac{M}{\rho} = \frac{4}{3}\pi N a^3 \quad . \quad . \quad . \quad . \quad (25)$$

in which M in grams is the mass of one mol of the substance, that is, 1 gm.-molecule, or the amount of the substance the mass of which in grams is given by the number which is the molecular weight of the substance: thus for H the molecular weight is 2, so that $M = 2$ grams: ρ is the density of the substance: and $N = 6{\cdot}06 \times 10^{23}$ is Avogadro's number (sometimes also known as Loschmidt's number), and is the number of molecules in the mass of one mol (see also Chapter II, page 56). For hydrogen gas at 0° C. and 760 mm. pressure, $\rho = 0{\cdot}898 \times 10^{-4}$ gm. per cubic centimetre, so that $\dfrac{M}{\rho} = 2{\cdot}25 \times 10^4$ c.c. and is the volume of 1 mol. The assumptions on which the expression (25) is based are now known to be erroneous for the majority of cases, and the magnitude of the dielectric constant ε is now known to be due to the composite effect

comprising two separate actions: (*i*) the distortion of the electronic orbits of the atoms when an electric field is impressed on the substance and (*ii*) the effect due to the existence of "electric doublets" or "dipoles" formed by the molecules of the substance (see Fig. 23). As regards (*i*), when an electric field is impressed on an insulator the electron orbits become displaced relatively to the nucleus, so that, instead of the centres of mass of the negative and positive charges being coincident as in the normal case when no electric field is present, the impressed field in effect separates these centres of mass so that the positive and negative charges together form a system having an electric moment and that such a distortion of the electric orbits does, in fact, take place, is proved by the modification of the spectrum when an electric field is present.

(*ii*) For relatively low frequencies the "molecular polarisation" is expressed by the quantity

$$R = \frac{\epsilon - 1}{\epsilon + 2}\frac{M}{\rho} \quad . \quad . \quad . \quad . \quad (26)$$

and in many cases this is very much larger than the quantity on the left hand side of the equation (25), which latter is in good agreement with the experimental results at very high frequencies such as those of light waves. For a long time the cause of these discrepancies was not understood, and it was only explained about 25 years ago when P. Debye showed that there was a second effect which contributed to the molecular polarisation as follows. When two atoms combine to form a molecule for example, when hydrogen (H) and chlorine (Cl) combine to form hydrochloric acid (HCl), it is not to be assumed that the electric charges of the respective atoms neutralise each action to the extent that they give rise to no external field. On the contrary, other physical considerations point to the conclusion that the H atom retains its positive charge and the Cl atom its negative character. Such a molecular structure behaves like an electrical dipole as shown in Fig. 23. When an electric field is impressed on this dipole it will become oriented in the field and, as experiment and calculation both show, will contribute a considerable portion to the total molecular polarisation. Debye has shown that when the dipole effect is taken into account, the molecular polarisation is completely accounted for. It is to be noticed in particular that the dipole contribution is strongly dependent upon the temperature. The molecular polarisation may then be expressed as follows:

$$\frac{\epsilon - 1}{\epsilon + 2}\frac{M}{\rho} = \frac{4\pi}{3}N\left(\alpha + \frac{\mu^2}{3kT}\right) \quad . \quad . \quad . \quad (27)$$

In this equation, μ is the electric moment of the dipole, k is the Boltzman constant, that is $\frac{R}{N}$, where R is the constant of a perfect gas, N is Avagadro's number, and T is the absolute temperature. If $\mu = 0$,

that is, if the molecule is non-polar, only the first term on the right-hand side of equation (27) will exist, and this equation then becomes of the same form as the Clausius Mosotti relationship, and this is the particular condition to which that relationship is applicable.*

The Energy of the Electric Field

Suppose in Fig. 24 V electrostatic e.g.s. units is the p.d. across the boundary surfaces A and B of an electric field, it being assumed for example, that this field has been established by transferring the quantity of electricity Q from the surface B to A. Suppose now that a further quantity δQ is transferred from B to A, this quantity being so small that the p.d. between the surfaces is not appreciably affected by the transfer. Then by definition of potential difference, the work done in making this transfer will be $V.\delta Q$ ergs. If C is the capacitance of the field and Q the charge when the p.d. is V, all these values being in electrostatic e.g.s. units, then

$$Q = V.C; \qquad \text{or } V = \frac{Q}{C}.$$

Hence the work done in effecting the transfer of the whole quantity from the surface B to A, that is, in establishing the field, will be

$$\int_0^Q V\,dQ = \int_0^Q \frac{Q}{C}\,dQ = \tfrac{1}{2}Q^2/C = \tfrac{1}{2}V.Q \text{ ergs} \quad . \quad . \quad (28)$$

and this energy is stored in the dielectric. Since the ratio

$$\frac{Q}{V} = C \text{ (which is a constant)}$$

defines the relationship between the Q and the p.d. V, this may be graphically represented by the straight line OB in Fig. 25, and it will be seen that the energy stored in the electric field as defined by the expression (28) is given by the area of the triangle OBC.

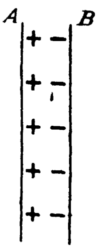


Fig. 24.

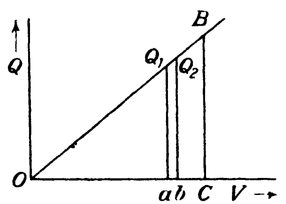


Fig. 25.

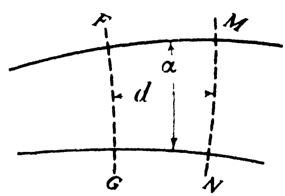


Fig. 26.

Energy Stored per Unit Volume of the Dielectric

In Fig. 26 is shown a unit tube of electric force, that is to say, one which extends from a charge $q = +1$ electrostatic unit on one boundary

* See *Engineering*, 1942, page 154.

conducting surface to a charge of $q = -1$ electrostatic unit on the other boundary conducting surface. Two equipotential surfaces FG and MN are also shown, the p.d. between these two surfaces being $v = 1$ electrostatic unit. The volume of the tube which is bounded by the two equipotential surfaces FG and MN may be termed a " unit cell " of the dielectric, and from page 117 it will be seen that the energy stored in this unit cell will be

$$u = \tfrac{1}{2}q.v = \tfrac{1}{2} \text{ erg} \quad . \quad . \quad . \quad . \quad (29)$$

If α sq. cm. is the mean cross-sectional area of the cell and d cm. the distance between the equipotential surfaces, the volume of the cell will be αd c.cm., so that

$$\text{Energy stored per c.cm. of the dielectric} = \frac{1}{2\alpha d} \text{ ergs} \quad . \quad (30)$$

Again, if E dynes is the electric force in the cell, then the p.d between the two equipotential surfaces FG and MN will be (see Chapter III, Expression (13))

$$v = E.d = 1 \quad . \quad . \quad . \quad . \quad . \quad (31)$$

But from Chapter III, expression (8), page 82,

$$E = \frac{4\pi\sigma}{\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where

$$\sigma = \frac{q}{\alpha} = \frac{1}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

hence, from expressions (31), (32) and (33)

$$\frac{1}{\alpha}\cdot\frac{1}{d} = \sigma E = \frac{E^2\epsilon}{4\pi} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and the density of the electric charge is

$$\sigma = \frac{\epsilon . E}{4\pi} \text{ electrostatic units} \quad . \quad . \quad . \quad (35)$$

From expressions (30) and (34).

Energy stored by the electric field in the dielectric is

$$\frac{\epsilon . E^2}{8\pi} \text{ ergs per cubic centimetre} \quad . \quad . \quad . \quad (36)$$

Dielectric Strain or Electric Displacement

Assuming that the application of the electric force E strains the dielectric in an analogous manner to the strain of a stretched spring (see also dipole theory, p. 116), electric energy will be stored in the dielectric when under the influence of an electric force just as energy is

stored in a stretched spring. For a stretched rod of elastic material, the energy stored is

$$\tfrac{1}{2} \times \text{Pull} \times \text{Extension}$$

or, the energy stored per unit length is

$$\tfrac{1}{2} \times \text{Pull} \times \frac{\text{Extension}}{\text{Length}} = \tfrac{1}{2} \times \text{Pull} \times \text{Strain} \quad . \quad . \quad (37)$$

and the energy stored per unit volume is

$$\tfrac{1}{2} \times \text{Intensity of pull} \times \text{Strain} \quad . \quad . \quad . \quad (38)$$

Now writing down the expression for the energy stored in the dielectric, viz (see page 118, expression (36))

$$\text{Stored energy} = \tfrac{1}{2}E.\frac{\varepsilon E}{4\pi} \text{ ergs per c.cm.}$$

$$= \tfrac{1}{2} \text{ Electric force} \times \text{Electric strain} \quad . \quad (39)$$

The quantity $\frac{\varepsilon E}{4\pi}$ is termed the *electric strain*: it has also been called the *electric displacement* or otherwise, the *polarisation in the dielectric* in the direction of E. In accordance with expression (35) this quantity is also the density of the electric charge.

Further, for the case of a stretched rod, the relationship holds:

$$\frac{\text{Intensity of pull}}{\text{Strain}} = \text{Young's Modulus of Elasticity.}$$

In the case of the electric field

$$\frac{\text{Electric force}}{\text{Electric strain}} = \frac{E}{\frac{\varepsilon . E}{4\pi}} = \frac{4\pi}{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

so that *the quantity* $\frac{4\pi}{\varepsilon}$ *is analogous to the Young's modulus of elasticity for elastic materials.*

In Chapter I, page 6, it has been seen that the propagation of electromagnetic waves (e.g. light) through open space is given by the relationship

$$c = \sqrt{\frac{1}{\varepsilon . \mu}} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

It is also known from the laws of mechanics that the velocity of propagation of wave motion is given by the relationship

$$\text{velocity} = k\sqrt{\frac{\text{elasticity}}{\text{density}}} \quad . \quad . \quad . \quad (42)$$

Further, since from expression (40) above, the quantity $\frac{4\pi}{\varepsilon}$ is anologous to elasticity, it follows from a comparison of the expressions (41) and (42) that magnetic permeability μ, is analogous to density.

Maximum Value of the Energy which can be Stored in a Dielectric

On page 118, expression (36) above, it is seen that the energy stored in a dielectric is

$$\frac{\varepsilon E^2}{8\pi} \text{ ergs per cubic centimetre,}$$

in which the electric intensity is $E = -\frac{dV}{dx}$ dynes when the pressure V is expressed in electrostatic units. If V is expressed in volts, then

$$E = \frac{1}{300}\frac{dV}{dx} \text{ dynes.}$$

Hence the energy stored in the dielectric is

$$\frac{\varepsilon E^2}{8\pi} = \frac{\varepsilon}{9 \times 10^4 \times 8\pi} [\text{Fall of potential in volts per cm.}]^2 \text{ ergs per c.cm.}$$

$$= \frac{\varepsilon}{9 \times 10^{11} \times 8\pi} [\text{Fall of potential in volts per cm.}]^2 \text{ joules per c.cm.}$$

so that the maximum value of the energy which can be stored in a dielectric is

$$0{\cdot}442 \times 10^{-13}\varepsilon \text{ [Dielectric strength in volts per cm.]}^2 \text{ joules per c.cm.}$$
$$= 5{\cdot}36 \times 10^{-13}\varepsilon \text{ [Dielectric strength in volts per cm.]}^2 \text{ ft. lb. per cubic inch.}$$

For example, taking the dielectric strength of air as 32,000 volts per centimetre, the maximum amount of energy which can be stored in air at normal temperature and pressure is

$$5{\cdot}36 \times 10^{-13}(32{,}000)^2 = 5{\cdot}5 \times 10^{-4} \text{ ft. lb. per cubic inch}$$

and for micanite, of which the dielectric strength is 32,000 volts per millimetre and the dielectric constant $\varepsilon = 5$, the maximum amount of energy which can be stored is

$$5{\cdot}36 \times 10^{-13} \times 5(320{,}000)^2 = 0{\cdot}28 \text{ ft.-lb. per cubic inch.}$$

Electric Stress in a Cable Dielectric

In Fig. 27 is shown diagrammatically a section of a single-core cable in which the radius of the sheath is R cm. and the radius of the core is r cm. Then if ε is the dielectric constant of the insulating material (see Table II), the capacitance of the cable will be, from expression (12), page 103,

$$C = \frac{\varepsilon}{2 \log_\varepsilon \frac{R}{r}} \text{ electrostatic units per centimetre length.}$$

If the core of the cable is charged to a potential V electrostatic units by means of a quantity q electrostatic units per centimetre length, then

$$q = V.C.$$

The electric stress in the dielectric at a distance a cm. from the axis is by expression (9), Chapter III, page 83,

$$E_a = \frac{1}{\epsilon}\frac{2q}{a} \text{ dynes} \quad . \quad . \quad . \quad . \quad (43)$$

The dielectric stress will therefore be a maximum at the surface of the core, viz. when $a = r$, that is, the maximum stress will be

$$E_r = \frac{2q}{\epsilon r} \text{ dynes} \quad . \quad . \quad . \quad . \quad (44)$$

From expressions (43) and (44) it follows that

$$E_a . a = E_r . r,$$

and since from expressions (41) and (43)

$$r = \frac{2q}{\epsilon . E_r} = \frac{2VC}{\epsilon . E_r} = \frac{V}{E_r} . \frac{1}{\log_e \frac{R}{r}} \text{ cm.}$$

If V is expressed in volts, then

$$E_r = -\frac{1}{300}\frac{dV}{dr} \text{ dynes, so that}$$

$$r = \frac{V}{U}\frac{1}{\log_e \frac{R}{r}} \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Where U is the stress in the dielectric at the surface of the core expressed in volts drop per centimetre measured along the radial direction. This quantity is the maximum stress to which the cable dielectric is subjected and the value which is to be assigned to U in the design of a cable is determined from the dielectric strength of the insulation material divided by an appropriate factor of safety. For a given value of V and U there

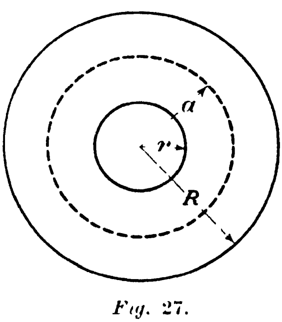


Fig. 27.

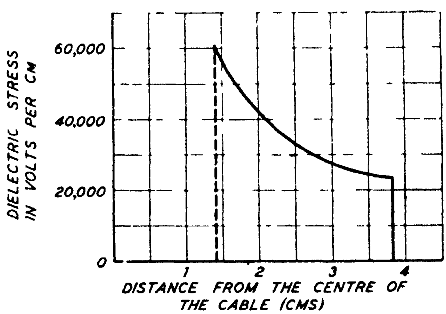


Fig. 28.

is one value of r which gives a minimum radius R for the sheath, and the condition for which R is a minimum is the condition that

$$\frac{dR}{dr} = 0$$

in the expression (45). This condition leads to the equations

$$r = \frac{V}{U} : \log_e \frac{R}{r} = 1 : R = e.r \quad . \quad . \quad . \quad (45)$$

where $e = 2{\cdot}72$ and is the base of natural logarithms.

EXAMPLE. Suppose the cable insulation is paper and that the permissible dielectric stress is $U = 60{,}000$ volts per centimetre, and let the working pressure be $V = 85{,}000$ volts. Then the radius of the core which will give a minimum thickness for the cable dielectric will be

$$r = \frac{V}{U} = 1{\cdot}42 \text{ cm.}$$

and the overall radius of the dielectric will be

$$R = e.r = 2{\cdot}72 \times 1{\cdot}4 = 3{\cdot}8 \text{ cm.}$$

The dielectric U_a stress at any radius a will be given by the equation

$$U_a.a = U.r$$

that is

$$U_a = \frac{U.r}{a} = \frac{V}{a}$$

and this relationship is shown in Fig. 28.

Chapter V

CURRENT DISTRIBUTORS AND NETWORKS

Equivalent Resistance of Conductors Connected Respectively in Series and in Parallel

(A) SERIES CONNECTION.—If several conductors are connected end to end—that is, *in series*, so that the same current flows through each conductor—the resistance of the combination is the ratio of the p.d. at the terminals of the series to the current through the series. Since the p.d. at the terminals is the sum of the p.d.'s at the end of the individual conductors the resistance of the series is the sum of the resistances of the individual conductors. Thus, if $R_1 : R_2 : R_3 : \ldots$ are the resistances of the individual conductors, the total resistance of the series is

$$R = R_1 + R_2 + R_3 + \ldots \qquad (1)$$

(B) PARALLEL CONNECTION.—If a number of conductors are joined so that one end of each conductor is connected to a common point A, and the other end of each conductor to a second common point B, the

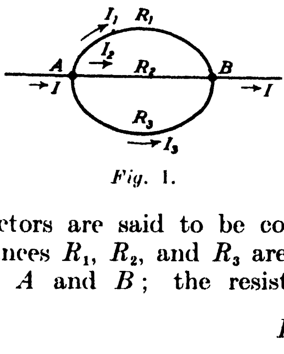


Fig. 1.

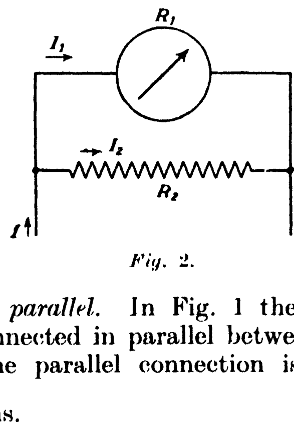


Fig. 2.

conductors are said to be connected in *parallel*. In Fig. 1 the three resistances R_1, R_2, and R_3 are shown connected in parallel between the points A and B; the resistance of the parallel connection is then

$$R = \frac{V}{I} \text{ ohms.}$$

Let $I_1 : I_2 : I_3$, respectively, be the currents through the individual branches, then

$$I = I_1 + I_2 + I_3$$

but
$$I = \frac{V}{R} : I_1 = \frac{V}{R_1} : I_2 = \frac{V}{R_2} : I_3 = \frac{V}{R_3}$$

hence,
$$\frac{V}{R} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

or

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A specially important example is that in which there are only two branches in parallel, such as when a *shunt* is used with a galvanometer (Fig. 2). Then the equivalent resistance R is given by the relationships

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \text{that is} \quad R = \frac{R_1 R_2}{R_1 + R_2}.$$

In this case $\quad \dfrac{I_1}{I} = \dfrac{V}{R_1} \cdot \dfrac{R}{V} = \dfrac{R}{R_1} \quad$ or $\quad \dfrac{I_1}{I} = \dfrac{R_2}{R_1 + R_2}$,

so that $\quad I_1 = I\dfrac{R_2}{R_1 + R_2}; \; I_2 = I\dfrac{R_1}{R_1 + R_2}; \; \dfrac{I_1}{I_2} = \dfrac{R_2}{R_1} \quad . \quad . \quad (3)$

Kirchhoff's Rules

The following two deductions from Ohm's Law, Chapter II, page 41, are very useful in dealing with a network of conductors.

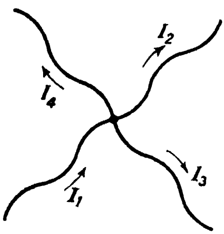


Fig. 3.

1. *If several conductors meet at a point the algebraic sum of the currents flowing to the point is zero, currents flowing to the point being reckoned positive and currents flowing from the point being reckoned negative.*

This is merely a statement of the fact that there is no accumulation of electricity at any point in a circuit in which a current is flowing and this result is really included in Ohm's Law

Thus, in Fig. 3

$$i_1 - i_2 - i_3 - i_4 = 0;$$

or

$$i_1 = i_2 + i_3 + i_4.$$

This result is useful in dealing with alternating current systems.

2. *If two or more conductors form a closed figure, the algebraic sum of the products of the resistance and current taken for each conductor (the current being reckoned positive the same way round the figure) is equal to the algebraic sum of the e.m.f.'s acting round the figure in the same direction that is,*

$$\Sigma IR = \Sigma E \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where the summation is to be taken round the closed figure.

This result may be deduced from Ohm's Law, and will be clear from the following example.

Suppose the arrangement of circuits is as shown in Fig. 4, and consider the mesh $ABCA$. From Kirchhoff's rule –

$$E_3 - I_3R_3 + I_2R_2 = I_1R_1.$$

Now $E_3 - I_3R_3$ is the amount by which the potential of B is *above* the potential of C, and I_2R_2 is the amount by which the potential of C

is *below* the potential of A. Hence $I_2R_2 - (E_3 - I_3R_3)$ is the amount by which the potential of B is *below* the potential of A. But I_1R_1 is the amount by which the potential of B is *below* the potential of A.

Hence $$I_1R_1 = I_2R_2 - E_3 + I_3R_3\,;$$
or $$E_3 = I_3R_3 + I_2R_2 - I_1R_1.$$

If a loop consists of three conductors, as in Fig. 5, in which no e.m.f. acts, then ΣIR for the closed loop is zero.

An example of the application of this result is found in connection with mesh connected alternating current systems.

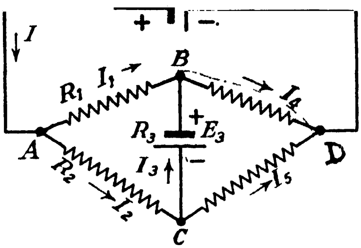


Fig. 4.

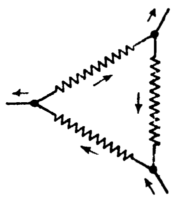


Fig. 5.

Wheatstone Bridge: (see also Chapter IX, page 311).

In Fig. 6 the two points A and B are connected by the branches ACB and ADB, of which ACB consists of the resistances R_1 and R_2 in series, and ADB the resistances R_3 and X in series.

Suppose a current enters at A and leaves at B, dividing between the two branches ACB and ADB.

The potential of the point C will be something intermediate between that of A and that of B. It will clearly be possible to find some point D in the branch ADB, such that the potential of D is the same as that of C. If such a point be found and a galvanometer be connected to the points C and D that is, to *bridge* the two branches - there will be no deflection of the galvanometer. This condition will be satisfied if—

$$I_1R_1 = I_2R_3, \text{ and } I_1R_2 = I_2X\,;$$

that is - $$\frac{I_1}{I_2} = \frac{R_3}{R_1} = \frac{X}{R_2}\,;$$

or $$X = \frac{R_2}{R_1}R_3.$$

If $$R_1 = R_2 \quad : X = R_3.$$

„ $$R_1 = 10R_2 \quad : X = \frac{1}{10}R_3.$$

„ $$R_1 = 100R_2 : X = \frac{1}{100}R_3.$$

If the values of $R_1 : R_2 : R_3$ are known and adjustable, the value of the unknown resistance X can be determined with great accuracy.

The above description gives the principle of this method of measuring resistances.

Actually the Wheatstone Bridge is made in a variety of forms, the resistances $R_1 : R_2 : R_3$ being built up in one box which sometimes also includes the galvanometer G. If very great accuracy is required, however, the galvanometer used is of the reflecting mirror type.

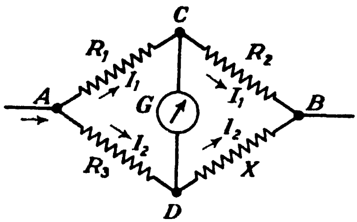


Fig. 6.

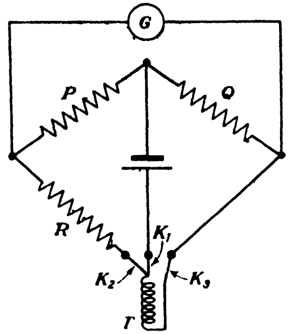


Fig. 7.

The Resistance Thermometer

The principle of the Wheatstone Bridge is of wide application, and one interesting example is as a " resistance thermometer " as illustrated in Fig. 7, this particular arrangement being known as the Siemens Three-Lead Bridge. The coil T is wound with pure platinum or pure nickel wire and is fitted with three leads, of which K_2 is connected in series with the arm R whilst the coil T and the lead K_3 form the adjacent arm. In this way, the lead resistances for the coil T are completely compensated since the resistances of K_2 and K_3 are supplied and made equal by the constructors of the coil T. The resistance arms P and Q are made equal and the galvanometer balance is obtained by adjusting the resistance R so that the resistance of the coil T is then equal to R.

The increase of resistance with the temperature of a pure metal wire provides a valuable and accurate method for the measurement of temperature. For a temperature range of $-200°$ C. to $+550°$ C. the practical importance of resistance thermometers has immensely increased in recent years. For the coil winding the only metals which are suitable are pure platinum and pure nickel wires. Commercial nickel wire is unsuitable since its degree of purity may vary over a wide range and the corresponding calibration charts when used for resistance thermometry may show largely divergent characteristics.

For nickel resistance thermometer wires the temperature coefficient

(see Chapter II, page 44, Table I) is $5{\cdot}48 \times 10^{-3}$ per 1° C. with an accuracy of about $\pm$ 16 parts in a million. In addition, for special purposes, nickel wire with a temperature coefficient of 6·4 to $6{\cdot}5 \times 10^{-3}$ can be obtained. For resistance thermometers using platinum wire the temperature coefficient is $3{\cdot}85 \times 10^{-3}$, (see also Chapter 6, page 163).

Current Distributors

A very important practical problem is the question of the maximum pressure drop at any point in a distribution system, to ensure, for example, that the percentage pressure drop shall not exceed the statutory limit. In Fig. 8*a* is shown the "go" and "return" conductors supplying a

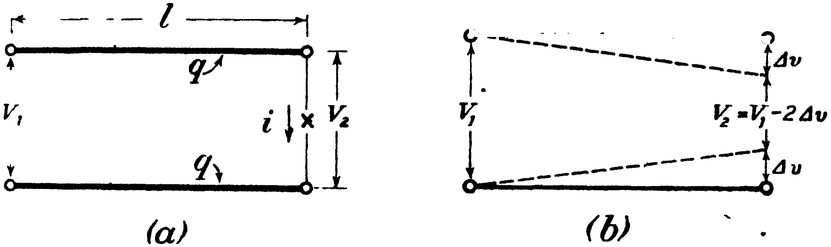


Fig. 8.

single consumer with a current of i amps. The cross-section of each conductor is q sq. mm. and the conductivity is λ in reciprocal ohms (i.e. siemens). If the transmission distance is l metres the pressure drop in each conductor will be Δv volts and the total pressure drop for the two conductors will be (Fig. 8*b*)

$$2\Delta v = 2\frac{i.l}{\lambda.q} \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For transmission and distribution problems it is convenient to use the metre as the unit of length and the square millimetre as the unit of cross-sectional area. The "conductivity", that is, the reciprocal of the specific resistance, may then be expressed as follows,

$\lambda = 56$ mm.²/m. reciprocal ohms (i.e. siemens) for copper

$\lambda = 35$ mm.²/m. reciprocal ohms (i.e. siemens) for aluminium.

The pressure at the consumer's terminals will then be $V_2 = V_1 - 2\Delta v$.

The foregoing treatment takes into account the drop in the return line as

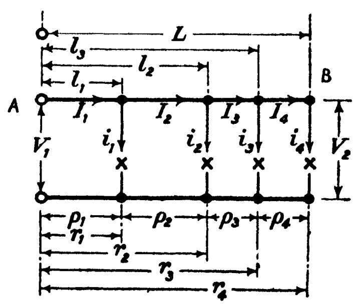


Fig. 9.

well as in the outward line. In many cases, however, the return line has a different cross-section from the outgoing line, and in what follows only the pressure drop in the outgoing line will be considered, and the corresponding pressure drop in the return line can always be found in a similar manner.

Suppose, now, that several consumers, $i_1, i_2, i_3, i_4 \ldots$ are connected to the line as shown in Fig. 9 and let $\rho_1, \rho_2, \rho_3, \rho_4 \ldots$ be the respective resistances of the line sections. Assuming both leads are of the same section and material, the total pressure drop at the distant end of the line will then be

$$\Delta v = \tfrac{1}{2}(V_1 - V_2) = (I_1.\rho_1 + I_2.\rho_2 + I_3.\rho_3 + I_4.\rho_4) \text{ volts} \quad . \quad (6)$$

where

$$I_1 = i_1 + i_2 + i_3 + i_4 : I_2 = i_2 + i_3 + i_4 : I_3 = i_3 + i_4 : I_4 = i_4$$

If these values are inserted in equation (6), the

$$\Delta v = [(i_1 + i_2 + i_3 + i_4)\rho_1 + (i_2 + i_3 + i_4)\rho_2 + (i_3 + i_4)\rho_3 + i_4\rho_4]$$
$$= [i_1\rho_1 + i_2(\rho_1 + \rho_2) + i_3(\rho_1 + \rho_2 + \rho_3) + i_4(\rho_1 + \rho_2 + \rho_3 + \rho_4)]$$

or, writing

$$r_1 = \rho_1 : r_2 = \rho_1 + \rho_2 : r_3 = \rho_1 + \rho_2 + \rho_3 : r_4 = \rho_1 + \rho_2 + \rho_3 + \rho_4$$

then

$$\Delta v = (i_1.r_1 + i_2.r_2 + i_3.r_3 + i_4.r_4) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

that is, in general,

$$\Delta v = \Sigma(i.r) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the distances from the supply end of the line to the individual consumers are, respectively, $l_1 : l_2 : l_3 : l_4$ metres, then

$$r_1 = \frac{l_1}{\lambda.q} : r_2 = \frac{l_2}{\lambda.q} \ldots$$

so that, from equations (7) and (8),

$$\Delta v = \frac{1}{\lambda.q}\{i_1.l_1 + i_2.l_2 + \ldots\} = \frac{1}{\lambda.q}\Sigma(i.l) \quad . \quad . \quad (9)$$

The equation (9) expresses the important practical result that the pressure drop Δv is obtained by superposing the pressure drops due to the individual currents. The products, such as $i.l$ in equation (9) are termed the "current moments" about the supply terminal A.

In Fig. 10 is shown a feeder with three consumers, the line being fed from the end A only. The lower diagrams show successively the pressure drop due to the load of each consumer and also the total resultant pressure drop due to the total load, the magnitude of the total drop being shown by Δv in Fig. 10e. If it is required to find the pressure drop at some intermediate point h distant l_h metres from the supply end A, it is convenient to imagine the line to be cut at h (Fig. 10a). Then the current i_3 will flow out of the severed end and the sum of the current moments up to the point h will be

$$(i_1.l_1 + i_2.l_2 + i_3.l_a)$$

and consequently the pressure drop at h will then be

$$\Delta v = \frac{1}{\lambda.q}(i_1.l_1 + i_2.l_2 + i_3.l_a) \text{ volts} . \qquad . \qquad . \quad (10)$$

The pressure drop is frequently given as a percentage of the supply pressure, viz. Δv per cent., the problem then being to find the necessary cross section q of the conductor where

$$\Delta v\% = \frac{\Delta v}{V}100 : \text{ that is } \Delta v = \frac{\Delta v\%\, V}{100} \qquad . \qquad . \quad (11)$$

and

$$q = \frac{\Sigma(i.l)}{\lambda.\Delta v} \text{ mm.}^2.$$

In order to deal with the current loading of a complicated network in which, in general, a large number of consumers are supplied, it is convenient to replace the consumers' currents which are connected to any one distributor by an equivalent current I'' which will have the same effect as regards the *maximum* pressure drop as the actually existing current loading. Thus, referring to the diagram of Fig. 10e, assume the current I'' is taken from the end of the line, the distance of which is L metres from the point of supply (see Fig 10a). Then for the pressure drop at the end of the line,

$$I''.L = \Sigma i.l, \text{ so that } I'' = \frac{\Sigma(i.l)}{L} \qquad . \qquad . \qquad . \quad (12)$$

Reference to Fig. 10e will show that the pressure drop due to the equivalent current I'' is the same as the actual drop at the end of the line, and for any intermediate point such as X the pressure drop Δv_x is given by the broken line, it being observed that this drop Δv_x is less than the drop at that point due to the actual current loading. Since, however, the practical problem is to ensure that the *maximum* pressure

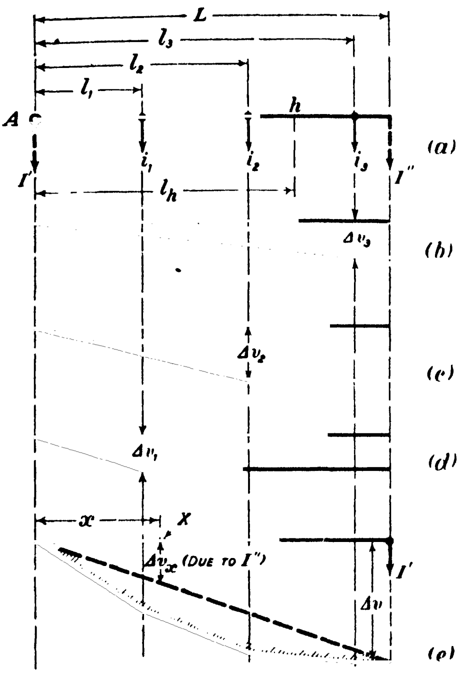


Fig. 10.

drop shall not exceed a given value, there is no particular purpose in calculating the actual pressure drops at points in the line other than at the point of maximum drop.

In order to complete the representation of the actual current loading by means of equivalent concentrated currents, it is necessary to assume that a current I' is also taken from the line at the supply end A and such that

$$I' = \Sigma i - I'' \qquad . \quad . \quad . \quad . \quad (13)$$

The current I' does not affect the pressure drop at the distant end of the line, but the sum of the two equivalent currents is then equal to the total of the actual load currents, that is, $I' + I'' = \Sigma i$. The necessity for introducing the equivalent current I' is seen in the following simple example.

Fig. 11 shows two lines of respective lengths, l_1 and l_2, connected in series and supplied from the terminal A, whilst the line l_2 supplied several consumers, as shown in the Fig. 11. The pressure drop in l *alone* will then be $\Delta e = \dfrac{I''l_2}{\lambda . q}$. The current in l_1 will be $\Sigma i = I' + I''$ so that the total pressure drop at the extreme end C of the line will be

$$\Delta e = \frac{(I' + I'')l_1 + I''l_2}{\lambda . q}.$$

It is of interest to derive directly, the expression for I' in a form which

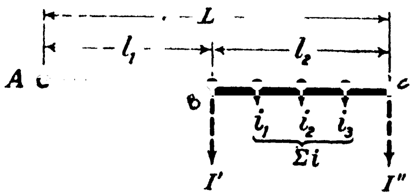


Fig. 11.

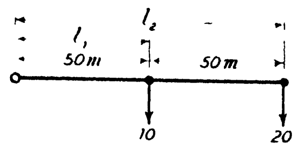


Fig. 12.

corresponds to that in which I'' has been expressed in equation (12) Referring to Fig. 9, it will be seen that

$$I' = \Sigma i - I'' = \Sigma i - \frac{\Sigma(i.l)}{L} \qquad . \quad . \quad . \quad . \quad . \quad (14)$$

$$= \frac{(i_1 + i_2 + i_3 + i_4)L - i_1.l_1 - i_2.l_2 - i_3.l_3 - i_4l_4}{L}$$

$$= \frac{i_1(L - l_1) + i_2(L - l_2) + i_3(L - l_3) + i_4(L - l_4)}{L}$$

that is

$$\left.\begin{aligned} I' &= \frac{\Sigma i(L - l)}{L} \\ I'' &= \frac{\Sigma(i.l)}{L} \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (15)$$

or, expressed in words,

$I'.L$ is equal to the sum of the current moments about the end B
$I''.L$,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, ,, A

EXAMPLE.—A 220-volt copper distributor is loaded as shown in Fig. 12. If the maximum permissible pressure drop is 2½ per cent., find the necessary cross-sectional area of the line.

Since from expression (9), $\Delta v = \frac{\Sigma(i.l)}{\lambda.q} = \frac{\Delta v\%}{100}V$ then $q = \frac{\Sigma(i.l)}{\lambda.\Delta v}$ where $\lambda = 56$ for a copper conductor,

so that,
$$\Delta v = \frac{2\cdot5}{100} \times 220 = 5\cdot5 \text{ volts},$$

and
$$q = \frac{(10 \times 50) + (20 \times 100)}{56 \times 5\cdot5} = 8\cdot1 \text{ sq. mm.}$$

The nearest larger standard size of conductor is 9·4 sq. mm., for which the percentage pressure drop will be $\frac{8\cdot1}{9\cdot4} \times 2\cdot5 = 2\cdot16$ per cent.

A Uniformly Distributed Load

Suppose in Fig. 13 the current load is uniformly distributed throughout the line and let j amperes per metre length be the magnitude of the load. Imagine that the line is cut at any point X distant x metres from the supply terminal A: then the current which will flow from the severed end will be $j(l - x)$ amperes as shown in Fig 13b. From the results on page 129 it is seen that the equivalent current I'' for the length of line x will be

$$I'' = \frac{\int_0^x (j\,dx)x}{x} = \tfrac{1}{2}j.x,$$

so that the equivalent current loading of the length x of the line will now be as shown in Fig. 13(c), and consequently the pressure drop at the point X of Fig. 13d will be

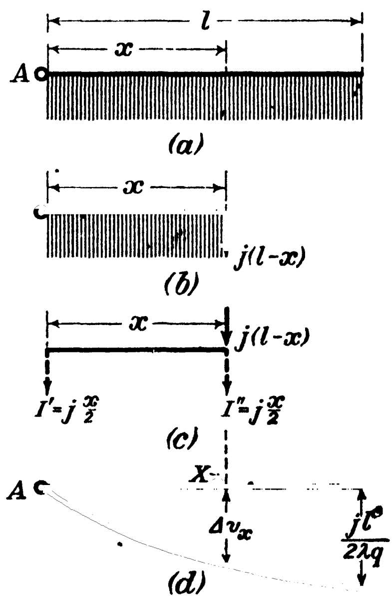


Fig. 13.

$$\Delta v_x = \frac{I''.x + j(l - x)x}{\lambda.q} = \left\{j\frac{x}{2} + j(l - x)\right\}\frac{x}{\lambda.q} \quad . \quad (16)$$

The maximum drop is then obtained by putting $x = l$ in this expression, so that

$$\Delta v = j \cdot \frac{l}{2} \cdot \frac{l}{\lambda . q} = \frac{j . l^2}{2\lambda . q} \quad . \quad . \quad . \quad . \quad (17)$$

From the form of the expression (16) it will be seen that the drop along the line is distributed as shown by the parabola of Fig. 13*d*.

EXAMPLE: A conductor supplies a uniformly distributed load of 10 kW., the length of the distributor being 400 metres and the maximum permissible pressure drop is 2 per cent., the supply pressure being 220 volts Find the necessary cross-sectional area for an aluminium distributor.

Since $V . I = 10{,}000$ watts, the total current supplied will be 45·5 amperes, so that the current supplied per metre run of the conductor will be

$$j = \frac{45 \cdot 5}{400} = 0 \cdot 113 \text{ ampere per metre.}$$

Also, the maximum permissible pressure drop is

$$\Delta v = \frac{2}{100} \times 220 = 4 \cdot 4 \text{ volts,}$$

so that
$$q = \frac{j . l^2}{2\lambda . \Delta v} = \frac{0 \cdot 113 \times 400^2}{2 \times 34 \times 4 \cdot 4} = 60 \text{ sq. mm.}$$

The nearest largest standard size is 65 sq. mm., for which the maximum pressure drop will be 1·93 per cent.

The Current is Fed into the Line at Both Ends

Two cases will be considered, viz. when (*i*) both ends of the line are maintained at the same pressure and (*ii*) both ends of the line are at different pressure.

CASE I *The Supply Pressure is the Same at Both Ends of the Line.* - For this condition the pressure drop distribution can be found from the results already obtained in connection with Fig. 14 by means of a simple device. Assume, first, that the line is severed from the supply terminal *B*, so that the whole of the consumers' current is drawn from the supply terminal *A*. Then the conditions will be exactly the same as those in the previous case which has been dealt with in connection with Fig. 10, so that the pressure drop at any point in the line will then be given by the polygon diagram *abce* Figs. 10 (*b*) and (*c*), whilst the equivalent currents are I'' and I', of which the equivalent current I'' produces the pressure drop along the distributor as defined by the straight line *abe*. That is to say, the pressure drop at the end *B* will be given by the amount *ef*.

Next suppose that the distributor is re-connected to the supply terminal *B* and that no current is being taken by any of the consumers but that the equivalent current I'' is being fed into the line from the terminal *B*. The pressure at terminal *B* must then clearly be raised

above that of the terminal A by the amount fg, where $fg = fe$. If the consumers are now connected to the line and if the equivalent current I'' is still supplied from the supply terminal B, the pressure drop fe will be just equal to the pressure rise fg, so that if the terminal B is maintained at the same pressure as the terminal A, the pressure drop distribution throughout the line will be given by the polygonal diagram of Fig. 14*d*. The current in each section of the line is then easily found and the maximum pressure drop in the line can be calculated, it being observed that *the maximum drop will occur at that tapping point to which the current flows from both ends of the line.*

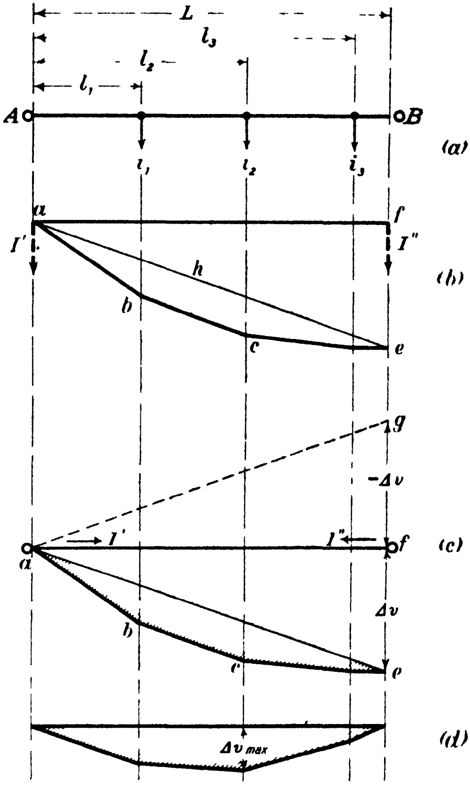


Fig. 14.

CASE II. *The Supply Terminals are Maintained at Different Potentials.* — This case can easily be dealt with by proceeding as already explained in connection with Case I. Then, if in Fig. 15 the potential of A is V_A and that of B is V_B, and if the resistance of the conductor is r ohms, there will be an out-of-balance current $I_a = \frac{V_A - V_B}{r}$ amperes which flow along the line from A to B so that the current in any one section will be given by the superposition of this out-of-balance current on the current distribution as found for the condition that both ends of the line are at the same potential. The maximum drop will be at the tapping point K which receives current from both ends of the line. The following example will make this clear.

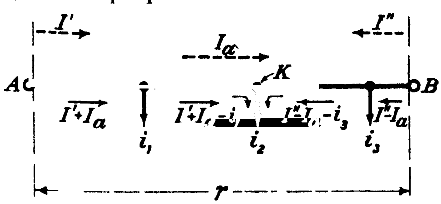


Fig. 15.

Example.—A distributor is fed from both ends, the respective pressures at the ends of the line being 220 and 228 volts. Find the maximum drop for an aluminium distributor of cross-section $q = 35$ sq. mm. when the current loading is as shown in Fig. 16.

The equivalent currents are

$$I'' = \frac{\Sigma(i.l)}{L} = \frac{(50 \times 100) + (100 \times 150)}{250} = 80 \text{ amperes}$$

and $\quad I' = \Sigma i - I'' = 150 - 80 = 70$ amperes.

The resistance of the distributor will be $r = \dfrac{250}{35 \times 35} = 0{\cdot}205$ ohm, so that the compensating current will be $I_a = \dfrac{228 - 220}{0{\cdot}205} = 39$ amperes

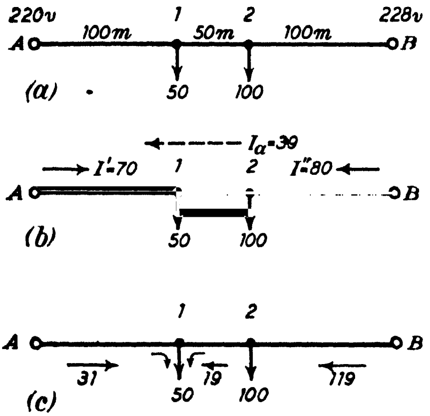


Fig. 16.

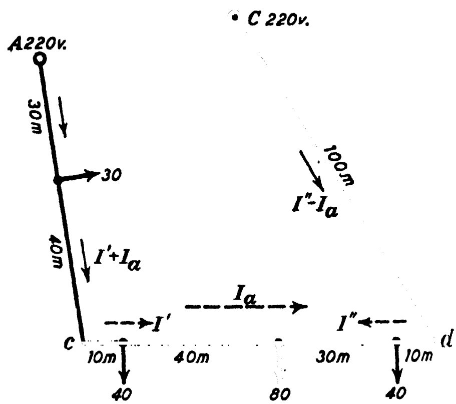


Fig. 17.

The current distribution in the individual sections will be as shown in Fig. 16*c*, from which it follows that the maximum pressure drop will be at the 50-ampere tapping point, i.e. point 1, and the pressure at this point will be

$$220 - \left(\frac{100}{250} \times 0{\cdot}205\right)31 = 220 - 2{\cdot}54 = 217{\cdot}46 \text{ volts.}$$

Example.—In Fig. 17 is shown a distributor system connected by means of feeder conductors to the supply terminals A and C, which are maintained at the constant pressure of 220 volts. The lengths of the conductors are shown in metres, and for convenience, it will be assumed that *the resistance of the conductors is* 1 *ohm per kilometre.*

Consider first the distributor *cd*, which is fed from each end, these ends being clearly at different potentials. The equivalent currents for the distributor *cd* will then be

$$I'' = \frac{(40 \times 10) + (80 \times 50) + (40 \times 80)}{90} = 84{\cdot}4 \text{ amperes,}$$

and $\quad I' = \Sigma i - 84{\cdot}4 = 75{\cdot}6$ amperes.

If I_a amperes is the out-of-balance current which flows along the distributor due to the potential difference at the two ends, and if V_c is the pressure at c and V_d the pressure at d, then

$$V_c = 220 - \{(I_a + 75{\cdot}6 + 30)0{\cdot}03 + (I_a + 75{\cdot}6)0{\cdot}04\}$$
$$= 213{\cdot}81 - 0{\cdot}07I_a$$

and $\quad V_d = 220 - \{(84{\cdot}4 - I_a)0{\cdot}1\} = 211{\cdot}56 + 0{\cdot}1I_a$

so that $\quad V_c - V_d = 2{\cdot}25 - 0{\cdot}17I_a$

also, since the resistance of the line $cd = 0{\cdot}09$

$$V_c - V_d = 0{\cdot}09I_a$$

and hence $\quad I_a = 8{\cdot}65$ amperes.

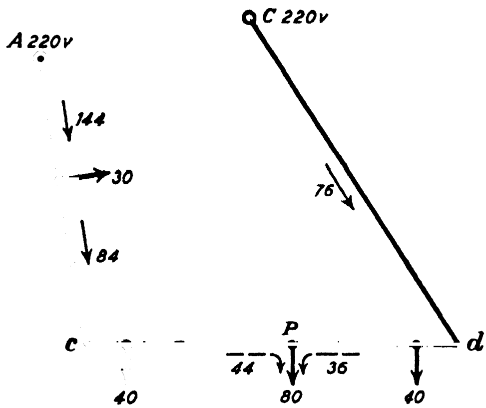


Fig. 18.

In the diagram of Fig. 18 is shown the current distribution throughout the line. It will be seen that the maximum pressure drop will occur at P and that the pressure of this point will be

$$V_P = 220 - 76 \times 0{\cdot}11 - 36 \times 0{\cdot}03 = 210{\cdot}56 \text{ volts,}$$

so that the maximum pressure drop will be

$$\Delta v\% = \left(\frac{220 - 210{\cdot}56}{220}\right)100 = 4{\cdot}3\%.$$

EXAMPLE.—In Fig. 19*a* is shown part of a network which is supplied from the terminals A and B, respectively. The pressure at each terminal is 220 volts and the lengths of the individual sections are given in metres, it being assumed that 1 *km. length of conductor has a resistance of* 1 *ohm.*

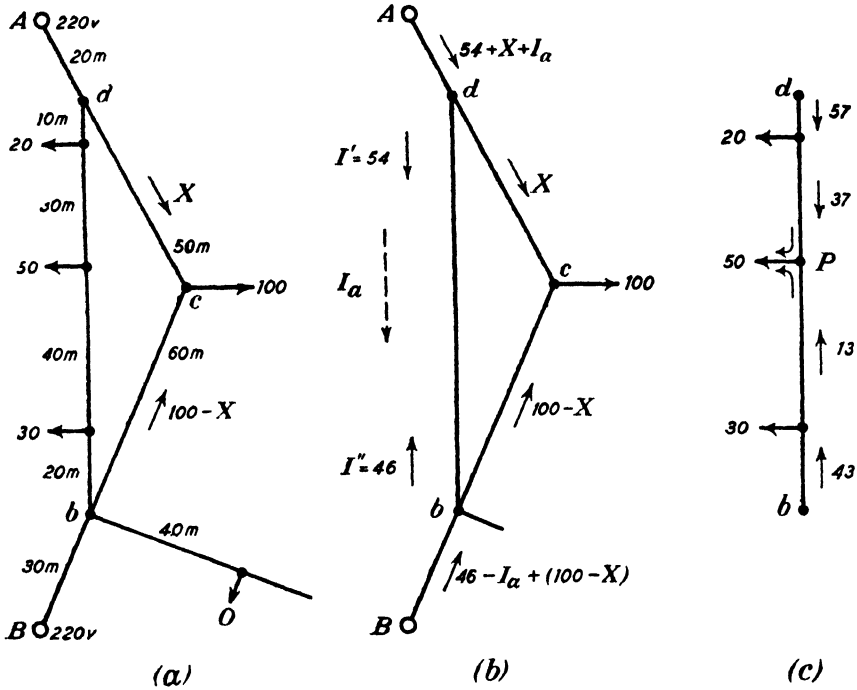


Fig. 19.

The equivalent currents for the distributor *ab* are (see Fig. 19*b*)

$$I'' = \frac{(20 \times 0{\cdot}1) + (50 \times 0{\cdot}4) + (30 \times 0{\cdot}8)}{100} = 46 \text{ amperes,}$$

$$I' = \Sigma i - I'' = (30 + 50 + 20) - 46 = 54 \text{ amperes.}$$

If, in Fig. 19*b*, I_a is the compensating current which will flow from *d* to *b* due to the p.d. between *d* and *b* that is, due to the pressure $V_d - V_b$, then

$$(I_a + 54 + X)0{\cdot}02 + X \times 0{\cdot}05 = (146 - I_a - X)0{\cdot}03 + (100 - X)0{\cdot}06$$

or

$$0{\cdot}05I_a + 0{\cdot}16X = 9{\cdot}3 \quad . \quad . \quad . \quad . \quad (18)$$

where X amperes is the current in the conductor *dc*. Also

$$(V_d - V_b) = I_a \times 0{\cdot}1 = (146 - I_a - X)0{\cdot}03 - (I_a + 54 + X)0{\cdot}02$$

that is

$$0{\cdot}15I_a = 3{\cdot}3 - 0{\cdot}05X \quad . \quad . \quad . \quad . \quad (19)$$

so that from (18) and (19)

$$I_a = 3{\cdot}0 \text{ amperes} : \ X = 57 \text{ amperes.}$$

Hence $I' + I_a = 57$ amperes will enter the distributor at the end d and $I'' - I_a = 43$ amperes will enter at the end b, so that the current distribution will then be as shown in Fig. 19c. The maximum pressure drop will therefore be at the point P and the pressure at this point will be

$$V_P = V_d - (57 \times 0{\cdot}01 + 37 \times 0{\cdot}03) = 216{\cdot}04 \text{ volts},$$

so that the percentage drop will be

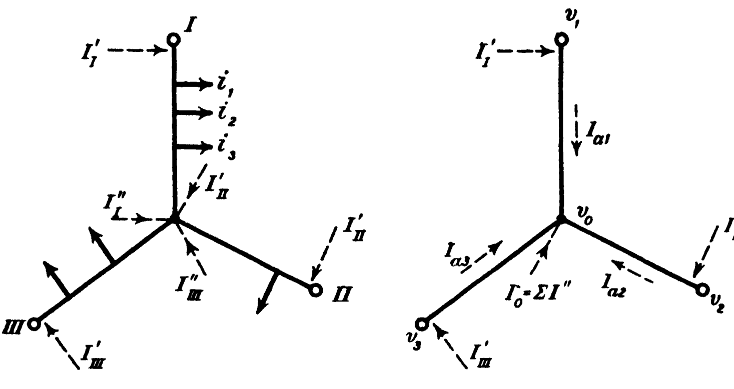
$$\Delta v_P\% = \frac{3{\cdot}96}{220} \times 100 = 1{\cdot}8\%$$

The pressure at C will then be

$$V_c = V_d - (57 \times 0{\cdot}05) = 214{\cdot}87 \text{ volts}$$

so that the percentage drop will be

$$v_c\% = \frac{5{\cdot}13}{220} \times 100 = 2{\cdot}33\%.$$



(a) (b)

Fig. 20.

The Pressure Drop at the Common Junction of Three Distributors

This is a very common system of connections in practice since every junction in a network provides an example of this type. The calculations are greatly simplified by replacing the actual current loading by the equivalent currents I' and I'' at the respective ends of the three component distributors, as shown in Fig. 20a. In Fig. 20b the load currents are assumed to be replaced by the current $I_0 = \Sigma I''$ flowing to the common junction and the currents $I_I' : I_{II}' : I_{III}'$ respectively flowing from the other ends of the distributors. If the pressures at the respective ends of the distributors are known and the current loading for each distributor is also known, then the total current flowing from each distributor to the junction can be calculated. It is then easy to find the compensation current for each distributor, viz $I_{aI} : I_{aII} : I_{aIII}$.

If $v_1 : v_2 : v_3$ be the respective pressures at the supply ends of the distributors and v_0 the pressure at the junction, then the respective compensation currents which will flow along the distributors from the supply points to the junction will be

$$I_{a1} = \frac{v_1 - v_0}{r_1} : I_{a2} = \frac{v_2 - v_0}{r_2} : I_{a3} = \frac{v_3 - v_0}{r_3}.$$

But since no resultant current either leaves or enters the junction the sum of these compensation currents must be equal to the sum $I_0 = \Sigma I''$

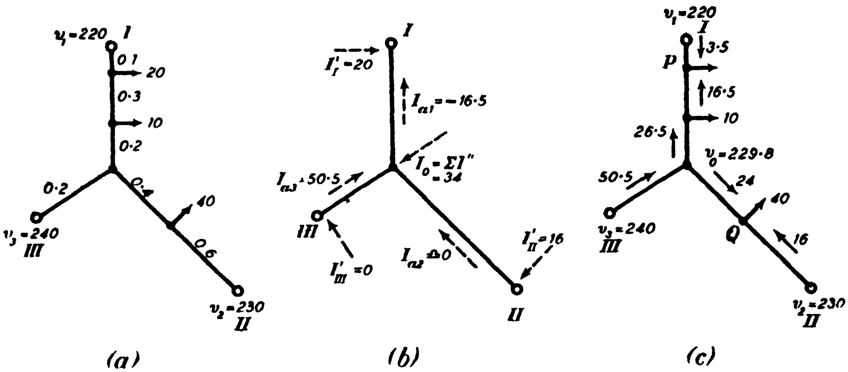


Fig 21.

of the equivalent currents which are assumed to be supplied to the junction, so that

$$\frac{v_1 - v_0}{r_1} + \frac{v_2 - v_0}{r_2} + \frac{v_3 - v_0}{r_3} = I_0$$

that is

$$\frac{v_1}{r_1} + \frac{v_2}{r_2} + \frac{v_3}{r_3} = v_0\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) + I_0$$

or

$$\frac{v_1}{r_1} + \frac{v_2}{r_2} + \frac{v_3}{r_3} - I_0 = v_0\frac{1}{r_0} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where r_0 is the resistance which is equivalent to the resistances $r_1 : r_2 : r$ in parallel. Since all the quantities in the equation (20) are known except v_0, this unknown can be at once determined.

EXAMPLE.—In Fig. 21*a* are shown the current loads on the respective distributors and the resistance in ohms of the individual sections are also shown. The supply pressures at the terminals *I*, *II*, *III* are, respectively, 240, 230, and 220 volts.

The respective equivalent currents for the three distributors are easily found by the method already explained on page 129, viz.

$$I_I'' = \frac{(10 \times 0{\cdot}4) + (20 \times 0{\cdot}1)}{0{\cdot}6} = 10 \text{ amperes}$$

$$I_I' = \Sigma i - 10 = 20 \text{ amperes.}$$

similarly $I_{II}'' = 24$ amperes $\quad : I_{III}'' = 0$

$I_{II}' = 40 - 24 = 16$ amperes : $I_{III}' = 0$

so that $I_0 = I_I'' + I_{II}'' + I_{III}'' = 34$ amperes (Fig. 21*b*)

$$\frac{1}{r_0} = \frac{1}{0{\cdot}6} + \frac{1}{1{\cdot}0} + \frac{1}{0{\cdot}2} = 7{\cdot}67 \text{ reciprocal ohms}$$

and $r_0 = 0{\cdot}1305$ ohm.

$$v_0 = 0{\cdot}1305\left\{\left(\frac{220}{0{\cdot}6} + \frac{230}{1} + \frac{240}{0{\cdot}2}\right) - 34\right\} = 229{\cdot}85 \text{ volts}$$

$$I_{aI} = -\frac{9{\cdot}85}{0{\cdot}6} = -16{\cdot}5 \text{ amperes} : I_{aII} \simeq 0 : I_{aIII} = 50{\cdot}5 \text{ amperes.}$$

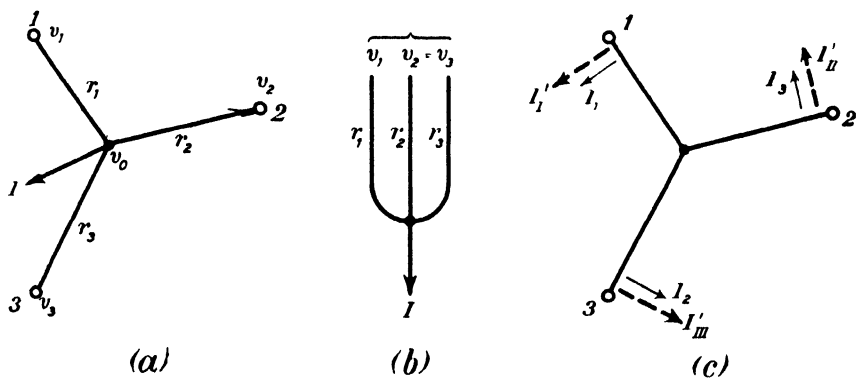


Fig 22.

The current distribution will then be as shown in Fig. 21*c* for the various sections of the distributors, from which it will be seen that the points P and Q are places of maximum pressure drop, viz.

$$\Delta v_P = 0{\cdot}1 \times 3{\cdot}5 = 0{\cdot}35 \text{ volts} : \Delta v_P\% = \frac{0{\cdot}35}{220} \times 100 = 0{\cdot}16\%$$

$$\Delta v_Q = 0{\cdot}6 \times 16 = 9{\cdot}6 \text{ volts} : \Delta v_Q\% = \frac{9{\cdot}6}{220} \times 100 = 4{\cdot}16\%.$$

Network which Supplies Current to a Junction

In the portion of a network shown in Fig. 22*a* a current I is assumed to be taken from the common junction and the resistance of the respective conductors which are connected together at this junction are $r_1 : r_2 : r_3$. If it is assumed in the first place that the pressure at the three supply

points are all equal, that is, $v_1 = v_2 = v_3$, then it may be assumed that all these supply points are connected together as shown in Fig. 22*b*, so that the three conductors are connected together in parallel and consequently the equivalent resistance r_0 is given by the relationship

$$\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

If v_0 denotes the pressure at the junction, the pressures v at the terminals 1, 2, 3, will be $v = v_0 + Ir_0$, and the current I will divide between the three conductors, so that

$$I_1 = I\frac{r_0}{r_1} : I_2 = I\frac{r_0}{r_2} : I_3 = I\frac{r_0}{r_3},$$

and nothing will be altered as regards current and pressure in the rest of the network to which the system is connected, if the terminals 1, 2 and 3 are assumed to be respectively loaded with the currents $I_1 : I_2 : I_3$ and these currents will be additional to the component equivalent currents $I_1' : I_2' : I_3'$, which correspond to any consumers' loads which may be taken from tapping points on the individual branches (see also Fig. 22*c*)

If the terminals 1, 2, and 3 are not all at the same potential, then compensating currents will flow in each of the conductors in addition to the currents which have been considered already. The compensating currents can be easily calculated by the method which has been explained in previous examples.

Transformation of a Star Network into an Equivalent Mesh

Fig. 23 shows four conductors of respective resistance $r_1 : r_2 : r_3 : r_4$, and connected to a common junction O, it being assumed that a current of I_3 amperes is being taken from the terminal 3. In Fig. 24 the same four terminals are assumed to be linked together by means of a mesh system of which the resistances of the respective links are marked in and it is assumed that the same current of I_3 amperes is taken from the terminal 3 as shown in Fig. 24.

It is required to find the relationship between the resistances of the conductors of the mesh and those of the equivalent star. That is to say, the two systems are supposed to be electrical equivalent so that the pressure at the terminals and the current from these terminals shall be the same in both systems.

For the star system of Fig. 23,

$$\frac{v_1 - v_0}{r_1} + \frac{v_2 - v_0}{r_2} + \frac{v_3 - v_0}{r_3} + \frac{v_4 - v_0}{r_4} = 0 \quad . \quad . \quad (21)$$

by Kirchhoff's rule since the total current flowing to the junction must be zero. Hence

$$\frac{v_1}{r_1} + \frac{v_2}{r_2} + \frac{v_3}{r_3} + \frac{v_4}{r_4} = v_0\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) = \frac{v_0}{r_0} \quad . \quad (22)$$

that is
$$v_0 = r_0\left(\frac{v_1}{r_1}+\frac{v_2}{r_2}+\frac{v_3}{r_3}+\frac{v_4}{r_4}\right).$$

This expression can be simplified somewhat as follows. Since the current flow will only depend upon the relative potentials of the terminals, nothing will be altered if one terminal (say 3) is termed zero potential and $v_1' : v_2' : v_4' : v_0'$ are the respective potentials of the other terminals and the junction with respect to the potential of the terminal 3. Then the current I_3 flowing from terminal 3 will be

$$I_3 = \frac{v_0'}{r_3} = \frac{r_0}{r_3}\left(\frac{v_1'}{r_1}+\frac{v_2'}{r_2}+\frac{v_4'}{r_4}\right) \quad . \quad . \quad . \quad . \quad (23)$$

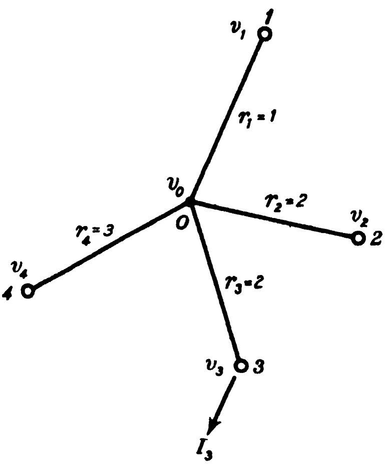


Fig 23.

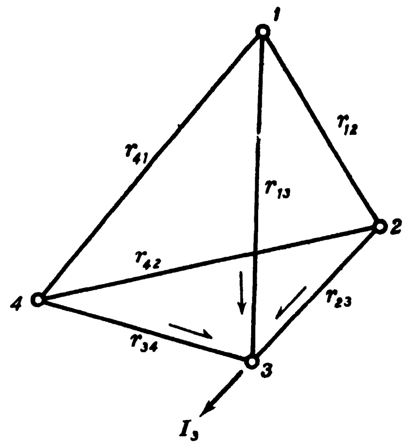


Fig. 24

Now consider the *mesh* system of Fig. 24 in which the current I_3 is flowing from the terminal 3. That is

$$I_3 = \frac{v_1'}{r_{13}}+\frac{v_2'}{r_{23}}+\frac{v_4'}{r_{34}},$$

so that for equivalence of the star and mesh systems, the following relationships must hold for the resistances of the individual conductors, viz

$$\left.\begin{aligned} \frac{r_0}{r_3 . r_1} &= \frac{1}{r_{13}} \\ \frac{r_0}{r_2 . r_3} &= \frac{1}{r_{23}} \\ \frac{r_0}{r_3 . r_4} &= \frac{1}{r_{34}} \\ \text{or, in general,} \qquad \frac{1}{r_{\nu\mu}} &= \frac{r_0}{r_\nu . r_\mu} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

EXAMPLE.—The star junction of four links shown in Fig. 25 is connected to terminals of which the respective potentials are $v_1 = 280$: $v_2 = 285$: $v_3 = 300$: $v_4 = 290$ volts. Applying the relationships already obtained in the foregoing, it will be found that the pressure at the junction will be

$$v_0 = r_0\left[\frac{280}{1} + \frac{285}{2} + \frac{300}{2} + \frac{290}{3}\right] = r_0(699\cdot2) \text{ volts}$$

where
$$\frac{1}{r_0} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{14}{6}$$

so that
$$r_0 = \frac{3}{7} : v_0 = 286\cdot8 \text{ volts.}$$

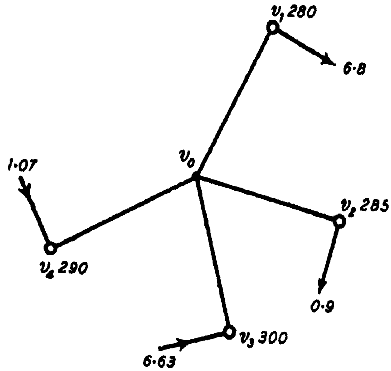


Fig 25

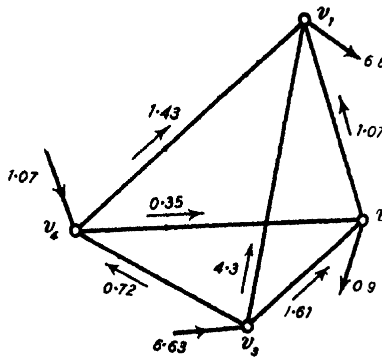


Fig. 26.

It is then a simple matter to determine the respective currents which will flow from the star into the terminals or from the terminals into the star, these currents being shown in Fig. 25.

The values of the respective resistances of the conductors in the mesh system of Fig. 26 can then be found by means of the equations already given. For example, $\frac{1}{r_{13}} = \frac{r_0}{r_1 . r_3} = \frac{3}{14}$, that is $r_{13} = 4\cdot67$, and similarly

$$r_{23} = 9\cdot33 : r_{34} = 14 : r_{41} = 7 : r_{12} = 4\cdot67 : r_{24} = 14,$$

from which the currents in the individual conductors are easily obtained and are marked in the diagram of Fig. 26.

It is to be observed that, whereas a star group can always be converted into an equivalent mesh grouping, the converse is not, in general true. Thus the mesh shown in Fig. 24 provides six equations out of which only four unknowns are required for the star, so that the problem is over-defined. A special case in which a mesh can be converted into

the equivalent star is that of a triangular mesh, as shown in Fig. 27, since from such a mesh three equations are obtained for the three unknowns of the star. Such a transformation from star to triangular mesh or from triangular mesh to star is of frequent practical application, and

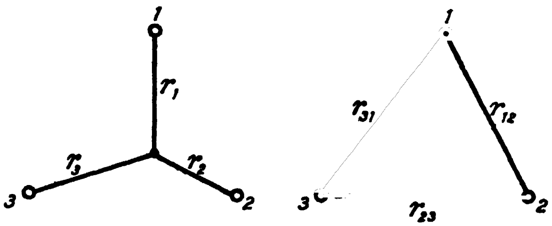


Fig. 27.

the required relationships for the resistances of the conductors of the corresponding systems are as follows. From the previous equations (24),

$$r_{12} = \frac{r_1 \times r_2}{r_0} = r_1 . r_2 \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) \quad . \quad . \quad . \quad (25)$$

or

$$r_{12} = \frac{r_1 . r_2}{r_3} + r_1 + r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Similarly, for the other resistances, hence :

$$\left. \begin{aligned} r_{12} &= \frac{r_1 . r_2}{r_3} + r_1 + r_2 \\ r_{23} &= \frac{r_2 . r_3}{r_1} + r_2 + r_3 \\ r_{31} &= \frac{r_3 . r_1}{r_2} + r_1 + r_3 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and these relationships give the equivalent mesh conductor resistances in terms of the star-arranged conductors.

In order to obtain the equivalent star from a given triangular mesh, the three equations corresponding to (24) are

$$\left. \begin{aligned} r_{12} &= \frac{r_1 . r_2}{r_0} \\ r_{23} &= \frac{r_2 . r_3}{r_0} \\ r_{31} &= \frac{r_3 . r_1}{r_0} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (28)$$

Adding these three equations together gives

$$r_{12} + r_{23} + r_{31} = \frac{r_1 . r_2 + r_2 . r_3 + r_3 . r_1}{r_0}$$

$$= \frac{r_1 r_2 r_3}{r_0}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = \frac{r_1 r_2 r_3}{r_0{}^2} . \qquad . \quad (29)$$

but, from (26)

$$r_{12} . r_{23} = \frac{r_1 . r_2 . r_3 . r_2}{r_0{}^2} \qquad . \quad . \quad . \quad . \quad (30)$$

so that from (27) and (28)

$$r_2 = \frac{r_{12} . r_{23}}{r_{12} + r_{23} + r_{31}} .$$

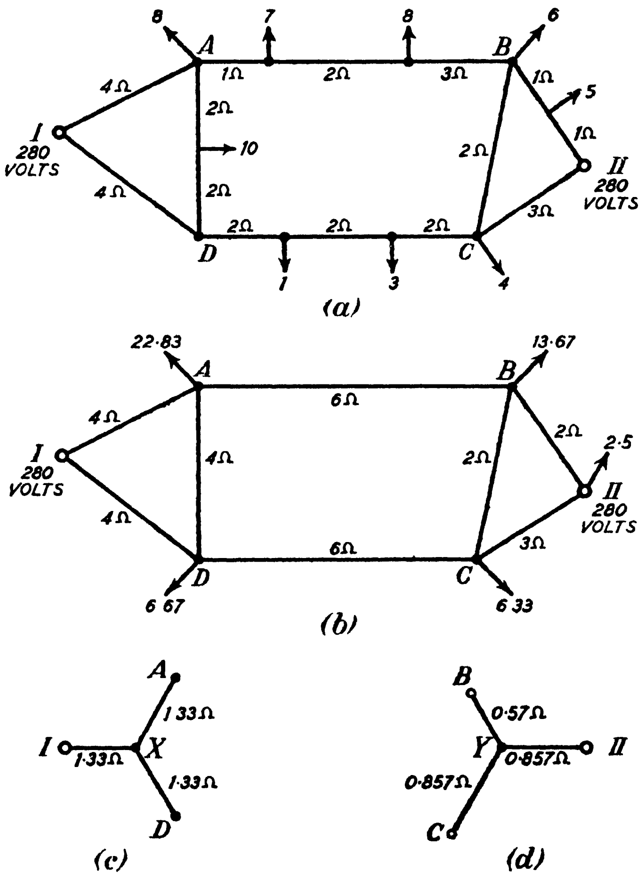


Fig. 28.

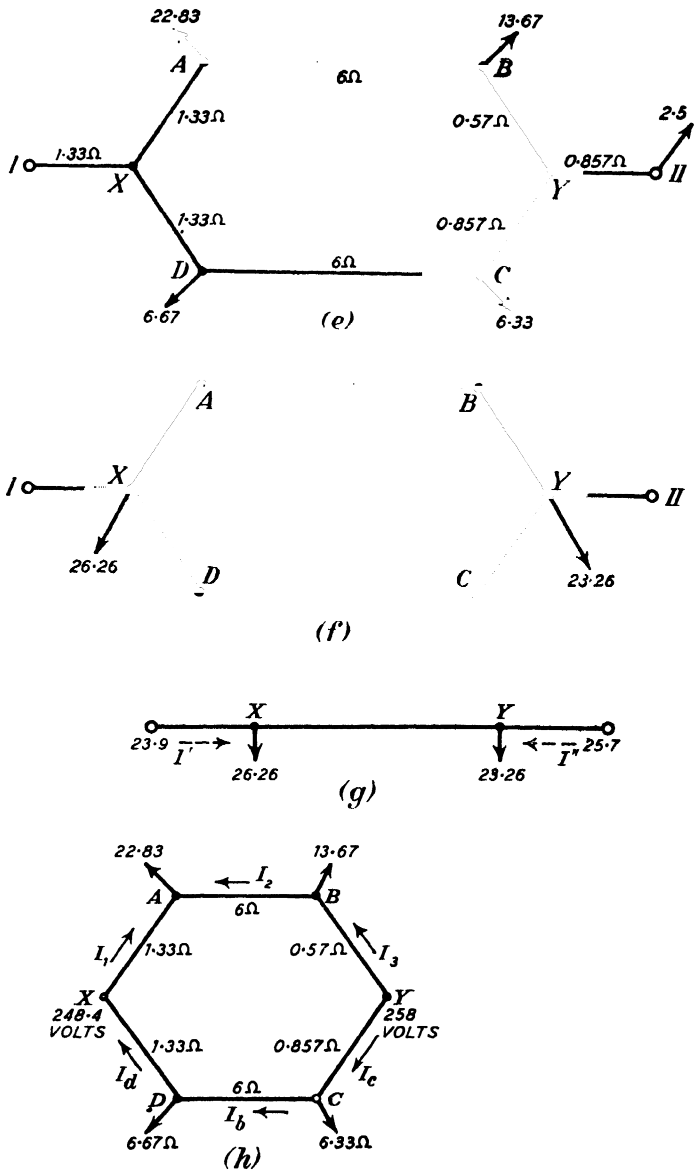

22·83
13·67
6Ω
1·33Ω
0·57Ω
2·5
0·857Ω
I
X
Y
II
6·67
(e)
6·33
26·26
(f)
23·26
23·9
25·7
(g)
248·4 VOLTS
258 VOLTS
6·67Ω
6·33Ω
(h)

Fig. 28.

Hence

$$\left.\begin{aligned} r_1 &= \frac{r_{31} \cdot r_{21}}{r_{12} + r_{23} + r_{31}} \\ r_2 &= \frac{r_{12} \cdot r_{23}}{r_{12} + r_{23} + r_{31}} \\ r_3 &= \frac{r_{32} \cdot r_{31}}{r_{12} + r_{23} + r_{31}} \end{aligned}\right| \qquad (31)$$

and these relationships give the equivalent star conductor resistances in terms of the triangular mesh arranged conductors (Fig. 27).

COMPREHENSIVE EXAMPLE.—In Fig. 28*a* is shown a loaded network, the solution of which is greatly facilitated by a judicious application of the principles which have been considered in the foregoing theorems.

It will be seen from Fig. 28*a* that the total current taken by the network is 52 amperes. The following procedure for finding the currents in the individual sections consists in a step-by-step reduction of the complex network of Fig. 28*a* to an increasingly simpler arrangement, viz. :

(*i*) Transfer the currents of the individual distributors to the junctions by finding the corresponding equivalent currents I' and I'', thus, for the loads on AB the equivalent currents will be 5·17 amperes at B and 9·83 amperes at A. Similar treatment for the other distributors gives the total currents at the junctions, respectively, as shown in Fig. 28*b*.

(*ii*) Convert the mesh AID into the equivalent star, thus obtaining the system shown in Fig. 28*c*, and in Fig. 28*d* is shown the equivalent star for the triangular mesh $BIIC$ of Fig. 28*b*. The whole system now becomes simplified to the arrangement shown in Fig. 28*e*.

(*iii*) Reduce the load on the star terminals of Fig. 15*e* to the equivalent loads at the respective star points X and Y, thus giving

$$I_X = \frac{(13{\cdot}67 \times 0{\cdot}57) + (22{\cdot}83 \times 6{\cdot}57)}{7{\cdot}9} + \frac{(6{\cdot}33 \times 0{\cdot}857) + (6{\cdot}67 \times 6{\cdot}857)}{8{\cdot}19}$$

$$= 26{\cdot}26 \text{ amperes}$$

$I_Y = 23{\cdot}26$ amperes.

The resultant equivalent system will then be as shown in Fig. 28*f*.

(*iv*) Replace the two parallel branches between X and Y in Fig. 28*f* by an equivalent single line, viz. one of resistance 7·9 ohms in parallel with 8·19 ohms, thus giving 4·03 ohms on the equivalent resistance, the consequent equivalent system being then as shown in Fig. 28*g*. If this loading is now reduced to the equivalent load at the terminals I and II the currents flowing into the line will be

$$I_I = \frac{(23{\cdot}26 \times 0{\cdot}857) + (26{\cdot}26 \times 4{\cdot}89)}{6{\cdot}22} = 23{\cdot}9 \text{ amperes :}$$

$$I_{II} = 25{\cdot}7 \text{ amperes}$$

(v) The potential of the point X is $280 - (23{\cdot}9 \times 1{\cdot}33) = 248{\cdot}4$ volts and the potential of the point Y is 258 volts.

The respective potentials of A and B (Fig. 28h) can then be found thus :

The p.d. between Y and $X = 9{\cdot}6$ volts,

$$\therefore\ 9{\cdot}6 = I_3 \times 0{\cdot}57 + I_2 \times 6 - I_1 \times 1{\cdot}33$$

and $\quad I_1 + I_2 = 22{\cdot}83 : I_3 = 13{\cdot}67 + I_2,$

so that $\quad I_1 = 18{\cdot}8 : I_2 = 4{\cdot}03 : I_3 = 17{\cdot}7$

and the potential of A will be $248{\cdot}4 - (18{\cdot}8 \times 1{\cdot}33) = 223{\cdot}4$ volts

" " " " B " " $258 \quad (17{\cdot}7 \times 0{\cdot}57) = 247{\cdot}9$ volts.

Similarly, for the potentials of C and D,

$$I_c = 7{\cdot}85 : I_b = 1{\cdot}52 : I_d = \quad 5{\cdot}15,$$

so that

The potential of C will be 25·126 volts

" " " D " " 241·55 "

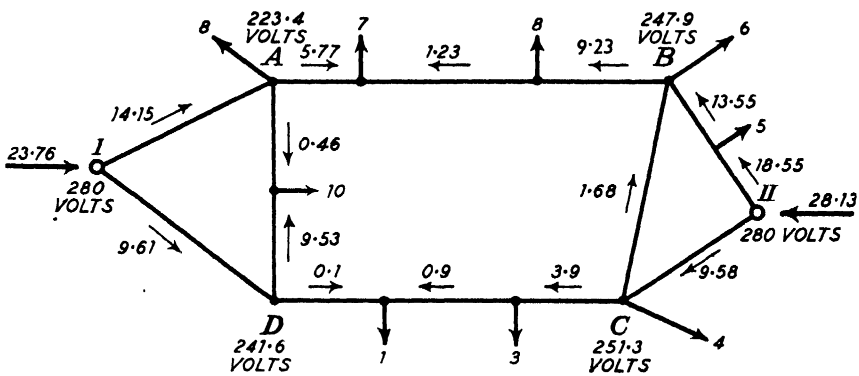


Fig. 29.

From these known values of the respective potentials the current distribution in the whole network is at once obtained, as shown in Fig. 29.

An alternative method for solving this network problem is to assume one of the conductors is cut at some suitable point as shown by XX in Fig. 30a and opened as shown in Fig. 30b, it being observed that since the terminals I and II are at the same potential they may be brought into coincidence.

The simultaneous equations for the individual closed circuits may then be written down as follows, viz.

$$\left.\begin{aligned} 280 - V_X &= 3I_a + 3I_x \\ 280 - V_X &= 4I_g - 3I_x \end{aligned}\right\} \text{that is, } 3I_a + 6I_x = 4I_g$$

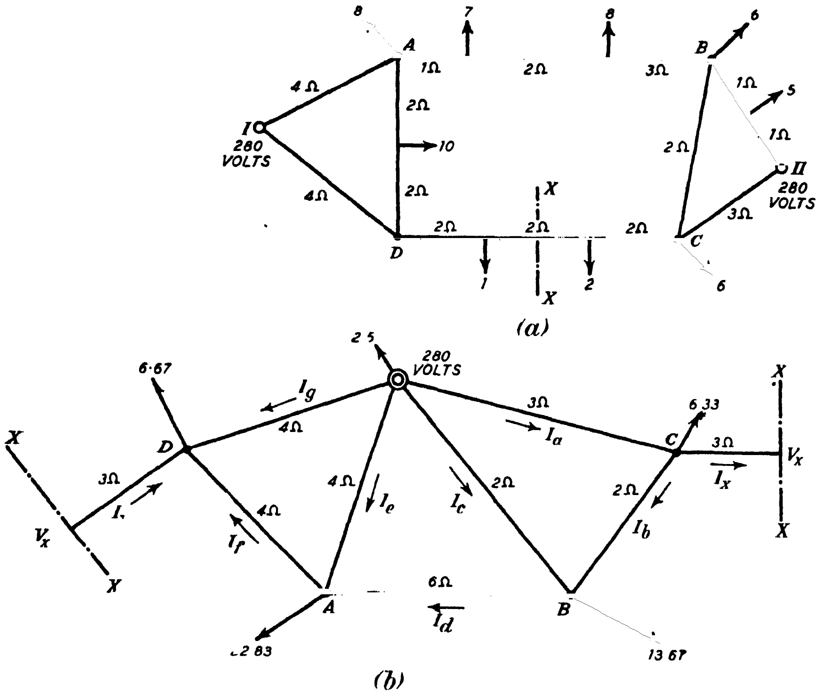


Fig. 30.

then

$$3I_a + 6.I_x = 4I_g \quad (1)$$
$$I_c + I_b = 13{\cdot}67 + I_d \quad (2)$$
$$I_e + I_d = 22{\cdot}83 + I_f \quad (3)$$
$$I_f + I_g + I_x = 6{\cdot}67 \quad (4)$$
$$3I_a + 2I_b = 2I_c \quad (5)$$
$$2I_c + 6I_d = 4I_e \quad (6)$$
$$I_e + I_f = I_g \quad (7)$$
$$I_a = 6{\cdot}33 + I_b + I_x \quad (8)$$

Solving these equations gives the complete current distribution throughout the network and it will be found that the two methods as exemplified by Figs. 28 and 30 lead to the same results.

The Economical Division of the Conductor Material for a Branched Distributor

The diagram of Fig. 31*a* shows a distributor in which the various consumers' currents have been referred to the junction points A and B and the end points 1 and 2, respectively. If the total drop from A to 1

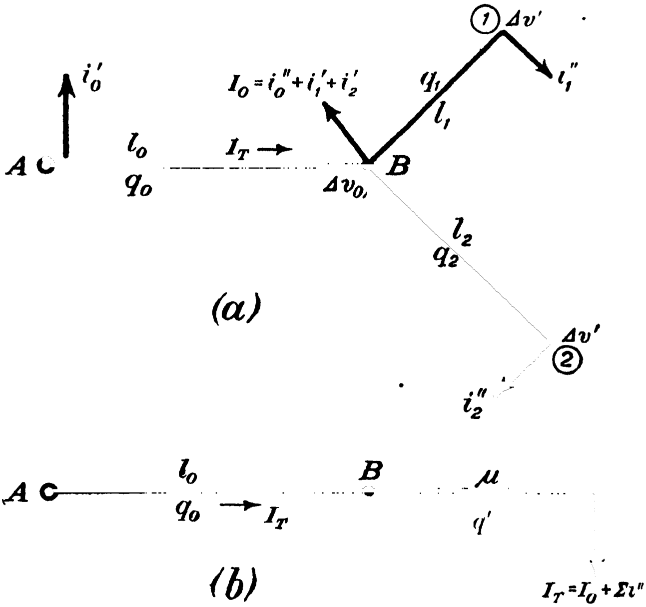


Fig. 31.

and A to 2 is to be not greater than Δv, and if the drop from A to B is Δv_0 and from B to 1 and from B to 2 is $\Delta v'$, then $\Delta v = \Delta v' + \Delta v_0$ and the dimensions of the conductors can be chosen so that the total drop Δv can be divided into the components Δv_0 and $\Delta v'$ as may be desired. For a given total current in the main line AB the minimum volume of copper will be obtained for one definite relationship between $q_0 : q_1$ and q_2. It is seen that

$$\Delta v' = \frac{I_1 l_1}{\lambda . q_1} = \frac{I_2 l_2}{\lambda . q_2} : \text{ that is, } \Delta v' = \frac{I_1 l_1^2}{\lambda . l_1 q_1} = \frac{I_2 l_2^2}{\lambda . l_2 q_2}$$

or

$$Q_1 = \frac{I_1 l_1^2}{\lambda . \Delta v'} : Q_2 = \frac{I_2 l_2^2}{\lambda . \Delta v'}$$

where Q_1 and Q_2 are the respective volumes of copper in the lines 1 and 2. The total volume of copper in these two lines is therefore

$$Q_b = \frac{I_1 l_1^2 + I_2 l_2^2}{\lambda . \Delta v'} \text{ or } Q_b = \frac{\Sigma(I_1 . l_1^2)}{\lambda . \Delta v'} \text{ for any number of spur lines.}$$

Now suppose these two lines are replaced by a single line of the same volume Q_b and having the same pressure drop $\Delta v'$, and let μ be the length of this equivalent line and I_T the current flowing in the main line AB. Then

$$Q_b = \frac{I_T\mu^2}{\lambda.\Delta v'} = \frac{\Sigma(I_1.l_1^2)}{\lambda.\Delta v'} \text{ so that } \mu = \sqrt{\frac{\Sigma(I_1.l_1^2)}{I_T}}$$

and the equivalent system will then be as shown in Fig. 31*b*. The total volume of copper for the whole distributor will then be

$$Q_T = Q_0 + Q_b = \frac{I_T l_0^2}{\lambda.\Delta v_0} + \frac{I_T\mu^2}{\lambda.\Delta v'}$$

so that $Q_T = \frac{I_T}{\lambda}\left\{\frac{l_0^2}{(K - \Delta v')} + \frac{\mu^2}{\Delta v'}\right\}$ where $K = (\Delta v_0 + \Delta v')$ and is a constant,

and for Q_T to be a minimum $\frac{dQ_T}{d(\Delta v')} = 0$,

that is
$$\frac{l_0^2}{(K - \Delta v')^2} = \frac{\mu^2}{(\Delta v')^2}$$

or
$$\frac{l_0}{(\Delta v_0)^2} = \frac{\mu^2}{(\Delta v')^2}$$

that is,
$$\frac{l_0}{\Delta v_0} = \frac{\mu}{\Delta v'}.$$

But
$$\frac{1}{\Delta v_0} = \frac{\lambda.q_0}{I_T l_0} : \frac{1}{\Delta v'} = \frac{\lambda.q'}{I_T\mu}$$

that is
$$q_0 = q' \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where q' is the cross-sectional area of the equivalent line of length μ.

Also
$$\left.\begin{aligned} q_0 &= \frac{(l_0 + \mu)I_T}{\lambda.\Delta v} : \Delta v_0 = \frac{I_T l_0}{\lambda.q_0} : \Delta v' = \frac{I_T\mu}{\lambda.q_0} \\ q_1 &= \frac{i_1'' l_1}{\lambda.\Delta v'} : q_2 = \frac{i_2'' l_2}{\lambda.\Delta v'} \\ \frac{\Delta v_0}{\Delta v} &= \frac{l_0}{l_0 + \mu} \end{aligned}\right\}$$

EXAMPLE.—A network is supplied from a 220-volt terminal A and loaded as shown in Fig. 32*a*. Find the cross-sections for the respective aluminium conductors for a total drop of not more than 3 per cent. and for a minimum volume of aluminium. The lengths of the conductor sections are marked in metres in Fig. 32*a*.

Transferring the loads shown in Fig. 32*a* to the ends of the individual distributors, viz.

$$i_0'' = \frac{15 \times 100}{150} = 10 \text{ amperes} \quad : i_0' = \frac{15 \times 50}{150} = 5 \text{ amperes}$$

$$i_1'' = \frac{10 \times 50 + 20 \times 100}{100} = 25 : i_1' = 5 \text{ amperes}$$

$$i_2'' = \frac{25 + 80}{80} = 25 \text{ amperes} \quad : i_2' = 0,$$

so that (see Fig. 32*b*)

$$I_T = I_0 + \Sigma i'' = i_0'' + i_1' + i_2' + \Sigma i''$$
$$= 15 + 50 = 65 \text{ amperes.}$$

For the equivalent line (Fig. 32*c*)

$$\mu = \sqrt{\frac{25 \times 100^2 + 25 \times 80^2}{65}} = 79 \text{ metres}$$

$$\Delta v = 0{\cdot}03 \times 220 = 6{\cdot}6 \text{ volts.}$$

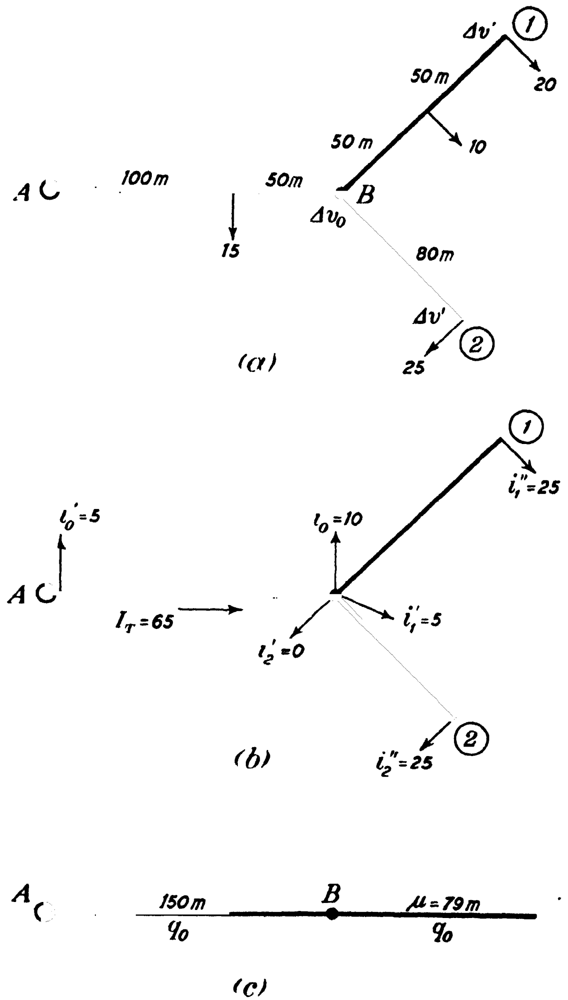


Fig. 32.

The conductivity of aluminium is $\lambda = 34{\cdot}8$, so that

$$q_0 = \frac{(150 + 79) \times 65}{34{\cdot}8 \times 6{\cdot}6} = 64 \text{ sq. mm.}$$

for aluminium conductors, which is a standard size. Hence:

$$q_0 = 64 \text{ sq. mm.} \quad \Delta v_0 = \frac{65 \times 150}{34{\cdot}8 \times 64} = 4{\cdot}3 \text{ volts}$$

$$\Delta v' = 6{\cdot}6 - 4{\cdot}3 = 2{\cdot}3 \text{ volts}$$

$$q_1 = \frac{25 \times 100}{34{\cdot}8 \times 2{\cdot}3} = 31{\cdot}5 \text{ sq. mm.} : q_2 = \frac{25 \times 80}{34{\cdot}8 \times 2{\cdot}3} = 25 \text{ sq. mm.}$$

$$q_1 + q_2 = 56{\cdot}5 \text{ sq. mm.} : q_0 = 64 \text{ sq. mm.}$$

Thévenin's Theorem

Before stating this important and comprehensive theorem the following preliminary considerations will be helpful. Suppose the diagram of Fig. 33 represents a portion of a network and let the pressure between

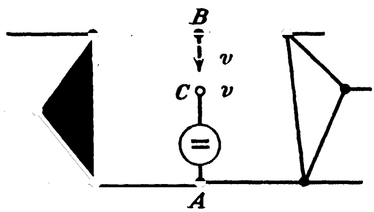


Fig 33

the two points A and B be v volts. If now a conductor AC is connected to A and if an e.m.f. v volts is operating in this conductor, then if C is connected to B no current will flow in the conductor AC since the p.d. between the points A and B is zero. If, however, for any reason, the p.d. v volts which exists in the network fails, then when C is connected to B the e.m.f. v will produce a current in the network of a magnitude which is given by the ratio

$$\frac{v}{\text{the resistance of the network as measured between the points } A \text{ and } B.}$$

Thévenin's Theorem states that if the points A and B of the actual network were to be connected together, the consequent current change in the network would be the same as would be produced by the pressure v volts acting through the junction A and B when these points are connected together. This theorem applies not only to d.c. networks but also to networks operating under steady a.c. pressures as well as to

transient phenomena in the network. Some of the implications of this theorem will be seen from the following numerical example.

In Fig. 34*a* is shown a network which is being supplied by an e.m.f. E volts and the resistances in ohms of the individual branches of the system are marked in on the diagram. The problem of finding the current distribution among the various conductors of the system can be solved by an application of Kirchhoff's rule and writing down the corresponding equations for the various closed loops comprised in the system. This method, however, is a tedious and laborious process and arithmetical errors are not easily avoided. By means of Thévenin's Theorem, however, the current in any selected conductor can be found as follows.

(*i*) Suppose, for example, that the conductor BC is removed from the system so that the circuit becomes as shown in Fig. 34*b*. The resistance from A to D of the loop $ABCD$ is

$$\frac{10 \times 15}{10 + 15} = 6 \text{ ohms},$$

so that the total resistance of the circuit to which the e.m.f. E is applied is $6 + 4 = 10$ ohms, and consequently the current I_G will be given by

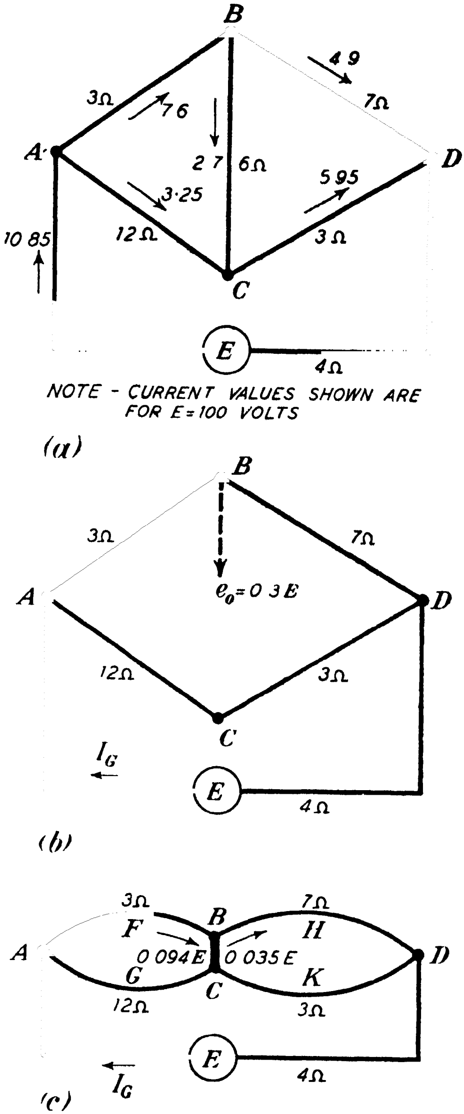


Fig 34.

$I_G = \frac{E}{10}$ amperes. Of this total current the amount $\frac{15}{10 + 15} I_G = 0{\cdot}06E$ amperes will flow in the branch ABD and the amount $0{\cdot}04E$ amperes

will flow in the branch ACD. The pressure drop along AB will therefore be

$$0{\cdot}06E \times 3 = 0{\cdot}18E \text{ volts,}$$

and the pressure drop along AC will be

$$0{\cdot}04E \times 12 = 0{\cdot}48E \text{ volts.}$$

Hence the pressure between the points B and C will be

$$e_0 = 0{\cdot}3E \text{ volts} \quad . \quad . \quad . \quad . \quad (31)$$

as shown in Fig. 34b.

(ii) Suppose, now, that the points B and C are brought into contact as shown in Fig. 34c. The equivalent resistance of the two looped circuits between A and D will then be

$$\frac{3 \times 12}{3 + 12} + \frac{7 \times 3}{7 + 3} = 4{\cdot}5 \text{ ohms,}$$

so that the current I_G will now be

$$I_G = \frac{E}{4 + 4{\cdot}5} = 0{\cdot}118E \text{ amperes.}$$

This current will divide in the loops so that

$$\begin{array}{llllll}
\frac{12}{15} I_G = 0{\cdot}094E & \text{amperes will flow in} & AB \\
\frac{3}{15} I_G = 0{\cdot}024E & \quad ,, \quad ,, \quad ,, \quad ,, & AC \\
\frac{3}{10} I_G = 0{\cdot}035E & \quad ,, \quad ,, \quad ,, \quad ,, & BD \\
\frac{7}{10} I_G = 0{\cdot}083E & \quad ,, \quad ,, \quad ,, \quad ,, & CD
\end{array}$$

Reference to Fig. 34c will now show that the current which will flow across the short circuit from B to C will be $0{\cdot}059E$ amperes. The resistance, therefore, of the short circuited network of Fig. 34c as measured between the points B and C will be given by

$$R_{SC} = \frac{0{\cdot}3E \text{ volts \{see expression (31)\}}}{0{\cdot}059E \text{ amperes}} = 5{\cdot}1 \text{ ohms.}$$

When the conductor BC of 6 ohms resistance is replaced in the system of Fig. 34b, the current which will flow in this conductor will be

$$I_{BC} = \frac{0{\cdot}3E}{R_{SC} + 6} = \frac{0{\cdot}3E}{11{\cdot}1} = 0{\cdot}027E \text{ amperes} \quad . \quad . \quad (32)$$

and this is the required value of the current in the link BC.

In a similar way the respective currents in two other links may be found and it then becomes a simple matter to write down the current in each of the other links. The values so obtained are shown in Fig. 34a for the assumption that $E = 100$ volts.

The Solution of Network Problems by Means of Determinants

A network comprising a chain of similar circuits as shown in Fig. 35 is representative of a large number of practical cases, and the method

of finding the current distribution among the individual "cells" of the linked system by the use of determinants will now be considered.

Let E be the e.m.f. acting round any one of the "cells" and suppose this e.m.f. to be positive when acting in a clockwise direction. Let I_n be the current flowing in the nth cell and this will also be considered to be positive when flowing in a clockwise direction round the cell. Writing down the equations for each cell for the conditions shown in Fig. 35, viz.

$$\left.\begin{aligned} E_1 &= I_1(R + r) - I_2 r \\ E_2 &= 0 = I_2(R + 2r) - I_1 r - I_3 r \\ E_3 &= I_3(R + r) - I_2 r \end{aligned}\right\} \quad . \quad . \quad . \quad (33)$$

or, rearranging the terms of these equations in numerical sequence,

$$\left.\begin{aligned} I_1(R + r) + I_2 - r + I_3 . 0 &= E_1 \\ I_1(-r) + I_2(R + 2r) + I_3(-r) &= 0 \\ I_1 . 0 + I_2(-r) + I_3(R + r) &= E_3 \end{aligned}\right\} \quad . \quad . \quad (34)$$

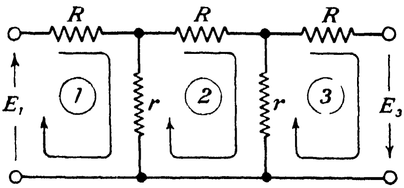


Fig. 35.

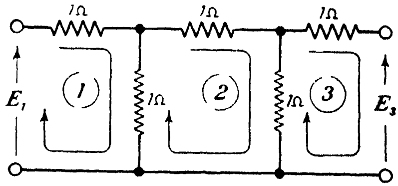


Fig 36

The currents in the individual cells may now be found from the equations (34) by means of the determinants as follows. The determinant D is given by

$$D = \begin{vmatrix} R + r & -r & 0 \\ -r & R + 2r & -r \\ 0 & -r & R + r \end{vmatrix} \quad . \quad . \quad . \quad . \quad (35)$$

The determinant D_1 is obtained by replacing in column 1 of D the column of e.m.f.s $E_1 : E_2 : E_3$, thus

$$D_1 = \begin{vmatrix} E_1 & -r & 0 \\ 0 & R + 2r & -r \\ E_3 & -r & R + r \end{vmatrix} \quad . \quad . \quad . \quad . \quad (36)$$

It can then be shown that the current I_1 is given by the equation

$$I_1 = \frac{D_1}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Similarly, the determinant D_2 is obtained by replacing in expression (35) the column 2 by the column of e.m.f.s $E_1 : E_2 : E_3$: thus

$$D_2 = \begin{vmatrix} R+r & E_1 & 0 \\ r & 0 & r \\ 0 & E_3 & R+r \end{vmatrix} \quad . \quad . \quad . \quad . \quad (38)$$

then

$$I_2 = \frac{D_2}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The determinant D_3 is obtained by replacing column 3 of expression (35) by the column of e.m.f.s $E_1 : E_2 : E_3$, and then

$$I_3 = \frac{D_3}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

EXAMPLE. The foregoing procedure will easily be understood by means of a simple numerical example, the conditions being as shown in Fig. 36, where the numerical values of the individual resistances are marked in ohms.

In this example it will be seen that E_3 acts in a counter-clockwise direction round the cell 3, so that it is negative. Assuming then that $E_3 = -\frac{1}{2}E_1$ the determinant D of expression (35) becomes

$$D = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(6-1) - 1(0+2) = 8$$

The determinant D_1 of expression (36) is

$$D_1 = \begin{vmatrix} E_1 & 1 & 0 \\ 0 & 3 & 1 \\ -\frac{1}{2}E_1 & 1 & 2 \end{vmatrix} = E_1(6-1) - \tfrac{1}{2}E_1(1) = 4\tfrac{1}{2}E_1$$

. . . .

. . . .

$$\begin{vmatrix} E_1 & 1 & 0 \\ 0 & 3 & 1 \\ -\frac{1}{2}E_1 & 1 & 2 \end{vmatrix}$$

so that

$$I_1 = \frac{D_1}{D} = \frac{4\frac{1}{2}E_1}{8} = \frac{9}{16}E_1 \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Also

$$D_2 = \begin{vmatrix} 2 & E_1 & 0 \\ 1 & 0 & 1 \\ 0 & -\frac{1}{2}E_1 & 2 \end{vmatrix} = 2(\tfrac{1}{2}E_1) - 1(-2.E_1) = -E_1$$

. . . .

. . . .

$$\begin{vmatrix} 2 & E_1 & 0 \\ 1 & 0 & 1 \\ 0 & -\frac{1}{2}E_1 & 2 \end{vmatrix}$$

so that

$$I_2 = \frac{D_2}{D} = \frac{1}{8}E_1 \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Further,

$$D_3 = \begin{vmatrix} 2 & -1 & E_1 \\ -1 & 3 & 0 \\ 0 & -1 & -\frac{1}{2}E_1 \end{vmatrix} = 2(-3 \times \tfrac{1}{2}E_1) - 1(\quad E_1 - \tfrac{1}{2}E_1) \\ = -1\tfrac{1}{2}E_1$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2 & -1 & E_1 \\ -1 & 3 & 0 \\ 0 & -1 & -\frac{1}{2}E_1 \end{array}$$

so that

$$I_3 = \frac{D_3}{D} = -\frac{3}{16}E_1 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

That the foregoing are the correct values for the respective currents in the individual cells in this simple example is easily verified by direct calculation from the equations (34).

In the foregoing numerical calculation each determinant has been duplicated, as shown by the second columns under the dots, in order to facilitate the evaluations of the terms of each determinant without confusion of signs. Thus, taking the general form of a determinant of three rows and three columns as follows:

$$D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) \\ + A_3(B_1C_2 - B_2C_1)$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{array}$$

The method of determinants applies, of course, to any mesh in which the e.m.f.s acting round the individual cells are known and also the resistance comprised by the individual cells are known.

Chapter VI

THERMO-ELECTRICITY : PIEZO-ELECTRICITY

Principles of Thermo-electricity

SEEBECK discovered in 1821 that in a closed circuit formed by two different metals, if the two junctions of the metals are at different termperatures, an electric current will flow round the circuit. For example, if the metals are respectively iron and copper soldered together to form a closed circuit, and if one junction is heated (see Fig. 1) a current will flow *from copper to iron* across the hot junction Such currents are termed " thermo-electric currents " and two different metals soldered together form a thermo-couple or thermo-junction.

Peltier showed in 1834 that when a current flows across the junction of two dissimilar metals it gives rise to an absorption or a liberation of heat. If the current flows in the same direction as the current which would flow when the junction is heated, then an absorption of heat will take place. If the current flows in the same direction as the current across the cold junction of a thermo-couple then heat is liberated. This effect is known as the " Peltier Effect ".

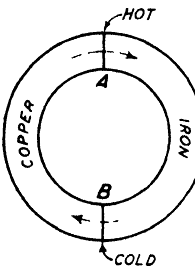


Fig 1

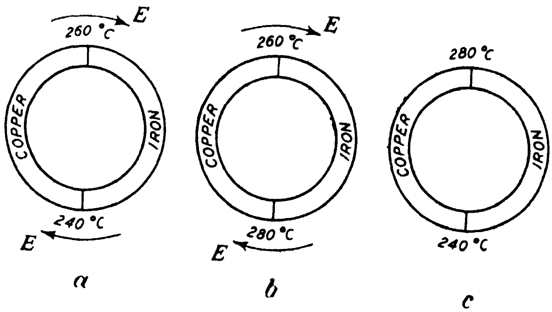


Fig. 2.

Cumming, in 1823, showed that there are thermo-electric circuits such that when the temperature of the hot junction is raised above a certain value, the thermo-electric e.m.f. of the circuit decreases. Thus, suppose that a thermo-electric circuit is made of copper and iron, and let the cold junction be kept at 20 C. As the temperature of the hot junction is raised the e.m.f. round the circuit increases until the temperature of the hot junction reaches a value of about 260 C., when the e.m.f. reaches a maximum value. If the temperature of the hot junction is still further increased the e.m.f. decreases and eventually

becomes zero and then reverses in direction. The temperature of the hot junction for which the e.m.f. is a maximum has a definite value for any given pair of metals and does not depend upon the temperature of the cold junction. This temperature, which, as already stated, is about 260° C. for an iron-copper couple, is termed the neutral temperature. From what follows, it will be seen that, if the mean temperature of the junctions is the same as the *neutral temperature, there will be no thermo-electric e.m.f. in the circuit.* This effect is shown by the diagrams of Fig. 2. Thus, in Fig. 2*a* the mean temperature of the two copper-iron junctions is 250° C., that is, less than the neutral temperature, and consequently the e.m.f. is directed from *copper to iron* across the hotter junction (260° C.). In Fig. 2*b* the mean temperature of the two junctions is 270° C., that is, greater than the neutral junction and consequently the e.m.f. is directed from *iron to copper* across the hotter junctions (280° C.). In Fig. 2*c* the mean temperature is equal to the neutral temperature so that there is no resultant e.m.f. round the circuit.

Fig. 3 gives a thermo-electric diagram for a series of different metals with lead as the datum metal. That is to say, the thermo-electric line for any given metal lies above that of lead for the mean temperature of the hot and cold junctions, the direction of the e.m.f. is from that metal to lead across the *cold* junction and if for the mean temperature the line lies below the lead line the direction of the e.m.f. is from that metal to lead across the *hot* junction. As an example of the application of this diagram suppose that, for a thermo-couple of iron and copper, the

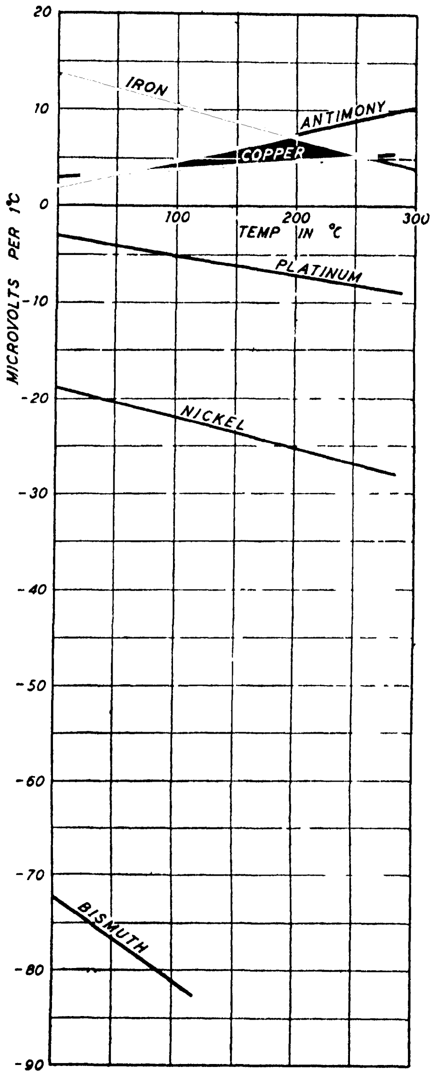


Fig. 3.

mean temperature of the hot and cold junctions is 100° C. From Fig. 3 it is seen that at this temperature the corresponding e.m.f. is 6·6 micro-volts and is directed from copper to iron across the hot junction. If the difference of temperature between the hot and cold junctions is 150° C., then the e.m.f. acting round the circuit will be

$$6{\cdot}6 \times 150 = 990 \text{ micro-volts.}$$

The metal lead is taken as the datum metal in the diagram of Fig. 3 because the "Thomson Effect" for lead is practically zero, that is to say, there is no e.m.f. developed in a lead wire between points in the wire which are at different temperatures. In all other metals an e.m.f. is developed between two points in the metal which are at different temperatures, and although this effect is relatively small, it cannot always be neglected.

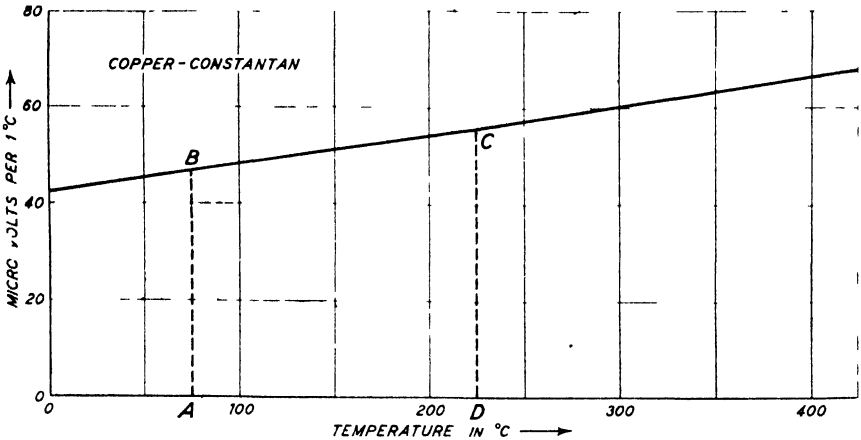


Fig. 4.

The alloy technically known as "constantan" (60 per cent. Cu : 40 per cent. Ni), when used as a thermo-couple with either copper, iron, or nichrome, has a relatively very high thermo-electric e.m.f., and consequently this alloy has been widely used for the measurement of temperature. The chief disadvantages of this alloy are : (*i*) In spite of the greatest care in preparing the melt it has not been found possible to manufacture the alloy so that the thermo-e.m.f.s of different melts are in sufficiently close agreement with one another. This necessitates the calibration of the measuring instrument each time a new couple is used (*ii*) Its application is limited to ranges of temperature which are well below 1,000° C. since otherwise oxidation sets in and rapid deterioration of the alloy takes place.

In Fig. 4 is shown the thermo-electric characteristic for a copper-

TABLE I

Open Circuit E.M.F. Cold Junction at 0° C.

Temperature C.	Iron Eureka		Copper Eureka	
	Millivolts	Millivolts per ° C.	Millivolts	Millivolts per ° C.
10	0·6	0·058	0·4	0·040
20	1·2	,,	0·8	1
30	1·7	,,	1·2	2
40	2·3	,,	1·6	3
50	2·9	,,	2·1	4
60	3·5	,,	2·5	5
70	4·0	,,	3·0	5
80	4·6	0·057	3·4	6
90	5·2	,,	3·9	7
100	5·8	,,	4·3	7
110	6·3	,,	4·8	8
120	6·9	,,	5·3	9
130	7·5	,,	5·8	9
140	8·0	,,	6·3	0·050
150	8·6	,,	6·8	1
160	9·2	,,	7·3	1
170	9·7	,,	7·8	2
180	10·3	,,	8·3	2
190	10·9	,,	8·8	3
200	11·5	,,	9·4	4
210	12·0	,,	9·9	4
220	12·6	,,	10·4	5
230	13·2	,,	11·0	5
240	13·7	,,	11·5	6
250	14·3	,,	12·1	6
260	14·9	,,	12·7	7
270	15·5	,,	13·2	8
280	16·1	,,	13·8	8
290	16·6	0·058	14·4	9
300	17·2	,,	15·0	9
310	17·8	,,	15·6	0·060
320	18·4	,,	16·2	0
330	18·9	,,	16·8	1
340	19·4	,,	17·4	1
350	20·1	,,	18·0	1
360	20·7	,,	18·6	2
370	21·3	,,	19·2	2
380	21·8	,,	19·8	2
390	22·4	,,	20·5	3
400	23·0	,,	21·1	3
410	23·6	,,	21·7	3
420	24·2	,,	22·3	3
430	24·7	,,	23·0	4
440	25·3	,,	23·6	4
450	25·9	,,	24·3	4
460	26·5	0·058	24·9	4
470	27·1	,,	25·5	4
480	27·6	,,	26·2	4
490	28·2	,,	26·8	4
500	28·8	,,	27·5	—

M

constantan couple and relates the e.m.f. in micro-volts per 1° C. with the temperature, from which it is seen that this relationship is a straight line for the range of temperatures given. If the hot and cold junctions of such a thermo couple are, respectively, given by the points D and A then the e.m.f. in micro volts which will be generated will be given by the area $ABCD$. If, for example, the cold junction is at 20° C. and the hot junction at 400° C., the mean temperature will be 212° C. Reference to Fig. 4 shows that at 212° C. the thermo-e.m.f. is about 56 micro volts per 1° C., so that for a temperature difference between the hot and cold junctions of 380° C. the thermo-e.m.f. of the couple will be

$$56 \times 380 = 22{,}000 \text{ micro-volts} = 22 \text{ milli-volts}.$$

Usually iron-constantan couples can be used up to 800° C., nickel-nichrome couples up to 1,100° C., and couples of platinum with a platinum rhodium alloy up to about 1,600° C.

In the Table I will be found the thermo-e.m.f. calibration of a representative copper-constantan couple and also an iron-constantan couple In Table II is given the calibration for a platinum-platinum-rhodium couple. Such couples can be manufactured to such a degree of identity that when a new couple is used to replace a defective one there is no necessity to re calibrate the measuring instrument.

Feussner has found that a couple of pure iridium and an alloy (60 per cent. rhodium : 40 per cent. iridium) can be used for the precision measurement of temperatures up to 2,000° C., and such couples can be duplicated with such uniformity that it is not necessary to re-calibrate the measuring instrument when renewing the couple. The following table gives the characteristics :

Temp C	200	400	600	800	1,000	1,200	1,400	1,600	1,800	2,000
E M F. mV.	1·1	2 2	3 3	4 4	5 5	6 6	7 65	8 7	9 8	10 85

TABLE II

Temperature ° C	*Thermo E.M F* *mV.*	*Temperature* ° C	*Thermo E.M F.* *mV.*
20	0 00	800	7 23
100	0 54	900	8 36
200	1 33	1,000	9 50
300	2 22	1,100	10 66
400	3 15	1,200	11 85
500	4 12	1,300	13 04
600	5·13	1,400	14·25
700	6 16	1,500	15 45
		1,600	16 62

It will be seen that this characteristic gives a straight line relationship between the temperature and the e.m.f. As regards the necessary

protecting sheath it has been found that corundum is satisfactory up to 1,700° C., spinell up to 1,950° C., magnesia and zirconium oxide up to 2,000° C.

For temperatures up to about 1,600° C. a widely used type of thermo-couple is the platinum : (platinum-rhodium) couple of which the characteristic is given in Table II, page 162.

Thermo-couples which are made from noble metals for the high temperature ranges are expensive, besides which, such metals have a high specific resistance so that the use of cheap measuring instruments which require a relatively large operating current can only be used to a limited extent. Such thermo-couples are extremely sensitive to the presence of metal vapour so that they must never be used in a metal protecting sheath but must have a gas-tight ceramic sheath.

Practical Applications

In almost every kind of industrial plant there are processes which require the satisfactory maintenance of a definite temperature. For example, the iron and steel industries comprise a multitude of furnaces of every kind which must operate at specified temperatures, and almost every chemical process requires the maintenance of precise temperatures in order that the process shall be correctly performed at the requisite efficiency. The electrical methods for measuring temperature which are commercially available are the radiation pyrometer, the thermo-couple, and the resistance thermometer. For temperatures from 800 down to as low a value as desired, resistance thermometers can be used (see Chapter V, page 126, for the principle of operation of the resistance thermometer). For temperatures greater than 800° C. up to the highest attainable values, the radiation pyrometer can be used. The principle of this appliance will be seen by reference to Fig. 5. A thermo-couple of small dimensions is sealed into a glass bulb which may be either evacuated or filled with some suitable gas. The radiation from the hot body of which the temperature is to be measured is focussed on to the thermo-couple and the corresponding value of the e.m.f. measured by a moving

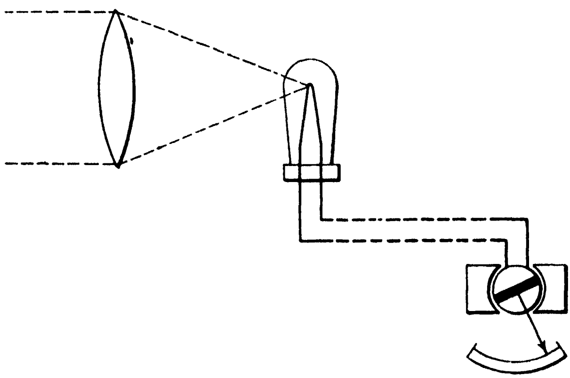


Fig. 5.

coil voltmeter. The advantage of this type of thermometer is that the thermo-couple is only subjected to low temperatures, but an essential requirement for accurate measurements is that the radiation should be from a "black" body. Gas-filled bulbs show, in general, a shorter time-lag than evacuated bulbs. With well-constructed units a time-lag of from 2 to 5 seconds may be obtained, and radiation pyrometers comprising a thermo couple would then be suitable for use in those cases in which such a time lag would be permissible.

The leads from the thermo-couple wires may have to be taken some distance to where the measurement instruments are installed, and this can be done by connecting copper leads to the respective ends of the wires which form the thermo couple itself. It is to be observed, however, that in this case, each of the junctions of these copper wires with the thermo-couple wires involves a new thermo-junction, and consequently it is essential that provision must be made to ensure that these junctions are maintained at a constant known temperature, otherwise the readings of the thermo couple may be seriously falsified. One method of doing this is to bury the junctions in the earth, in which case only small temperature fluctuations will be obtained. A more frequently used method is to provide a small thermostatic control, by means of which the junctions of the copper leads to the thermo couple wires can be maintained at the requisite known temperature.

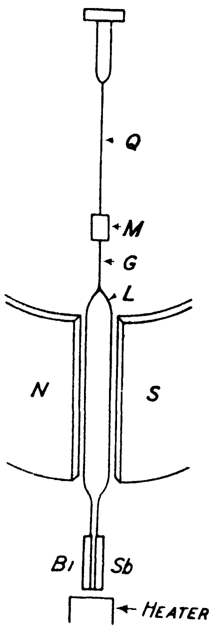


Fig 6.

It has now been found possible to construct thermo couples such that the wires of which the couple are formed are so selected that for temperatures under 150° C. no thermo-electric e.m.f. is developed, and consequently, if the cold junction temperature is less than this value, no fluctuations of the cold junction temperature will affect the reading obtained from the hot junction. These thermo couples are built for measuring temperatures up to 1,250° C.

In Fig. 6 is shown a diagrammatic view of a thermo-galvanometer in which a loop of silver wire L is suspended by means of a quartz fibre Q between the poles pieces N, S, of a permanent magnet. The loop is surmounted by a glass stem G carrying a mirror M, whilst its lower ends are connected to a bismuth antimony thermo-couple. A "resistance heater" consisting of a fine filament of high specific resistance, such as a platinised quartz fibre, is fixed immediately under the thermo-couple The current to be measured is passed through the heater and the radiated heat so generated produces an e.m.f. in the thermo-junction.

and hence a current in the loop L. The deflection of the loop L is measured by means of a spot of light reflected from the mirror to a distant scale. The instrument is sufficiently sensitive to measure a current of 20 micro amperes, whilst currents as small as 2·2 micro amperes may be detected.

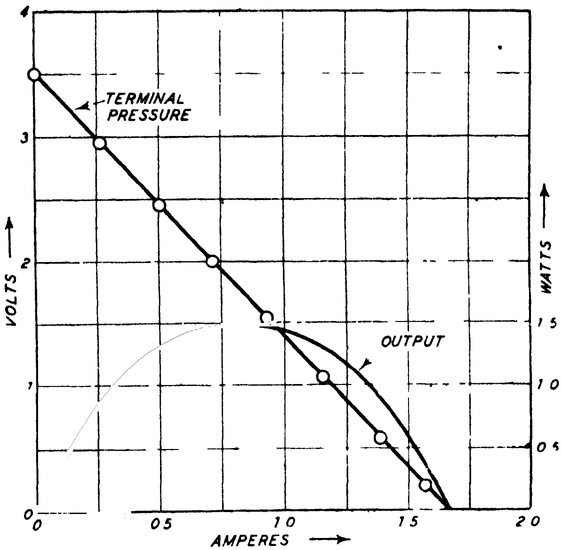


Fig. 7.

For some purposes for which great constancy of electrical supply pressure is necessary, a thermo-electric generator can be used which comprises a number of thermo couples connected in series.* In Fig. 7 the straight line shows the terminal p.d. and the parabolic curve the output of such a generator of which the thermo couples were iron-constantan and heated by gas. The internal resistance in the generator was 2 ohms and the parabolic form of the output curve is defined by the equation

$$\text{Output} = E.I - I^2R_i \text{ watts},$$

where E volts is the thermo e.m.f., I amperes is the current and R_i ohms is the internal resistance.

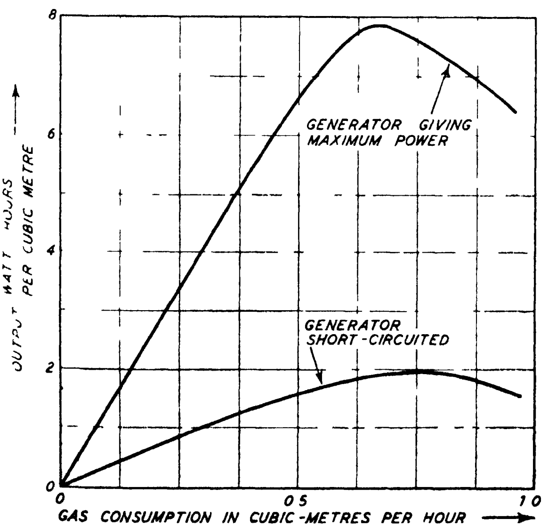


Fig. 8.

In Fig. 8 is shown the relationship of the output (curve C) and

* See *The Electrical Review*, Vol. 101, 1927, Nov. 18, p. 847.

the power consumed at short circuit (curve D) as a function of the gas consumption. The thermal efficiency is very low, viz. for the gas-heated generator about 0·04 per cent. For a coke-heated generator of 0·5 kW output the thermal efficiency was from 3 to 4 per cent.

Thermo-Converter

This instrument is a heat-transformer consisting of an electrically heated thermo-couple (Fig. 9) and it provides a valuable means for measuring alternating currents of small magnitudes. The a.c. to be measured is caused to heat the thermo-couple, thus generating a direct

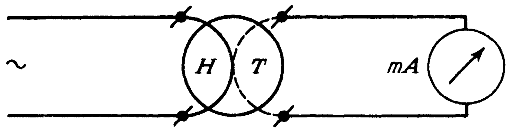


Fig. 9

current, the magnitude of which will depend upon the a.c. heating current. The direct current so generated is passed through a permanent magnet, moving-coil instrument.

The thermo-couple which is used for this purpose may be of nickel and chromium, or of iron and constantan. The soldered joint of the couple is also soldered to the heating wire which carries the a.c. current so that this heating wire produces the heat immediately at the thermo

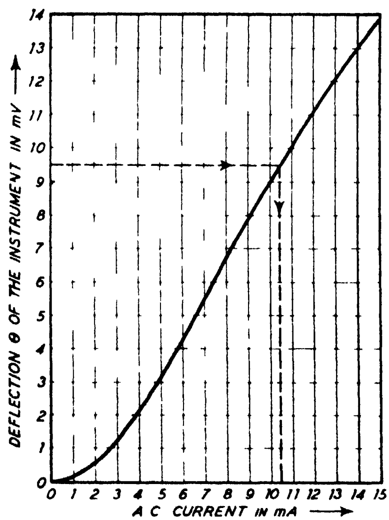


Fig. 10.

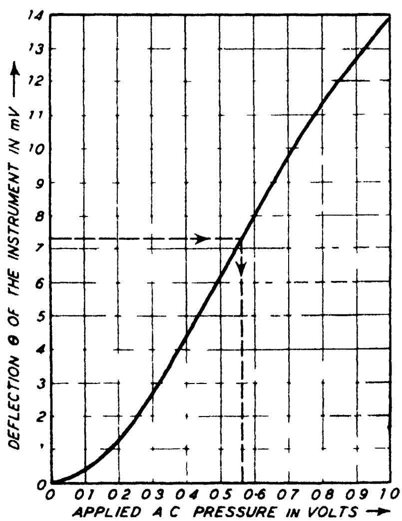


Fig. 11.

junction. For current strengths up to 100mA. the thermo-converter is enclosed in an exhausted glass bulb in order to avoid harmful dissipation of the heat. The general arrangement of the device will be clear from Fig. 9, the source of the a.c. which it is desired to measure is connected to the heating wire *H* and the measuring instrument is connected to the thermo-couple *T*.

In Figs. 10 and 11 are shown the calibration curves for a thermo-converter from which the values of the current through the heating wire and the applied pressure at the heating wire input terminals, respectively, can be measured. The ordinates in each case give the deflection of the instrument pointer in mV. and the abscissa show what current and pressure, respectively, must be applied to the heater wire in order to produce a given deflection of the pointer of the measuring instrument.

Piezo-electricity

In 1782 Hauy discovered that an electric charge was developed on the surface of certain crystals when subjected to mechanical pressure. In 1880 the brothers I. and P. Curie found that faces of certain pyro-electric crystals became electrically charged when the crystal was subjected to pressure or tension. These investigators also showed that if such crystals were placed in an electric field they would become mechanically deformed. Such phenomena are now termed "piezo-electrical effects".

A crystal is termed "piezo-electric" if it has one or more polar axes. By a "polar axis" is meant an imaginary straight line in the crystal of which the beginning and end are not equivalent, that is to say, are not interchangeable, so that if the crystal is turned through 180° about an axis which is at right angles to the polar axis, the original aspect of the crystal is not attained. A good example of this is shown in Fig. 12, in which the polar axis of a pentaerythrit crystal is marked x.

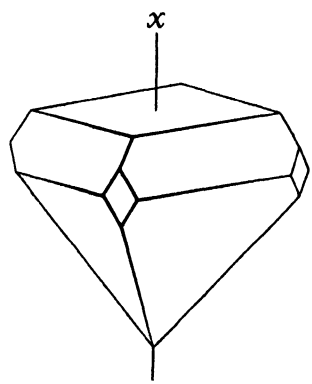


Fig. 12.

A quartz crystal is shown in Fig. 13 and such a crystal has three polar axes which are respectively marked x_1, x_2, x_3, and each of these axes passes through a pair of opposite edges of the six-sided prism. The axis marked z is the optical axis and is an axis of symmetry for optical purposes. The practical significance of the polar axes lies in the fact that when the crystal is subjected to an elastic stress the greatest value of the corresponding electric charge is always found at the ends of a polar axis. The consequence is that when plates or rods are to be cut from the crystal for piezo-electric purposes, care is taken that

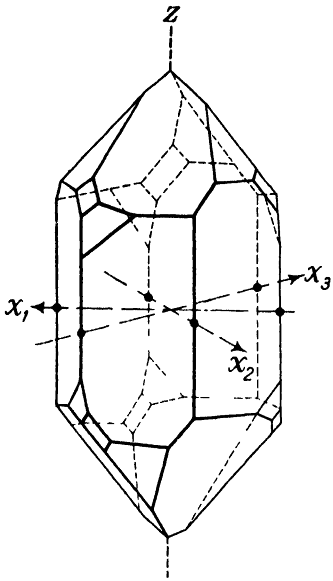


Fig. 13.

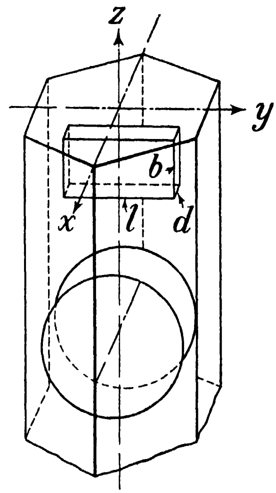


Fig. 14.

the main pair of parallel surfaces are at right angles to one of the polar axes. For quartz, the most suitable places for cutting out such slabs are shown in Fig. 14. The polar axis which is at right angles to the main pair of parallel faces is the electrical axis x, whilst the axis y, which is at right angles to the electrical axis, is the neutral axis. The axes x and y form with the optical axis a rectangular three-dimensional system of co-ordinate axes.*

From the mathematical treatment of the problem of the relationship between the compressive or tensile stress in the electrical axis of a piezo crystal and the corresponding electrical quantity to which that stress will give rise, the following equations have been derived:

$$q = e . P_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$q = - e . \frac{S_x}{S_y} P_y \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where e is the piezo electric modulus, P_x is the total compressive or tensile force in the direction of the electrical axis, and P_y the total force in the direction of the neutral axis y. The magnitude of the piezo modulus for quartz has been found to be

$$e = 6{\cdot}9 \times 10^{-8} \text{ c.g.s. units} \quad . \quad . \quad . \quad (3)$$

Equation (1) expresses the important fact that a mechanical force P_x

* *Elektrotechnische Zeitschrift*, Aug. 4, 1938, p. 819.

in the direction of the polar axis gives rise to an electric charge on both faces of the slab which are at right angles to the polar axis, and that the magnitude of this charge does not depend upon the surface area of the faces. This is known as the " direct piezo-electric effect ". Equation (2) states that if a mechanical force P_y is applied in the direction of the neutral axis, the faces which are parallel to this axis will become charged with a quantity of electricity which is proportional to the total force applied and also proportional to the ratio,

$$\frac{\text{Area } S_x \text{ of each of the electrically charged surfaces}}{\text{Area } S_y \text{ of the surface to which the mechanical force is applied}}.$$

This result is known as the " cross piezo-electric effect ". Both effects have the polar characteristic that a reversal of the direction of the pressure results in a reversal of the sign of the corresponding induced charges.

EXAMPLE.—Suppose the main faces of the slab are fitted each with a metal plate electrode so that the arrangement may be regarded as a condenser of capacitance C and the p.d. which will be developed between the electrodes will be given by the expression

$$V = \frac{\epsilon . P_x}{C + C_0} \qquad . \quad . \quad . \quad . \quad . \quad (4)$$

where C_0 is the capacitance of the electrostatic voltmeter which is used to measure the p.d. If the total capacitance $C + C_0 = 10$ electrostatic units, and if the applied pressure is $P_x = 2$ kg., then from equation (1) the quantity of electricity with which the faces of the crystal will become charged is

$$q = 6{\cdot}9 \times 10^{-8} \times 2{,}000 \times 987 = 0{\cdot}136 \text{ electrostatic c.g.s. units}$$

so that the p.d. which the voltmeter will measure will be

$$V = \frac{q}{C + C_0} = \frac{0{\cdot}136}{10} \text{ electrostatic units} = 4{\cdot}08 \text{ volts.}$$

If however, the supplementary capacitance C_0 of the voltmeter were not connected in parallel with the crystal, and if the capacitance C is calculated from the dimensions of the crystal (see Chapter IV, page 109), that is

$$C = \frac{\varepsilon . b . l}{4\pi d},$$

where $\varepsilon = 4{\cdot}5$ and is the dielectric constant of quartz,
d cm. is the thickness of the quartz slab,
b ,, ,, breadth ,, ,, ,,
l ,, ,, length ,, ,, ,,

then for $d \times b \times l = 1 \times 1 \times 1$ cm.³

$$C = \frac{4 \cdot 5}{4\pi} = 0 \cdot 36 \text{ electrostatic c.g.s. units,}$$

and
$$V = \frac{q}{C} = \frac{0 \cdot 136}{0 \cdot 36} = 0 \cdot 378 \text{ electrostatic units} = 113 \cdot 4 \text{ volts}$$

Since however, the capacitance C of the quartz slab is only about 0·4 pF and is consequently negligibly small in comparison with the capacitance C_0 of the voltmeter, it follows that the theoretically maximum value of the p.d. can never be realised for practical purposes.

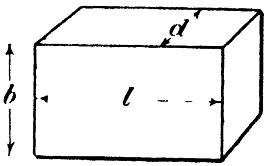


Fig. 15.

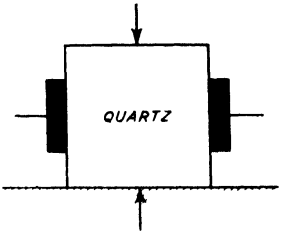


Fig 16.

Next, suppose that a quartz slab of dimensions

$$l \times b \times d = 5 \times 1 \times 0 \cdot 5$$

as shown in Fig. 15 is used and a total force of $P_y = 2$ kg. is applied in the direction of the neutral axis, then the corresponding value of the quantity of electricity with which the faces will become charged will be

$$q = -6 \cdot 9 \times 10^{-8} \times \frac{5}{0 \cdot 5} \times 2{,}000 \times 981$$
$$= -1 \cdot 36 \text{ electrostatic units,}$$

so that for the total capacitance $C + C_0$ of the quartz slab and measuring instrument the p.d. realisable for practical purposes will be

$$V = \frac{q}{C + C_0} = \frac{1 \cdot 36}{10} = 0 \cdot 136 \text{ electrostatic units,}$$

that is $V = 40 \cdot 8$ volts.

The "direct piezo electric effect" is reversible, that is to say, if an electric field is impressed on the crystal in the direction of the electric axis, that is, if a p.d. V is applied to the faces of the slab by means of two electrodes (Fig. 16), an elastic change of dimension u will be produced in the direction of the electric axis and an elastic change of dimension v in the direction of the neutral axis where

$$u = + e . V \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

$$v = - e . V \frac{S_x}{S_y} \qquad . \quad . \quad . \quad . \quad . \quad (6)$$

These equations state that, if an electric field is impressed on the crystal in the direction of the electric axis, an elastic charge of dimension will be produced in the direction of that axis, the amount of which will be independent of the surface area of the faces. There will also be a simultaneous change of dimension in the direction of the neutral axis the magnitude of which will be proportional to the ratio,

$$\frac{\text{Area of faces parallel to the neutral axis}}{\text{Area of faces parallel to the electric axis}}.$$

Both effects are polar in nature, that is, the extension becomes a compression when the sign of the impressed field is changed.

Making use of the numerical data of the previous example and assuming that the applied p.d. is $V = 100$ volts $= \frac{1}{3}$ electrostatic unit, and considering the crystal of dimensions $l \times b \times d = 5 \times 1 \times 0{\cdot}5$ cm.^{3}, then

$$u = 6{\cdot}9 \times 10^{-8} \times \tfrac{1}{3} = 2{\cdot}3 \times 10^{-8}\ \text{cm.} = 2{\cdot}3 \times 10^{-4}\mu$$

(1μ = one micron = 10^{-6} metre)

and since $\dfrac{S_x}{S_y} = \dfrac{5}{0{\cdot}5} = 10 : v = \dfrac{6{\cdot}9}{10^8} \times \tfrac{1}{3} \times 10 = 23 \times 10^{-4}\mu$,

that is to say, the change in dimension in the direction of the neutral axis is 10 times that in the direction of the electric axis. The practical applications of these effects can be divided into two groups, (*i*) when the effects are static and (*ii*) when the effects are dynamic. The static effects are usually associated with quantitative measurements, e.g. the measurement of the induced charge due to the application of a mechanical and consequently the determination of the magnitude of the applied force from the magnitude of the induced electric charge. The dynamic effect is utilised by causing the crystal to perform elastic oscillations.

One interesting example of the application of the static effect is the measurements which have been carried out in connection with the French railway system in which determinations were made of the forces which are developed between the rails and the wheels of the vehicles. The problem of such measurements is an extremely complicated one and the use of the piezo-electric effects has provided a notable addition to the knowledge of the subject.

The dynamic effect is of the utmost importance in the realm of high-frequency technique. P. Langevin in 1917 was the first to produce oscillations of a quartz crystal by means of an impressed alternating current field. In 1922 W. G. Cady demonstrated that quartz plates and rods when placed in a high-frequency alternating electric field could be caused to execute powerful oscillations, and he commenced to develop the applications of this principle for the purposes of generating and receiving high-frequency currents. In consequence of the production of the mechanical change of dimensions in both the electric and the neutral

axes, corresponding oscillations will also occur in the direction of each of these axes.

For oscillations in the direction of the electric axis, the natural frequency will be given by the equation

$$f_{el} = \frac{1}{2d}\sqrt{\frac{E_x}{\rho}} \text{ hz. (for } d \text{ in centimetres)} \qquad (7)$$

and for oscillations in the direction of the neutral axis the natural frequency will be

$$f_n = \frac{1}{2l}\sqrt{\frac{E_y}{\rho}} \text{ hz. (for } l \text{ in centimetres)} \qquad (8)$$

where E_x and E_y are the respective moduli of elasticity and ρ is the density.

For quartz $\rho = 2{\cdot}65$ gm. per cm.^{3}
$E_x = 8{,}711$ kg. per mm.^{2}
E_y 7,871 kg. per mm.^{2}

If these values are substituted in equations (7) and (8), respectively, then

$$f_{el} = \frac{1}{d} \times 2{\cdot}91 \times 10^5 \text{ hz.}$$

and

$$f_n = \frac{1}{l}\, 2{\cdot}68 \times 10^5 \text{ hz.}$$

For example, if a quartz plate is 0·5 mm. thick, that is, if

$$d = 0{\cdot}05 \text{ cm.}$$

then

$f_{el} = 5{\cdot}82 \times 10^6$ hz. and the wave-length is

$$\lambda = \frac{\text{speed of light}}{f_{el}} = \frac{3 \times 10^8}{5{\cdot}82 \times 10^6} = 51{\cdot}5 \text{ m.}$$

For a quartz rod of length $l = 5$ cm.,

$$f_n = 53{,}600 \text{ hz.}$$

and the wave-length is $\lambda = 5{,}590$ m.

In addition to the fundamental frequencies which are defined by the foregoing expressions, high harmonics may also be produced, and of particular importance in this connection is the excitation of the oscilla tion of quartz plates in the direction of the electric axis. It is possible, for example, in the case of a quartz plate 10 mm. thick to obtain an oscillation harmonic of 200 times the frequency of the fundamental oscillation. The magnitude of the amplitude of the oscillation for quartz in practice is about 10^{-3} mm., the limit being fixed by the mechanical strength of the quartz.

The damping of the oscillations of a quartz plate in air is extremely small, the logarithmic decrement being about 10^{-4}. The limiting value

of the fundamental frequency which a quartz plate can be maintained satisfactorily in oscillation is about 30×10^6 hz., the corresponding thickness of the quartz plate being 0·095 mm. For tourmaline crystals the modulus of elasticity in the direction of the electric axis is 16,119 kg. per mm.^{2} and the density is 2·94 gm. per cm.^{3}, from which it follows that the fundamental frequency of a tourmaline plate oscillating in the direction of its electric axis is

$$f_{el} = \frac{1}{d} \times 363{,}620,$$

so that for a given frequency a tourmaline plate will be about 38 per cent. thicker than a quartz plate. The highest frequency so far obtained for a tourmaline plate is $1 \cdot 50 \times 10^8$ hz.

The quartz-crystal oscillator when operated under the most favourable conditions is recognised as the most accurate known marker of time intervals. Pendulums and tuning-forks could possibly be operated with comparable precision, although greater difficulties would probably be experienced in doing so. Usually, standard piezo-oscillators operate at frequencies of about 100 kHz., and lower frequencies are obtained by multivibrators. Frequencies down to about 1,000 Hz. are usually obtained in this way.

Formerly, the difficulty of measuring frequencies of very high values was the measurement of the time. The best standard clocks could not develop a greater accuracy than about one to two-thousandths of a second per day. Consequently, the idea was conceived of measuring time by frequency and the first published account of such a " clock " was made in 1929. As a result of many years' research, quartz clocks can now be constructed which for long periods of time maintain an accuracy of 1 part in 100,000,000, that is, at the rate of 0·3 second per annum. Such a quartz clock is now the time and frequency standard to which all measurements can be referred. The quartz oscillator of the clock generates a frequency of 60,000 hz. and the temperature of the crystal is automatically maintained constant to 0·001 C. For short periods such a clock can maintain an accuracy of 1 part in 10^9, that is, at the rate of 0·03 second per annum. For further information, reference should be made to the *Bureau of Standards Journal of Research*, Vol. 21, 1938, page 367 (see also Chapter I, page 18).

Chapter VII

MAGNETISM : MAGNETIC MATERIALS : MAGNETIC TESTING

The Atomic Structure of Iron

IN Chapter II some account has been given of the modern view of the structure of the atom and reference was made to the fact that the theory of atomic structure can be extended to embrace an explanation of the phenomenal magnetic characteristics of the three elements iron, cobalt, and nickel, the atomic numbers of which are respectively 26, 27, and 28.

Spectroscopic analysis has shown that not only do the electrons of an atom revolve round the nucleus like planets round the sun, but they also like planets, spin on their axes, and it is to this spin effect that the magnetic properties of an atom are believed to be due.* To a relatively minute degree, every element exhibits magnetic properties, but iron, cobalt, and nickel are magnetic to such an extraordinary degree that they are in a class by themselves and are termed "ferro-magnetic" metals. In Fig. 1 is shown a diagrammatical view of the supposed structure of the atom of iron. The nucleus contains 26 protons and there are normally 26 electrons revolving round the nucleus, these electrons being grouped in four concentric shells In the outer shell there are two electrons, the spins of which are in opposite directions so that they neutralise each other's magnetic effect. These electrons in the outer shell are the "valency electrons" and are responsible for the ordinary chemical properties of the atom. In each shell, except the third from the centre, there is an equal number of electrons having positive and negative spins. In the third shell from the centre, however, there are four unbalanced spins, and it is to these unbalanced spins

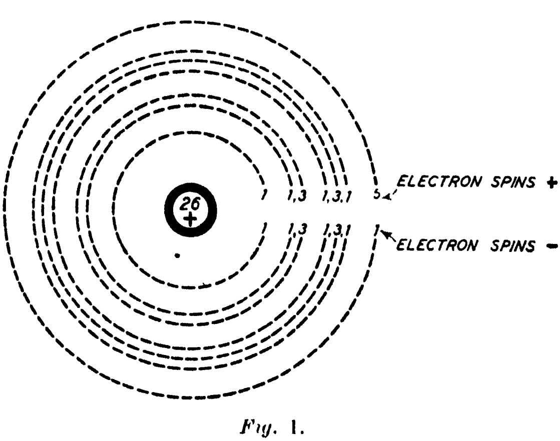


Fig. 1.

* Bozorth, *Electrical Engineering*, New York, November, 1935.

that the magnetic properties of the atom of iron are believed to be fundamentally due.

Fundamental Facts of Magnetism

The study of the complicated phenomena of magnetism is considerably clarified by realising that the magnetic state of a substance can be described from two different standpoints, (*i*) the conception of magnetic poles, and (*ii*) the conception of magnetic induction as associated with an electric current. These two aspects of magnetism will now be briefly considered.

(*i*) A bar magnet is assumed to have two equal quantities of magnetism of opposite signs, viz. $+ m$ and $- m$, concentrated respectively at a point near each end of the bar. These two points are termed the "poles" of the magnet and the strength of each pole is said to be m units. If two different bar magnets have, respectively, pole strengths m_1 and m_2, then the force which will act between the poles of the two magnets will be defined by the expression

$$H = \frac{m_1 . m_2}{\mu . r^2} \text{ dynes} \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the magnetic "permeability" of the medium in which the magnets are placed, and r cm. the distance between the poles m_1 and m_2. This relationship is Coulomb's "Law of Inverse Squares" for magnetic poles. For a vacuum $\mu = 1$ and for air and gases generally, μ is very nearly equal to unity. If the magnet poles are of equal strength and such that $H = 1$ dyne when $r = 1$ cm., and if the medium is air or a vacuum, then $H = 1$; $m_1 = m_2 = m$; $r = 1$: $\mu = 1$, so that $m^2 = 1$, that is, $m = 1$, and this leads to the definition of the c.g.s. unit magnetic pole (see also Chapter I, pages 2 and 4).

A unit magnetic pole is such that, when placed in air at a distance of 1 cm. from an equal pole, the mutual force with which each acts on the other, is 1 dyne.

The expression (1) and the consequent definition of a unit pole magnetic should be compared with the corresponding results for electric quantities given in Chapter III, page 76. Actually, it is not possible to isolate a magnetic pole in the sense that electric quantities can be isolated, since magnetic poles always appear in pairs. Experimentally, however, a good approximation to an isolated pole can be obtained when a long thin iron wire is magnetised in the direction of its axis. If the distance between the two poles of a magnet is l cm. and if m c.g.s. units is the strength of each pole, then

$$M = m . l \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where M is termed the "magnetic moment", and this conception of magnetic moment may be considered to be the most generalised statement of magnetic polarisation.

The space in the neighbourhood of a magnet is termed a "magnetic field" and its strength or "intensity" is denoted by the symbol H, viz.

*The intensity at any point in a magnetic field is the mechanical force in dynes which would act on a unit + pole if placed at that point and the direction of the magnetic force is the direction in which the unit + pole would tend to move.**

The unit of magnetic field intensity was formerly termed the "gauss", but by international agreement the name "oersted" was substituted for this quantity in the year 1930. The direction and the strength of a magnetic field can be graphically represented by means of "lines of force" in a precisely similar way to that which has been described already in detail for the case of an electric field (see Chapter III, pages 78, 79 and 80.)

Suppose a magnetic pole of strength + 1 unit is at the centre of a (geometrical) sphere in air, the radius of the sphere being 1 cm. Then in accordance with the foregoing definition, the magnetic intensity at any point on the surface of the sphere will be unity, that is to say, one unit line of magnetic force will cross each sq. cm. of the surface of the sphere. Since the area of the spherical surface is 4π sq. cms. it follows that a total of 4π unit lines of magnetic force will cross this surface or, in other words, a unit magnetic pole in air, gives rise to 4π unit lines of magnetic force (see also Chapter III, p. 81).

If a plane surface of S sq. cm. area is placed at right angles to the lines of force of a magnetic field of uniform intensity, then

$$\Phi = H.S \text{ maxwell} \qquad (3)$$

where Φ is the magnetic flux through the area S.

(*ii*) The second method of describing magnetic phenomena depends upon the mutual action between a magnet and an electric current. In the year 1820 Oersted discovered that in the neighbourhood of a wire which carries an electric current a magnetic field exists. Thus, if a long straight coil of insulated wire (i.e. a "solenoid") such that the diameter of the coil is small in comparison with its axial length, and if the current of I amperes is flowing in the coil, then at places within the solenoid and not far removed from the mid-point of the axis, the intensity of the magnetic force will be given by the expression (see also Chapter VIII, page 218)

$$H = \frac{4\pi}{10} \cdot \frac{I.W}{l} \text{ oersted} \qquad (4)$$

where W is the total number of turns in the solenoid winding,

l cm. is the length of the solenoid.

* It is to be noted here that a + pole is a N-seeking pole, e.g. that end of a compass needle which points towards the north pole.

The product IW is the number of "ampere-turns" of the solenoid and $\frac{IW}{l}$ is the number of ampere-turns per centimetre length of the solenoid, so that the expression (4) may be stated as follows:

$$H = \frac{4\pi}{10} \text{ (ampere-turns per centimetre)} \quad . \quad . \quad . \quad (5)$$

in oersted, so that

$$\left.\begin{aligned} 1 \text{ oersted} &= 0{\cdot}796\ IW \text{ cm.} \\ 1\ IW/\text{cm.} &= 1{\cdot}256 \text{ oersted} \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

If a substance, e.g. iron, is brought into a magnetic field it will become magnetised and will exhibit, temporarily and qualitatively, all the properties of a permanent magnet. The degree of magnetisation which the substance acquires will depend upon the strength of the field, the shape of the substance, and, in particular, the nature of the substance. The phenomena of magnetisation in this case can also be considered from the same two stand-points as have been stated in the foregoing.

(1) From the point of view of magnetic polarisation it may be said that each element of volume of the substance which is placed in the magnetic field is subjected to the polarisation effect of the field. That is to say, the substance when under the influence of the magnetic field acquires an "induced" pole strength m, and if the distance between the poles is l cm. the corresponding magnetic moment will be $M = m.l$. If V c.cm. is the volume of the substance, then the induced magnetic moment per unit volume is

$$J = \frac{M}{V} = \frac{m.l}{V} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, if a cylindrical iron rod of cross-section S sq. cm. and length l cm. is placed in the magnetic field with its axis in the direction of the field, then

$$J = \frac{m.l}{S.l} = \frac{m}{S} = \text{pole strength per square centimetre} \quad . \quad (8)$$

where J is termed the "intensity of magnetisation".

The ratio of the intensity of the magnetisation J to the magnetising force H is the "magnetic susceptibility"

$$\chi = \frac{J}{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a vacuum $\chi = 0$, but in the case of all substances, the value of χ is not zero.

Just as was seen to be the case for a quantity of electricity (see Chapter III, page 77), a unit magnetic pole gives rise to 4π unit lines of magnetic force as explained already on page 77, i.e., if the induced pole strength is m units per sq. cm. cross-section of the magnet, then each

square centimetre of the cross-section will give rise to $4\pi m$ induced unit lines of force, and these will be superimposed on the original lines of force H in the neighbourhood of the magnetised substance. Instead of considering the lines of magnetic induction separately from the inducing lines of force to which they are due, it is to be observed that for a whole range of magnetic phenomena it is the resultant of the magnetising force and the induced magnetisation which is the salient quantity, particularly for practical calculations relating to electrical engineering appliances and machines. The flux of magnetic induction Φ, that is, the total number of lines of force which pass through the cross-section of S sq. cm., is

$$\Phi = \mu . H . S \quad . \quad . \quad . \quad . \quad (10)$$

where H oersted is the intensity of the magnetising force and μ is a factor which defines the magnetic " permeability " of the substance as compared with that of free space.

The density of the magnetic induction, that is, the number of lines of force which cross each square centimetre of cross-section of the area which is at right angles to the lines of force, is termed the " induction density ", or, more simply, the " induction " B, so that

$$B = \frac{\Phi}{S} = \mu H \quad . \quad . \quad . \quad . \quad (11)$$

As a matter of definition, the total flux Φ is assumed to be the algebraic sum of H and $4\pi J$, that is,

$$B = H + 4\pi J \quad . \quad . \quad . \quad . \quad . \quad (12)$$

so that

$$\left.\begin{aligned} \mu &= \frac{B}{H} &: \chi &= \frac{J}{H} \\ \mu &= 4\pi\chi + 1 &: \chi &= \frac{\mu - 1}{4\pi} \end{aligned}\right\} \quad . \quad . \quad . \quad (13)$$

For the comparison of some aspects of the magnetic quality of different materials the " reluctivity " κ is used, this quantity being the reciprocal of the permeability, viz. $\kappa = \frac{1}{\mu}$.

For all theoretical investigations the quantities which are the more appropriate and generally used are J and χ rather than B and μ. The most common practical application of the foregoing expressions, however, involves the relationship between the magnetising ampere-turns of a magnetic circuit and the corresponding value of the flux Φ, as will be seen for the considerations in what follows. Reference should also be made to Chapter VIII, pages 219 and 222, for a more detailed study of the formulae which are used in what follows.

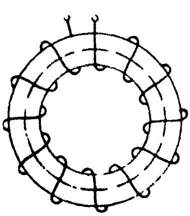


Fig. 2.

(ii) Fig. 2 shows a uniformly wound coil of insulated

wire the axis of which is circular. If the mean length of the axis is l cm. and if there are W turns in the coil, then when a current of i amperes flows in the coil, the total number of ampere-turns will be iW. From expressions (4) and (10) the flux through the coil will be

$$\Phi = H.S = \frac{4\pi}{10}\frac{iW}{l}S \quad . \quad . \quad . \quad . \quad (14)$$

since the medium which is enclosed by the coil is assumed to be air. If, now, the coil is assumed to be filled with a magnetic substance, for example, if the coil is wound on an iron ring of which the magnetic permeability is μ, then the flux through the coil will be

$$\Phi = \mu HS = \frac{4\pi}{10}\frac{iW}{l}S\mu$$

that is

$$\Phi = \frac{\frac{4\pi}{10}iW}{\frac{l}{S\mu}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The expression (15) was first used by Hopkinson in 1886. The quantity $\frac{4\pi}{10}iW$ is termed the "magneto-motive force" (m.m.f.) round the closed magnetic circuit (in this case, the iron ring), and the quantity $\frac{l}{S\mu}$ is termed the "reluctance" of the closed magnetic circuit. The equation (15) may therefore be written

$$\text{Magnetic flux} = \frac{\text{Magneto-motive force}}{\text{Reluctance}} \quad . \quad . \quad . \quad (16)$$

If this relationship is compared with the Ohm's law relationship for the electric circuit, viz.

$$\text{Electric current} = \frac{\text{Electro-motive force}}{\text{Resistance}},$$

it will be seen that the magnetic reluctance is analogous to electric resistance, magneto-motive force to electro-motive force, and magnetic flux to electric current. The great practical significance of the relationship (15) is the fact that it can be applied to the calculation of a magnetic circuit which is built up of a number of materials of different dimensions and different magnetic qualities, in which case the total reluctance of the magnetic circuit as expressed by the quantity $\frac{l}{S\mu}$ in equation (15) may be obtained by adding the respective reluctances of the individual component parts of the magnetic circuit, viz.

$$\frac{l}{S\mu} = \frac{l_1}{S_1\mu_1} + \frac{l_2}{S_2\mu_2} + \frac{l_3}{S_3\mu_3} + \ldots \quad . \quad . \quad . \quad (17)$$

and in Chapter VIII, page 221, a numerical example of the calculations involved in this type of problem will be found.

It is to be noted that the practical values of the permeability μ vary over a very wide range according to the nature of the substance and the intensity of the magnetising force H. Substances for which μ is less than 1 are termed *diamagnetic*, the most notable of which is bismuth for which at a temperature of 18° C.

$$\chi = -1{\cdot}4 \times 10^{-6}, \text{ so that } \mu = 4\pi\chi + 1 = 0{\cdot}999982.$$

Substances for which μ is a little greater than 1 are termed *paramagnetic*, e.g. oxygen gas, for which at a temperature of 16° C.

$$\chi = 0{\cdot}123 \times 10^{-6}, \text{ so that } \mu = 1{\cdot}00000155.$$

Substances for which μ is very large are termed *ferromagnetic* e.g. iron, cobalt, and nickel. As the temperature increases to definite values, the respective ferromagnetic substances become paramagnetic and the corresponding temperature at which the change takes place is known as the "Curie point", the values of which for the three ferromagnetic metals are as follows:

Iron .	about 770° C.
Cobalt .	,, 1,000° C.
Nickel .	,, 360° C.

so that for temperatures between about 770° C. and 1,000° C. cobalt is the only known ferromagnetic substance.

The Magnetisation Curve and the Hysteresis Loop

The curve which relates the magnetising force H and the induction B is known as the magnetisation curve or, more explicitly, the "null" magnetisation curve, since it passes through the origin of co-ordinates. In Fig. 3 are shown representative curves for dynamo stampings and

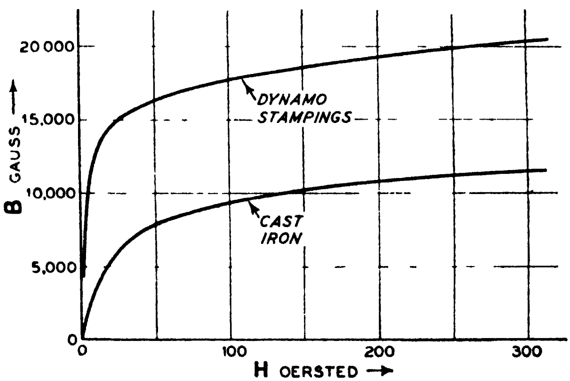


Fig. 3.

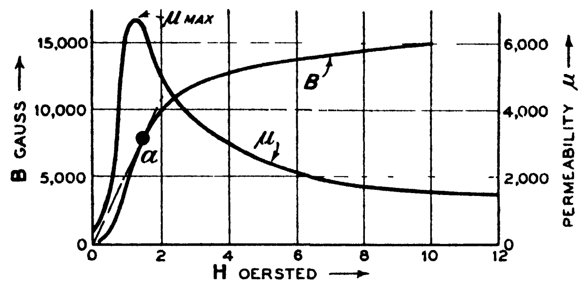


Fig. 4.

for cast iron, respectively. These curves show that the permeability $\mu = \frac{B}{H}$ varies over a very wide range of values, and in Fig. 4 is shown the relationship between the permeability μ and the magnetising force H for another typical magnetisation curve. It will be seen that for very small values of the magnetising force H, that is, in the neighbourhood of the origin, the permeability is small and the value at the origin, that is the slope of the magnetisation curve at the origin, is termed the initial permeability μ_0. As H increases μ also increases to a maximum value μ_{max}, falling again to a small value for very high values of B, that is, for the "saturated" condition of the magnetised substance.

If, commencing from the completely demagnetised condition of the substance, the magnetising force H is increased from zero to a maximum value OC', Fig. 5, this magnetising process will be defined by the "null" magnetisation curve such as that shown by OA in Fig. 5. If the magnetising force is then decreased and passed through the zero value to a maximum negative value OD which is equal to the previously attained maximum positive value OC', the curve such as AGB will be obtained. If H is then again reduced in magnitude and, passing through the zero value, is increased to the previously attained positive maximum value, it will be found that the closed loop $AGBFA$ has been obtained and this closed curve is known as the "hysteresis loop". The area of this loop represents the energy which has been converted into heat during the cycle of magnetisation, and consequently this area represents a definite loss of energy known as the "hysteresis loss", viz.

$$U_h = \frac{1}{4\pi}\{\text{area of hysteresis loop}\}, \text{ ergs per cubic centimetre}$$

that is

$$U_h = \frac{1}{4\pi}\int H dB \text{ ergs per cubic centimetre} \quad . \quad . \quad . \quad (18)$$

where the integration is carried out for the complete cycle of magnetisation (see also, Chapter VIII, page 242).

When the demagnetisation curve AG cuts the ordinate axis, the induction OF gives the remanent induction B_r which exists when the magnetising force is removed. Where the demagnetisation curve AG cuts the abscissa axis, the negative value of the magnetising force OG is termed the "coercive force", H_c, since it is the negative force which is necessary to completely demagnetise the substance. The demagnetisation curve AG is of primary importance in defining the quality of a substance suitable for the manufacture of permanent magnets.

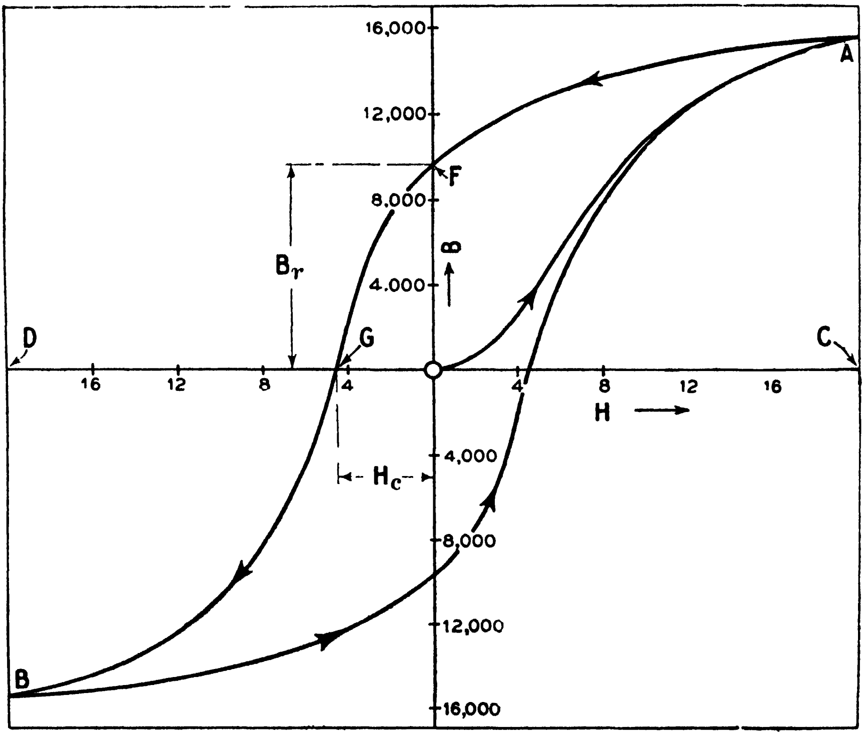


Fig. 5.

On the basis of a large number of experimental results, Gumlich has found that the following relationship exists between the previously defined characteristic quantities,

$$\mu_{max.} = \frac{1}{2}\frac{B_r}{H_c} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

that is to say, the maximum permeability is directly proportional to the remanent induction and inversely proportional to the coercive force.

This relationship gives accurate results for substances having very small values of the coercive force. For values of H_c up to 27·5 oersted the error is not greater than about 5 per cent.

Unless otherwise stated, the coercive force H_c and the remanent induction B_r refer to the condition that the maximum magnetising force OC' (Fig. 5) is sufficient to magnetise the substance to its saturation induction density.

In Fig. 6 are shown the hysteresis half-loops for (*a*) soft iron with small carbon content (C = 0·01 per cent.), (*b*) nickel, and (*c*) cobalt.

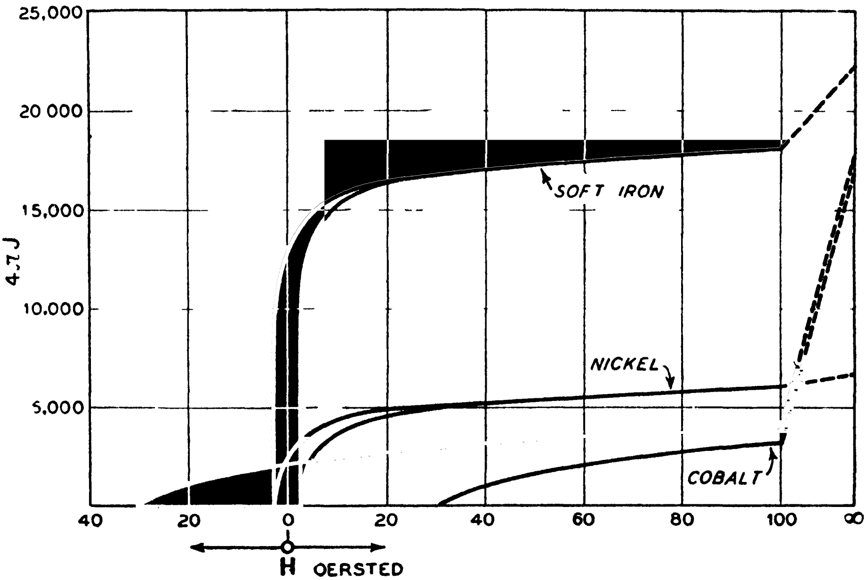


Fig. 6.

The null magnetisation curves are not shown in order to preserve the clarity of the diagrams. The ordinate values for these curves are the values of $4\pi J$. It is to be observed that the soft iron reaches a maximum value of $4\pi J_\infty$ is about 21,000, whilst for a magnetising force of H = 10 oersted the value of $4\pi J$ is about 30 per cent., and for H = 100 oersted about 10 to 15 per cent. below the saturation value. Saturation is practically complete for values of a magnetising force from 1,000 to 2,000 oersted. The descending branch of the hysteresis loop cuts the ordinate axis at B_r = 13,000, and the coercive force is from 0·9 to 1·0 oersted. The maximum permeability reaches a value of 5,000 to 10,000. Materials which have similar characteristic features to the foregoing are termed "magnetically soft".

The following table gives the leading characteristic data for various qualities of iron.

TABLE

Material	*Max. Permeability* μ_{max}	*Permeability μ for H* — 0	0·01	0·10	0·50	1·0	*Coercive Force H_c Oersted*	*Hysteresis Loss in Ergs per c.cm. per Cycle*
Dynamo Steel twice annealed	14,800	320	351	872	—	—	0·37	
Swedish Charcoal Iron	2,600	-			1,000	2,000	0·92	2,700
Annealed Electrolytic Iron	11,500		—		11,200	9,600	—	1,440
Electrolytic Iron melted	25,800			—	23,600	14,000	0·20	660
in vacuo	41,500			1,700	27,000	14,600	0·17	500
Electrolytic Iron melted *in vacuo* with addition of 0·15 per cent. Si	50,000				27,000	14,500	0·09	290
Armco Iron	7,000	250	260	320	1,000	1,300	0·72	2,106
Armco Iron annealed at 1,480 C. in presence of *H*	190,000	6,000	50,000	110,000			0·025	190

A very pure form of iron is that which is prepared from iron carbonyl, $Fe(CO)_6$. The iron powder so obtained is formed into sheets by sintering and rolling. The following characteristic data were obtained after annealing in the presence of hydrogen.

Initial permeability	μ_0	2,000 to 3,000
μ for H 0·005		3,500 to 4,000
Max. permeability	μ_{max}	$\simeq$ 15,000
Coercive force	H_c	0·21
Remanence	B_r	5,550

The following table shows the effect of different heat treatments on the magnetic quality of electrolytic iron. The results refer to the test conditions that the maximum value of the magnetising force was $H_{max.} = 150$.

Heat Treatment	*Coercive Force H_c Oersted*	*Remanence B_r Gauss*	*Maximum Permeability.* $\mu_{max.}$
Before any heat treatment	2·83	11,450	1,850
After heating *in vacuo* for 24 hours at 800 C. and slowly cooling	0·375	10,850	14,400
After 5th heating to 920 C. and rapidly cooling	0·225	5,000	11,600
After 13th heating at 830 C. and rapidly cooling	0·155	850	4,800

For very weak fields, e.g. for soft iron in field strengths up to $H = 0{\cdot}4$ oersted, the relationship between B and H has been shown to be defined by the expression

$$B = 183H + 1{,}382H^2 \qquad . \quad . \quad . \quad (20)$$

so that the permeability is

$$\mu = \frac{B}{H} = 183 + 1{,}382H$$

or

$$\mu = \mu_0(1 + a.H)$$

where μ_0 is the initial permeability. This expression represents a straight-line function between the permeability and the magnetising force and is usually termed the Rayleigh formula. In Fig. 7 is shown the value of μ as a function of H for Armco iron, that is, a very pure form of commercial iron produced by the American Rolling Company.

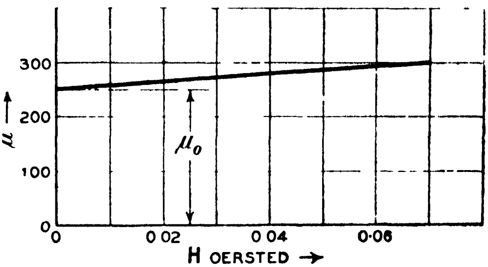


Fig. 7.

As the result of a large number of tests, Steinmetz found that the hysteresis loss per cycle could be represented by the expression

$$U_h = \eta . B^{\alpha}_{max} \text{ ergs per c.c. per cycle} \qquad . \quad . \quad . \quad (21)$$

in which B_{max} is the induction corresponding to the tip of the hysteresis loop and η is the "hysteresis constant" or "Steinmetz coefficient", the value of which depends upon the magnetic quality of the material concerned. This relationship holds over a wide range of values of the induction B_{max}, and in the case of iron it has been found that α varies from 1·5 to 2, according to the chemical composition of the iron, whilst it is larger for high values of the induction density than for low values. Thus, for inductions B_{max} between 2,000 and 7,000, $\alpha = 1{\cdot}6$, and for higher values of the induction α increases so that for $B_{max} = 15{,}000$, α has the value from 2·6 to 3·2. The magnitude of the factor η is given in the following table for a number of the more important magnetic materials.

Material	η
Pure Iron . . .	0·003
Mild Steel Castings	0·003 to 0·009
Cast Iron . . .	0·013
Hard Cast Steel . .	0·025 to 0·028
Nickel	0·013 to 0·040
Hard Tungsten Steel	0·058

Fig. 8 (*a* *d*) shows a group of four typical null magnetisation curves and the associated hysteresis half-loops. Each member of this group represents a set of different characteristic properties of practical importance and may be summarised as follows:

In Fig. 8 (*a*) the null curve rises rapidly for small values of the magnetising force and then bends back to give an almost rectangular form. That is to say, the curve denotes high permeability whilst saturation is obtained with a relatively low value of the magnetising force. The rising and the falling branches of the hysteresis loop lie very close together, the remanence is high, the coercive force is small, and the hysteresis

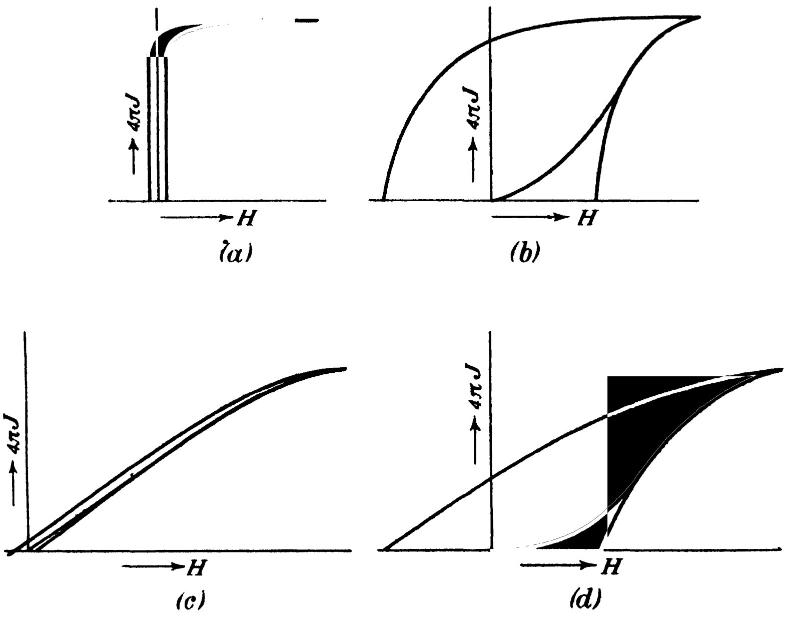


Fig. 8.

loss is small. Such a characteristic diagram is representative of "permalloy" type of magnetically soft materials as required for electromagnetic machines and transformers.

Fig. 8 (*b*) shows a hysteresis half loop with the same saturation value as shown in Fig. 8 (*a*) and the same high remanence value, but the coercive force is now very large and the hysteresis loss is also very large. Further the null magnetisation curve is very flat and consequently the permeability is low. This type of magnetic characteristic curves may be termed the "permanent magnet" type.

Fig. 8 (*c*) gives a hystersis loop for which low coercive force and low

remanence are the characteristic features. The hysteresis loss is small as in Fig. 8 (*a*), but the whole loop is displaced in the sense that saturation is only obtained with a high magnetising force and consequently the permeability is low.

Fig. 8 (*d*) characterises a magnetic material of low remanence and high coercive force: the hysteresis loss is large and the null magnetisation curve is very flat especially in the range of low intensity of magnetisation.

Further references to members of this group of magnetic characteristics will be made in what follows.

Effect of Temperature on Magnetic Quality

The general nature of the change of the magnetic characteristics associated with a change of temperature is similar for all the ferro-

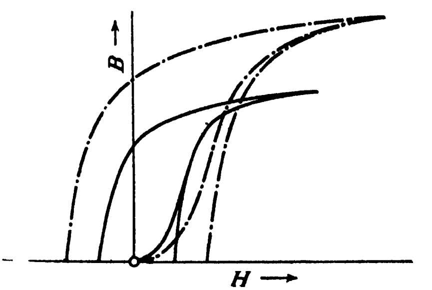


Fig. 9.

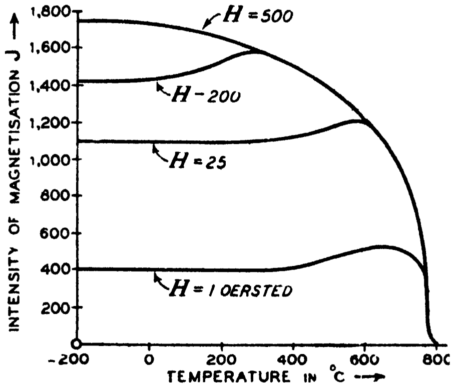


Fig. 10.

magnetic metals and their alloys. There are, however, many complicated differences in detail which are difficult to fit into a general comprehensive survey owing to the fact that changes of temperature produce changes of molecular and atomic structure, the effects of which are superimposed on the effects of temperature common to all the ferromagnetic substances.

The general nature of the effects of temperature may be seen by reference to Fig. 9, which shows the null magnetisation curves and the hysteresis half-loops for the same substance at two different temperatures. The larger loop refers to the room temperature and the smaller to a very much higher temperature, and the following characteristic features may be distinguished:

(*i*) For High Values of the Magnetising Force *H*, the saturation value of magnetisation, the permeability, the coercive force, the remanence, and the area of the hysteresis loop decrease as the temperature increases. If these individual quantities are plotted as functions of the temperature, a group of curves will be obtained which, when referred

to the absolute zero or to the room temperature as a datum, become at first gradually less and then decrease rapidly until at a definite temperature, known as the "Curie point" or the "magnetic change point", they fall to zero.

(*ii*) FOR SMALL VALUES OF THE MAGNETISING FORCE H quite different results are obtained. Reference to Fig. 9 shows that corresponding to the reduction of the coercive force and the size of the hysteresis loop, the null magnetisation curve for the higher temperature is much steeper than for the lower temperature, and consequently the magnetic quantities which appertain to this steeper region of the magnetisation curve, that is, for low values of the magnetising force, viz. the initial permeability μ_0, and the maximum permeability μ_{max}, increases as the temperature increases and then fall rapidly to zero at the Curie point.

In between these two extreme conditions as defined in (*i*) and (*ii*) there is a range of medium values of the magnetising force for which

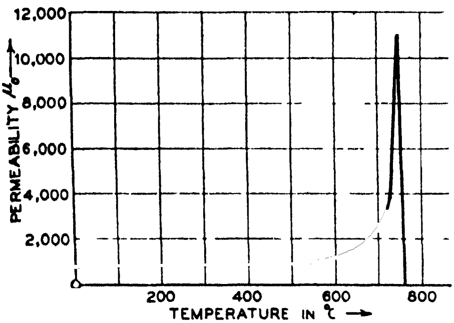


Fig. 11.

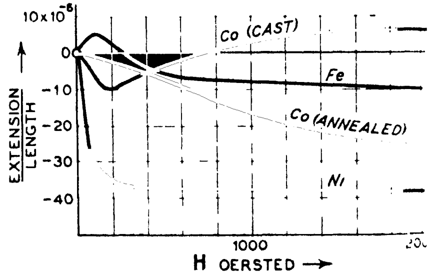


Fig 12.

the magnetic qualities remain constant over a wide range of temperatures up to values near the Curie point, after which they fall rapidly to zero.

These general effects are shown in Fig. 10, which refers to tests on large single crystals of iron. In Fig. 11 is shown the relationship of the initial permeability μ_0 and the temperature for soft iron.

At temperatures above the Curie point, that is to say for ferromagnetic substances when in the paramagnetic state, the Curie-Weiss law is followed, viz.

$$\chi(T + 273\ ^\circ C) = \text{a constant} \quad . \quad . \quad . \quad (22)$$

that is, the susceptibility χ is inversely proportional to the absolute temperature.

Effect of Elastic Stress on Magnetic Quality

It is now known that a very intimate relationship exists between the magnetic characteristics of a substance and the mechanical stress

to which it is subjected. The development of the "stress theory" has proved to be a notable advance in clarifying the complicated phenomena associated with magnetisation. Briefly, it may be said that in every polycrystalline substance, intercrystalline stresses exist due to impurities or due to the change of the crystal dimensions in accordance with the "magneto-striction effect". When a ferromagnetic substance is subjected to a magnetising field, energy has to be expended in overcoming these internal stresses, and there is reason to believe that if a substance were completely free from internal stress the magnetisation cycle would be of the type illustrated in Fig. 8 (*a*) and the associated coercive force would be vanishingly small. Departure from this ideal form of mag-

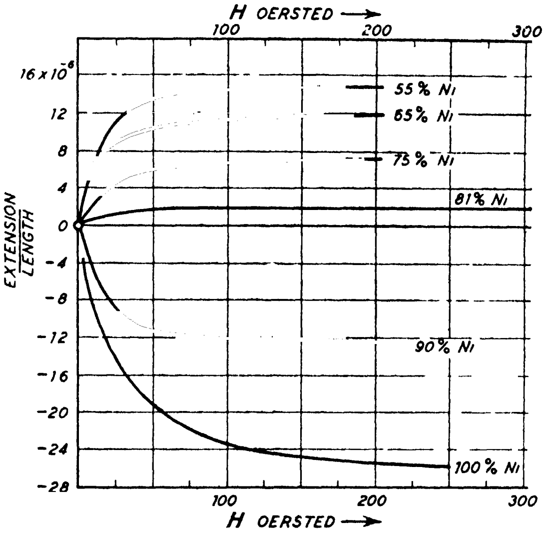


Fig 13

netisation curve is due to the distortion of the crystal lattice, and McKeehan has shown that the very small hysteresis loss associated with the nickel-iron alloy "permalloy" is also characterised by a vanishingly small "magneto-striction" effect.

Matteucci discovered that if a bar of magnetised iron is stretched its magnetisation is thereby increased, and this effect was further investigated by Villari, who found that Matteucci's result only holds if the bar is weakly magnetised. If the bar is strongly magnetised the effect of stretching is to decrease the magnetisation. That is to say, there is a certain critical value for the magnetising force such that stretching the bar has no effect on its magnetisation. If the magnetising force is *below* this critical value, *stretching* the bar *increases* the magnet-

isation, whereas if the magnetising force is *above* this critical value, *stretching* the bar *decreases* the magnetisation.

Joule appears to have been the first to observe that a bar of iron changes its length when magnetised, and Shelford Bidwell carried out a large number of exact investigations on this subject. In Fig. 12 are shown the magnitudes of the magneto-striction effect for the ferro-magnetic metals. It will be seen that the change of length invoked is extremely small, yet it is a characteristic magnitude for the individual metals. It is of interest to note that nickel contracts under all values of the magnetising force, whilst in the case of iron the effect changes sign in the neighl ›urhood to from 100 to 300 oersted.

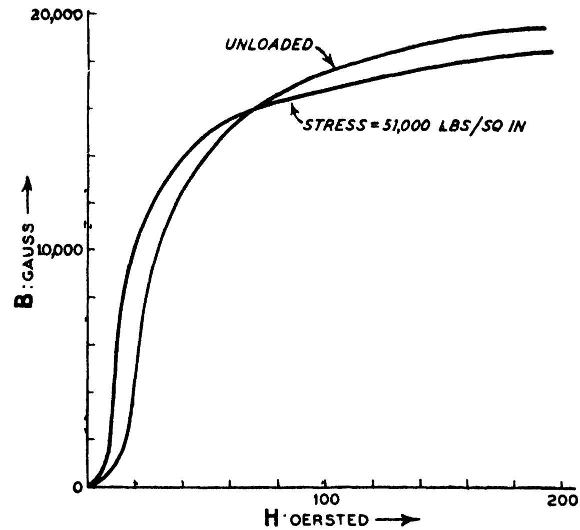


Fig. 14.

As has been mentioned in the foregoing, the nickel-iron alloys are of special importance, and particularly the one having 78·2 per cent. of nickel. In Fig. 13 are shown the magneto-striction effect for a number of such alloys of different percentage of nickel content. It will be seen that for a nickel content greater than about 81 per cent. contraction takes place for all values of the magnetising force, whilst for a percentage of nickel less than about 81, extension takes place for all values of the magnetising force. In the neighbourhood of the permalloy composition, therefore, the magneto-striction effect becomes zero.

The reciprocal relationship between the magneto-striction effect (i.e. the change of length due to magnetisation) and the change of magnetisation due to loading is summarised in the following table for an iron rod

Effect of Magnetisation on the Length	*Effect of Mechanical Stress on the Magnetisation*
A rod of iron becomes increased in length when magnetised in a weak field.	Stretching a weakly magnetised rod of iron increases its magnetisation.
A rod of iron becomes decreased in length when magnetised in a strong field.	Stretching a strongly magnetised bar of iron decreases its magnetisation.

In Fig. 14 is shown the effect of loading a steel rope wire to a stress of 51,000 lb. per square inch, and in Fig. 15 the corresponding effect on the permeability.

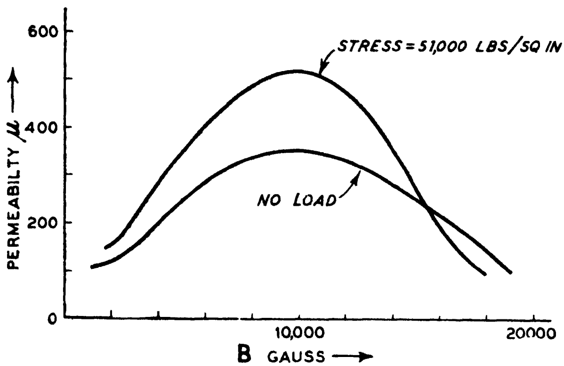


Fig. 15.

Nickel-Iron Alloys

Alloys having special characteristics when magnetised by weak fields are of great importance for purposes of telephony and telegraphy. They are required for the magnet cores of relays and recording apparatus for which only very weak currents are available. Their cardinal importance, however, is for the purpose of increasing the distributed self-induction of cables, and it may be said that some of the problems of long-distance communication technique could only be solved when the alloys had been discovered which possessed the requisite magnetic characteristics. Thus the damping of the travelling electric waves in a cable can be reduced by increasing the distributed self-induction of the cable, as was first shown by Heaviside, and, in consequence, the distortion is reduced and a higher speed of transmission becomes possible. For this purpose two methods have become available (*i*) the use of " Pupin coils ", which consist of inductancs of suitable magnitude connected in series with the cable at regular intervals ; (*ii*) the Krarup method of wrapping round the cable core throughout its length a wire or band of suitable magnetic

material, thus providing an additional distributed inductance to the cable (see Fig. 16).

The necessary magnetic characteristics of material suitable for these methods of "loading" a cable are:

(*i*) The highest possible initial permeability.
(*ii*) Small hysteresis loss and high specific electric resistance.
(*iii*) The permeability must be as constant as possible over a range of magnetising forces from 0 to 0·1 oersted (see, for example, Fig. 8 (*c*).
(*iv*) The maximum possible stability of the magnetic characteristics, that is to say, the permeability and the hysteresis loss must be as far as possible independent of externally superimposed magnetic fields, whether due to direct current or alternating current.

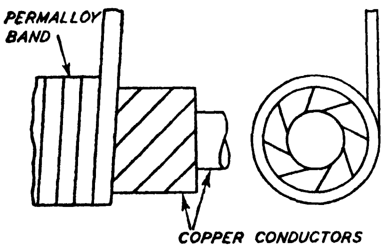


Fig. 16.

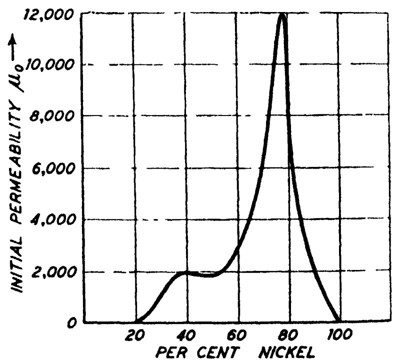


Fig. 17.

The special magnetic qualities of nickel-iron alloys and, in particular, the high permeability obtainable, were first pointed out by Panebianco in 1910, but no further progress was made until 1923, when Arnold and Elmen, as the result of exhaustive investigations with a view to the production of suitable material for the Krarup winding, discovered the remarkable characteristics of the range of nickel-iron alloys.

In Fig. 17 is shown the initial permeability, μ_0, of a nickel-iron alloy as a function of the nickel content, and it will be seen that in the neighbourhood of 78 per cent, nickel content the initial permeability reaches values of about 12,000 and nickel-iron alloys of nickel content corresponding to this high peak value of the curve of Fig. 17 are known as "permalloy". Fig. 18 shows the hysteresis loops for armco iron and permalloy respectively, whilst Fig. 19 shows the permeability of these two materials as a function of the induction B. In Fig. 20 the permeability of permalloy is given for very low values of the magnetising force, and

the sharp curve shown by this diagram is a disadvantageous feature of permalloy as regards its applications for certain purposes, as pointed out in what follows.

For practical purposes two main types of nickel-iron alloys come chiefly into consideration, viz. permalloy with about 78 per cent. nickel content, and the alloy with about 50 per cent. nickel content and variously known by the trade names of Invariat, Hipernik, and Copernik. Permalloy is characterised by its extreme values of high permeability and low hysteresis loss; thus, permalloy C, which is the softest mag-

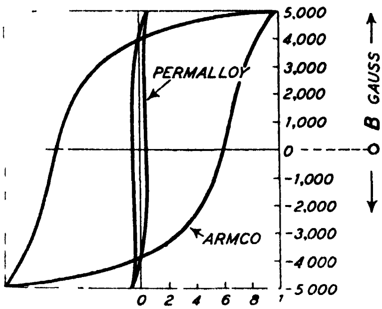


Fig. 18.

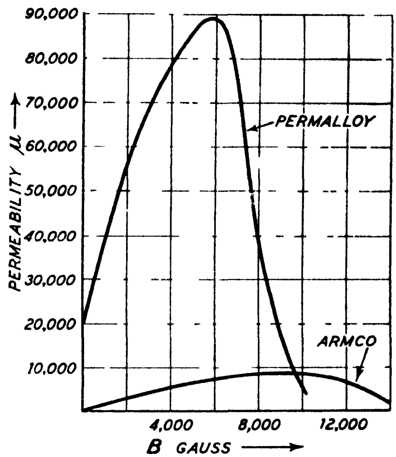


Fig. 19.

netic material, is used for telephones and other similar apparatus and has the following leading characteristics:

Initial permeability	μ_0	6,000
Maximum permeability (i.e. for H 0.036 oersted)	μ_{max}	100,000
Saturation induction density	B_{sat}	9,000 gauss
Coercive force	H_c	0.035 oersted
Hysteresis loss for B_{max} 5,000 gauss . . .		50 ergs per c.c. per cycle
Specific electrical resistance . . .		21 microhms per cm. cube

The advantages of the 50 per cent. nickel alloys are that, notwithstanding the somewhat lower values of μ_0 and μ_{max}, the permeability shows great constancy for low values of the magnetising force and, what is of particular importance, the saturation value B_{sat} is appreciably higher, so that for a given operating condition, a somewhat smaller transformer core is possible. This 50 per cent. nickel alloy has the further advantages that it is less sensitive to heat treatment and to mechanical handling. The leading characteristics are:

Initial permeability . . .	$\mu_0 = 3{,}000$
Maximum permeability . .	$\mu_{max.} = 70{,}000$
Saturation induction density .	$B_{sat.} = 15{,}500$ gauss
Hysteresis loss for $B_{max} = 10{,}000$	. 220 ergs per c.c. per cycle
Remanence	$B_r = 7{,}300$ gauss
Coercive force	$H_c = 0{\cdot}05$ oersted
Specific electrical resistance .	. 46 microhms per cm. cube

A purely binary alloy of nickel and iron, however, is comparatively little used in practice. Usually, some additional element is included which in each case gives some special characteristic feature to the alloy. Such additional elements give either a somewhat more easily worked

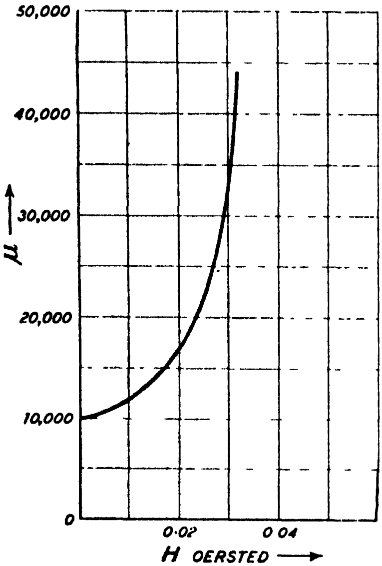


Fig. 20.

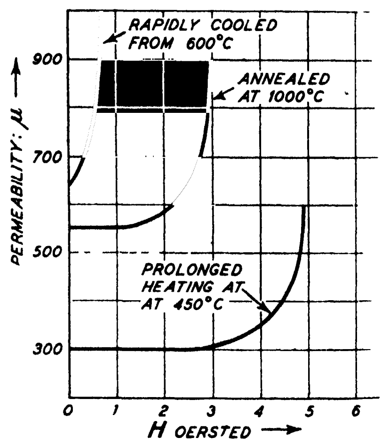


Fig. 21.

material or give an increased specific electrical resistance, or give a greater constancy of the permeability. Among such additional elements which are appropriate for one or other of these purposes are copper, manganese, chromium, vanadium, silicon, tungsten, cobalt, aluminium, and silver. The so-called "Mumetal" for example, has the following composition: 74 per cent. Ni : 20 per cent. Fe : 5 per cent. Cu : 1 per cent. Mn. The characteristic features of this material are stated to be that after cooling from 600° C. the initial permeability is 800 : after cooling from 700° C. the value of μ_0 is 2,600 : and after cooling from 900° C. the initial permeability is $\mu_0 = 7{,}000$, with corresponding change of the constancy of the permeability. The specific electrical resistance is

$0{\cdot}25\Omega\,m/mm^2$. A similar material with 70 per cent. Ni : 15 per cent. Fe : 15 per cent. Cu : shows the following of the permeability

$H = 0{\cdot}001$ oersteds	$\mu = 6{,}300$
$H = 0{\cdot}01$,, ...	$\mu = 6{,}800$
$H = 0{\cdot}5$,, ...	$\mu = 7{,}400$

However important it may be to obtain the highest possible value of the permeability for weak magnetising fields, yet for many purposes of telephony and telegraphy technique the foregoing nickel-iron alloys are not suitable because, associated with the high values of the permeability, is the fact that the permeability varies with the strength of the magnetising field, that is to say, the permeability curve such as in Fig. 20 shows a sharp bend. For certain purposes, it is essential that the permeability shall remain as constant as possible when the magnetising force varies. If, for example, the permeability varies with the magnetising force, telephony by means of cables over long distances would become practically impossible owing to the severe distortion which would take place in the wave-form of the speech. A typical magnetic curve which shows constant permeability over a wide range of magnetising forces is shown in Fig. 8 (*c*). One method for solving this problem is the choice of a suitable composition for the alloy, and a successful result is obtained with the tertiary system, nickel-iron-cobalt, the so-called " Perminvar ". In Fig. 21 are given curves which show how successful this material is for the purpose in view.

Magnetic Material for Pupin Coils

The purpose of Pupin coils has been referred to already on page 191, and the most suitable constructional methods for these coils has been the subject of prolonged investigations. The earliest type of the magnet core was simply built up from transformer sheet material or from tape, or from wire of diameter usually from 0·1 to 0·15 mm. Instead of using magnetically soft material a change-over was soon made to hard steel wire, and although this wire has a relatively small value for the initial permeability it has the advantage that there is only a relatively small increase of permeability with increasing field strength. The next stage in the development was to subdivide the core by one or more air spaces, and in this way a very small gradient was obtained for the magnetisation curve, so that the permeability varied very little with the field strength of externally impressed fields.

A notable improvement of the core characteristics was obtained by means of the " cross-wire " principle of building up the core, the magnetic wire of the core being arranged in a direction at right angles to the magnetic field in the core. Eventually, the modern construction of the Pupin coil was arrived at in which use is made of the so-called powder or dust core magnetic material. The powdered magnetic material is intimately mixed with insulating material (which also acts as a binding

medium) and the whole is subjected to high compression. By this means the eddy current loss becomes extremely small, although naturally the absolute value of the permeability is very small in comparison with that of the magnetic material when in the solid form. In the following table is given a comparison of the magnetic properties of different types of construction of Pupin coils.

Magnetising Field Strength H Oersted	*Wire wound so as to form a Ring Core*				*Powder Core*				*"Cross-Wire" Ring Core*	
	Soft Iron Wire		*Hard Steel Wire*		*Mechanically Powdered Material*		*Electrolytic Iron*		*Hard Steel Wire*	
	B	μ	B	μ	B	μ	B	μ	B	μ
0·5	57	125	39	78	26	52	20	40	—	
1·0	160	160	83	83	53	53	41	41	11	11·4
3·0	1,170	390	300	102	165	55	129	43	34	11·5
5·0	3,200	639	665	133	295	59	225	45	58	11·6
10·0	6,240	624	3,630	363	750	75	510	51	118	11·8
15·0	7,140	483	6,370	425	1,500	100	840	56	180	12·0
20·0	7,900	395	7,700	385	2,280	114	1,200	60	241	12·0
40·0	8,800	220	9,520	238	4,640	116	2,720	65	470	11·8
60·0	9,300	155	10,260	171	6,180	103	3,780	63	695	11·6
80·0	9,600	120	-		7,200	90	4,880	61	—	
0·0	5,630		6,900		2,270		1,130	—	44	
1·0	5,070		6,760		2,120		1,050		32	
3·0	2,600		6,310		1,770		880		10	
5·0	3,100		5,680		1,400		720		— 28	
10·0	6,200		990		360		240		102	
- 15·0	7,200		5,970		610		480	-	- 165	

This table defines the null magnetisation curve and the demagnetising curve in each case and it will be seen that the core which is built up of wire wound in ring form has a relatively high permeability, but the variation of permeability with the magnetising field strength is much too large for the purpose in view. The "cross-wire" core shows small permeability but great constancy of permeability with varying field strength, whilst the powder core shows relatively high permeability and great constancy of the permeability for a wide range of values of the magnetising field strength.

Materials for Permanent Magnets

The essential magnetic characteristics for the materials of which permanent magnets are to be made are that they shall have the highest possible value of the remanence induction B_r and the highest possible value of the coercive force H_c. In accordance with the "stress theory"

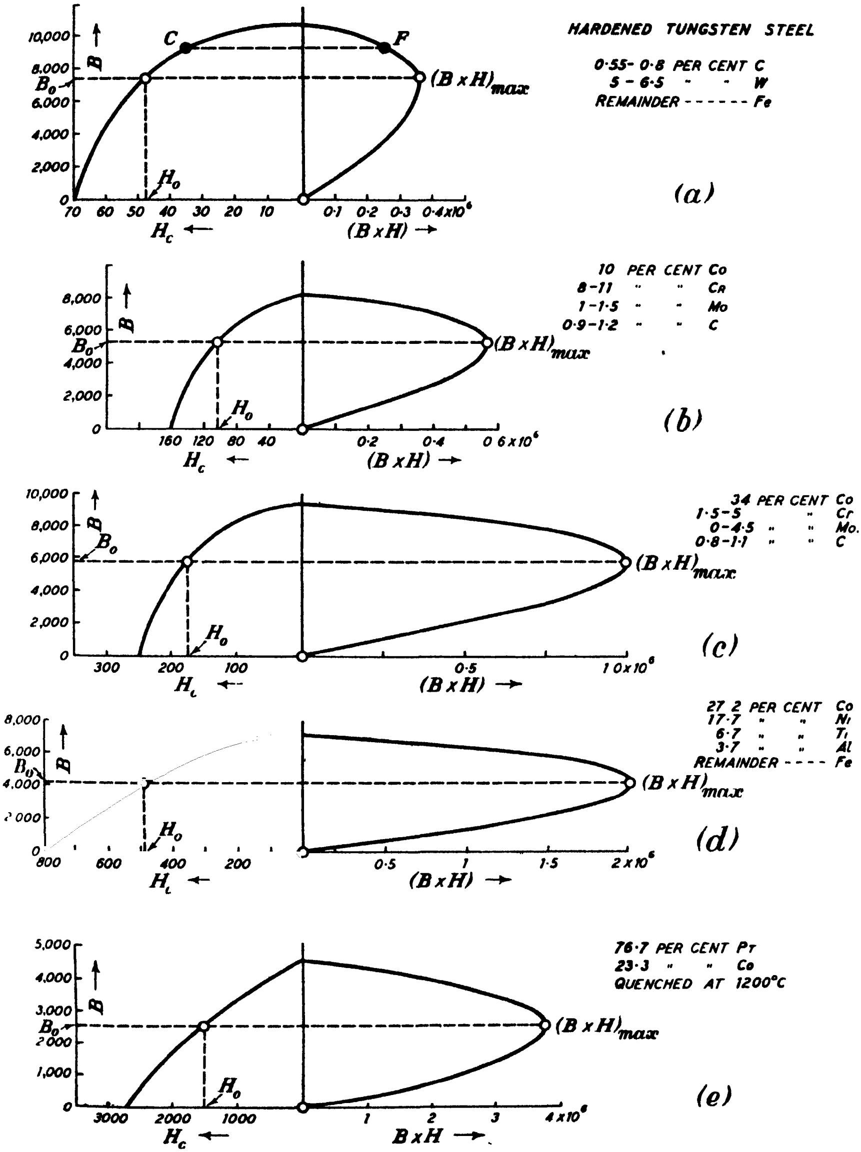

HARDENED TUNGSTEN STEEL
0·55- 0·8 PER CENT C
5 - 6·5 " " W
REMAINDER ------ Fe
(a)
10 PER CENT Co
8-11 " " Cr
1-1·5 " " Mo
0·9-1·2 " " C
(b)
34 PER CENT Co
1·5-5 " " Cr
0-4·5 " " Mo.
0·8-1·1 " " C
(c)
27 2 PER CENT Co
17·7 " " Ni
6·7 " " Ti
3·7 " " Al
REMAINDER ---- Fe
(d)
76·7 PER CENT Pt
23·3 " " Co
QUENCHED AT 1200°C
(e)
C
F
B
B_0
H_0
H_c
(B x H)
$(B \times H)_{max}$
B x H

Fig. 22.

of magnetisation, which was referred to on page 189, these requirements imply the greatest possible heterogeneity of the structure of the material and the consequent greatest possible internal stresses. The typical characteristic for a permanent magnet material is shown in Fig. 8 (*b*) and the portion of the demagnetisation curve which lies in the quadrant to the left of the positive ordinate axis defines the working range for such materials. The measurements necessary to obtain the demagnetisation curve can be made by means of the " bar and yoke " method which is described on page 202.

In Figs. 22 (*a* *e*) are shown the demagnetising curves for a number of permanent magnet materials. If for any point such as C (Fig. 22 (*a*)) the product $B \times H$ is found and then plotted (point F) in the quadrant to the right of the positive ordinate axis, a curve will be obtained which reaches a maximum value as indicated by $(B \times H)_{max.}$ the corresponding component values being denoted by B_0 and H_0, respectively. It can be shown that this quantity $(B \times H)_{max.}$ is a measure of the quality of the material for use for manufacturing permanent magnets. Further, the quantity $\frac{1}{8\pi}(BH)_{max.}$ gives the amount of energy in ergs per cubic centimetre of the material which a magnet of suitable form can maintain in a space outside the magnet.

In Fig. 23 are shown the values of $(B \times H)_{max}$ as a function of the maximum magnetising force H_{max} for iron-cobalt-nickel steel; 34 per cent. cobalt-steel; 10 per cent. cobalt-steel; hardened spring-steel; and a platinum-cobalt alloy.

Magnets from 2 to 2·5 cm. thick can be made fairly homogeneous throughout the section, but for greater thicknesses the magnet quality rapidly deteriorates on account of the non-uniformity of the hardening process, and consequently the values of the coercive force and the remanent induction fall to a serious extent from the characteristic values of the particular material which is used.

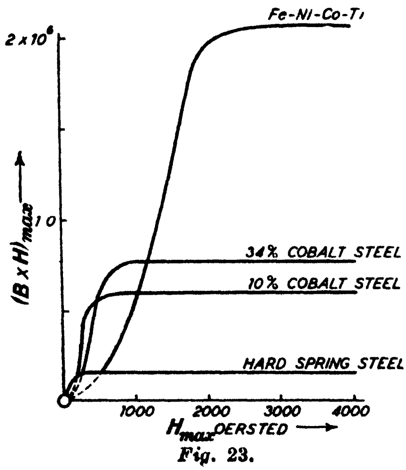


Fig. 23.

As an example of the practical significance of a high value for the quantity $(B \times H)_{max.}$ Fig. 24 shows magnetos made from tungsten steel and from 34 per cent. cobalt-steel respectively, from which it will be seen that although cobalt-steel is much more expensive than tungsten steel, the smaller dimensions of the cobalt-steel magnet enables it to compete successfully with the tungsten-steel type.

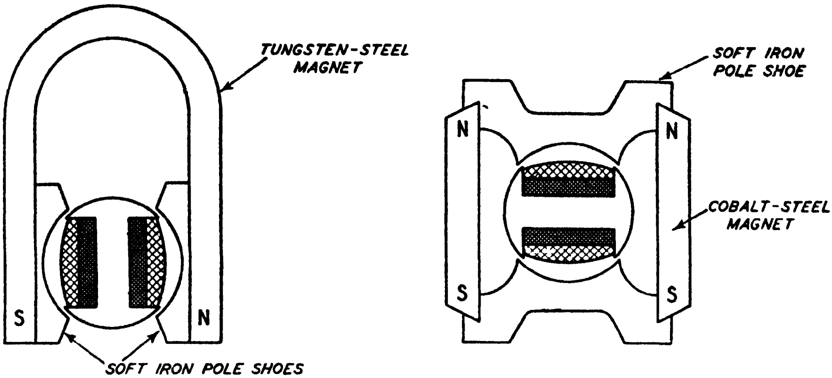


Fig. 24.

Magnetic Testing

In what follows, three of the more generally useful methods of magnetic testing will be briefly described.

(1) THE SAMPLE IS IN THE FORM OF A CLOSED RING.—This method is normally suitable for tests up to a maximum value of the magnetising force of H_{max} from about 250 to 350 oersted. One of the chief advantages of this method is that no magnetic measurements are necessary to determine the value of the magnetising force. Further, since there are no free magnetic poles produced when the ring is magnetised there will be no corrections necessary for the so-called "self-demagnetising effect".

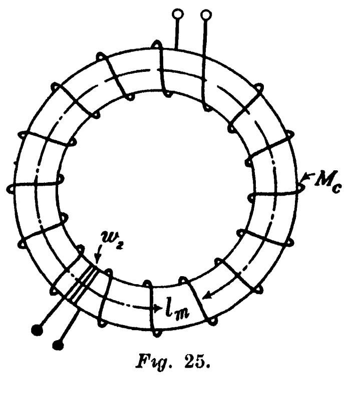


Fig. 25.

Fig. 25 shows diagrammatically the ring sample wound with a uniformly distributed magnetising coil M_c of w_1 turns and a concentrated search coil of w_2 turns. The length of the mean magnetic path in the ring is l_m cm., as shown in the diagram. If a current of I_1 amperes flows in the magnetising coil, then the m.m.f. round the mean path is related to the magnetising force H along the path by the equation

$$\frac{4\pi}{10} I_1 w_1 = H . l_m$$

so that

$$H = 1{\cdot}257 \frac{w_1}{l_m} I_1 \text{ oersted} \quad . \quad . \quad . \quad . \quad (23)$$

that is

$H = 1{\cdot}257$ **(ampere-turns per centimetre length of the mean magnetic path).**

Fig. 26 shows diagrammatically the circuit connections for obtaining (*i*) the null magnetisation curve, (*ii*) the hysteresis loop, (*iii*) the calibration of the ballistic galvanometer (see page 206).

TEST (*i*). *The Null Magnetisation Curve.*—For this test the switch Q is kept closed so that the resistance R_4 is short-circuited, and the switch S_1 is placed on contact n. By adjusting the resistances R_1 and R_2 the current in the magnetising coil M_c of the test sample is set by means of the ammeter A_M to the necessary value, so as to produce the required maximum value H_{max} of the magnetising force as calculated from equation (23). With the ballistic galvanometer short-circuited by the key K the current in the magnetising coil is reversed several times by

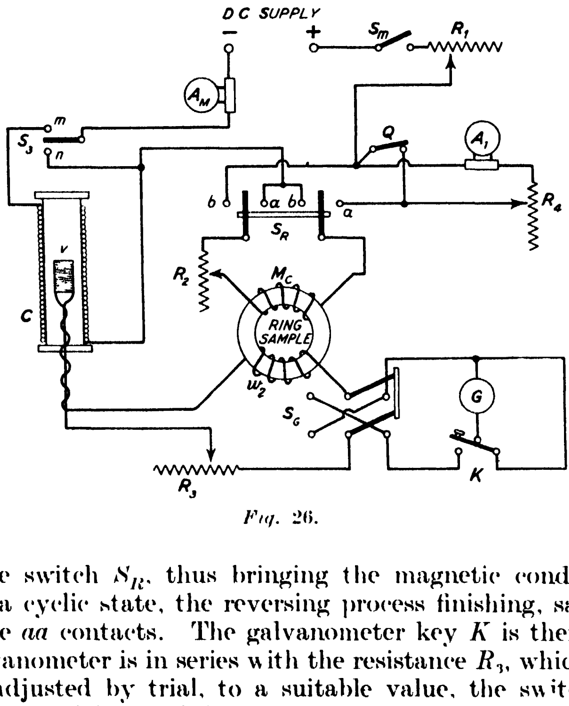


Fig. 26.

means of the switch S_R, thus bringing the magnetic condition of the sample into a cyclic state, the reversing process finishing, say, with the switch on the *aa* contacts. The galvanometer key K is then pressed so that the galvanometer is in series with the resistance R_3, which has previously been adjusted by trial, to a suitable value, the switch S_G being closed, say, on the right-hand side contacts. The switch S_R is then quickly moved over to the contacts *bb* and the galvanometer deflection noted After the galvanometer has come to rest, the switch S_R is moved quickly back to the contacts *aa* and the galvanometer deflection again noted The galvanometer switch S_G is then moved over to the left-hand side contacts and two further readings of the galvanometer deflection taken by repeating the two respective reversal operations of the switch S_R so that in this way *four* galvanometer readings are obtained for the *one*

value of the current in the magnetising coil M_c, the mean value θ of these four readings being then taken as the true deflection corresponding to that particular value of the magnetising current.

The magnetising current is now reduced by a suitable amount and, after reversing the current several times in order to bring the magnetic condition of the sample again into the cyclic state, the mean value of the four readings of the galvanometer deflection which corresponds to this new value of the magnetising current is taken, and the procedure is then repeated with successively smaller values of the magnetising current. The method by which the galvanometer deflection θ is converted into the corresponding value of the induction B is explained on page 207 : that is to say, Test (*iii*), is carried out by means of the air-core solenoid C and the search coil c shown in Fig. 26.

Test (*ii*). *The Hysteresis Loop.* For this test, the resistances R_1 and R_2 are adjusted with the switch Q closed so that the magnetising current I_{max} corresponds to the required value of H_{max} for the hysteresis loop, and after reversing the switch S_R several times in order to bring the condition of the sample into the cyclic state, the value of B_{max} is obtained, as has been explained already in the foregoing description of Test (*i*), and the corresponding values of H_{max} and B_{max} will then fix the position of the tip of the hysteresis loop which is shown in Fig. 27 by the point A.

With the setting of the resistances R_1 and R_2 maintained throughout the test, the switch S_R is placed on the contacts *aa* and the short-circuiting switch Q is opened. The resistance R_4 is then adjusted so that the current shown by the ammeter A_1 corresponds to some suitable value H_1, see Fig. 27, of the magnetising force. The switch Q is now closed and, after reversing S_R several times to ensure that the magnetic condition of the sample is again brought into a cyclic state, the reversing process is finished with the switch left on the contacts *aa*. The galvanometer key K is now pressed and the switch Q quickly opened, thus reducing the current in the magnetising coil by the predetermined amount to correspond to the magnetising force H_1. From the deflection of the galvanometer due to the opening of the switch, the resultant *reduction* of the induction, $B_{max} - B_1$, is found in accordance with the procedure explained in the foregoing description of Test (*i*), so that the new value of the induction B_1 and the new value of the magnetising force H_1 together define the point P on the descending branch of the hysteresis loop as shown in Fig. 27. By repeating the procedure with a new adjustment of the resistance R_4 so as to give a smaller value of the magnetising force, other points on this branch of the hysteresis loop are obtained, the point V being given when the current in R_4 is indefinitely large, that is to say, the point V is obtained by simply moving the reversing switch from the contacts *aa* to the off position.

For points on the portion VG of the descending branch of the loop

the general procedure, as already explained in the foregoing, is followed, with the exception that the reversing process which is carried out with the switch Q closed is now to be finished with the switch left on the *bb* contacts and, having previously adjusted the resistance R_4 to a suitable value, when, after finishing the reversing process, the switch S_R is on the *bb* contacts, the switch Q is opened and the switch S_R moved quickly over to the *aa* contacts, so that in effect the magnetising current in the winding M_c is changed from the full value I_{max} in one direction to something less than the full value in the opposite direction, and in this way, points on the descending branch of the loop, such as D, are obtained. From the branch AVG, the opposite branch GV_1A is then easily drawn by plotting, for example, $Pp = P_1p_1$ and so on, for other points such as V_1, D_1, F_1.

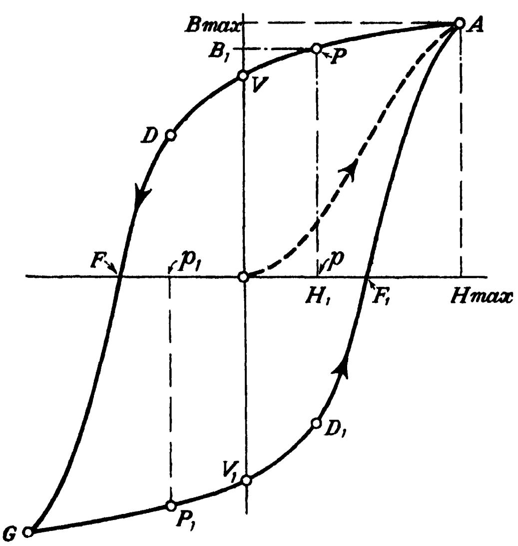


Fig. 27.

(2) The Bar and Yoke Method.—In many practical cases it is not convenient to prepare, in the form of a closed ring, a sample of the material which it is desired to test, and more frequently the sample is available in the form of a straight bar or rod. A convenient feature of this method of testing is that the magnetising coil and the search coil are comprised by the yoke system, and it is consequently unnecessary to specially wind both magnetising coil and search coil for each new sample, as is the case when the sample is in the form of a closed ring.

The sample may conveniently be about 30 cm. long, and in the form of a rod, having a diameter of about 0·6 cm.: in the case of a bar a section of about 0·5 × 0·5 sq. cm. is suitable. The sample is arranged in combination with a special magnetic yoke system to form a closed magnetic circuit as shown in Fig. 28, the sample being firmly clamped between the two halves of the yoke. In this way the self-demagnetising effect of the ends of the sample is eliminated. It is essential that the yoke shall be of high permeability, of a high specific resistance, and small magnetic viscosity. A suitable material is soft iron alloyed with from 3 to 4 per cent. silicon, or better still, a 50 per cent. nickel-iron

alloy. The cross-section of the yoke must be sufficiently large, and for a sample of the cross-section given in the foregoing, the section of the yoke should be about 2 × 50 sq. cm. Care must be taken to ensure that good magnetic contact shall be maintained between the yoke and sample throughout the end portions which are calmped between the two yoke halves.

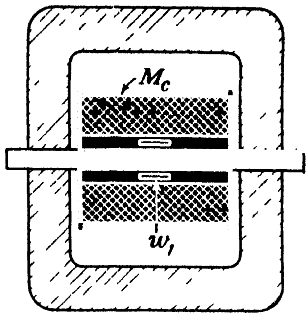


Fig. 28.

The magnetic testing circuit is the same as that shown in Fig. 26, and the general procedure for carrying out the test is identical with that described for the ring sample of the Method (1), the magnetising coil M_c and the search coil w_1 of Fig 28 corresponding respectively to the coils M_c and w_2 of Fig. 26.

(3) Method of Measuring the Magnetic Potential between Two Points on the Surface of the Test Sample.

The methods of magnetic testing which have been considered in the foregoing pages give the values of the induction B by direct measurement. The "magnetic potential" method, however, provides a means for the direct measurement of the magnetic intensity H. In many cases, H can be calculated from the m.m.f. of the magnetising coil as has been explained already in the foregoing, viz. :

$$H = \frac{4\pi}{10}\frac{IW}{l} = \frac{4\pi}{10} iw \text{ oersted} \quad . \quad . \quad . \quad (24)$$

where IW is the total number of ampere-turns in the magnetising coil and iw is the number of ampere turns per cm. length of the magnetic circuit. This simple calculation gives satisfactory results when the magnetic circuit is of uniform cross-section and of the same material throughout its path as for example, is the case when a ring sample of uniform section is being tested. Such a calculation fails, however, when the path of the magnetic circuit comprises a sudden change of permeability as for example, when it contains an air-gap or when there is a large diminution of section in some part of the path. In such cases, the magnetic intensity H will be different at different parts of the magnetic circuit because free magnetic poles will be developed, the effect of which would not be included in the calculation. In such cases therefore, it is desirable to be able to determine the magnetic intensity by direct measurement and this can be done by means of the "magnetic potential meter".

This appliance comprises a long flexible coil of wire the cross-section of this coil being constant throughout its length. The coil section may be rectangular or circular, as may be found to be convenient, but in any case, the section must be small in comparison with the non-uniformi-

ties of the magnetic field under test. The winding of the coil is so arranged that the two leads which are to be connected to the ballistic galvanometer, emerge from the middle of the coil. Such a coil may then be used for measuring directly, the magnetic potential between any two points of a magnetic circuit. That is to say, for the measurement of the m.m.f. between any two such points (see Fig. 29), by suddenly withdrawing the coil out of the field and observing the consequent "throw" of the ballistic galvanometer.

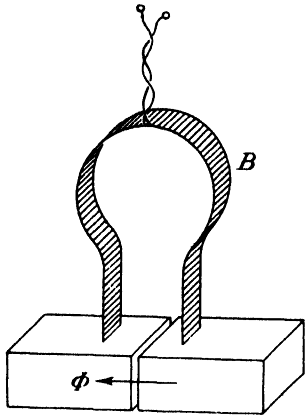


Fig. 29

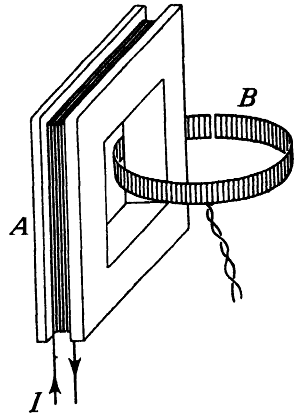


Fig. 30.

In order to obtain the calibration constant of this appliance, it is necessary to make use of a coil A (Fig. 30) of a known number of turns and which is carrying a known current. The magnetic potential meter flexible coil B (Fig. 30) is then arranged to link with the calibrating coil A so that the two ends of the flexible coil frame come together as shown in Fig. 30. The known current in the coil A is then reversed in direction and the consequent ballistic throw Θ of the galvanometer is noted. If the calibrating coil A has w turns and the magnitude of the current which is reversed is I amperes, then the calibration constant of the equipment is,

$$\text{m.m.f.} = \frac{0{\cdot}4\pi I w}{\Theta} \begin{Bmatrix} \text{per } 1^\circ \text{ deflection of} \\ \text{the galvanometer} \end{Bmatrix} \quad . \quad . \quad (25)$$

If then, the equipment is used to measure the value of H for any given sample, and the galvanometer deflection so obtained, is θ, then the measured value of the magnetic potential will be

$$\text{m.m.f.} = \frac{\theta}{\Theta}(0{\cdot}4\pi w I) \text{ ergs (i.e. oersted-cm.)} \quad . \quad . \quad (26)$$

Procedure for carrying out an actual Test. — Suppose in Fig. 31 a semicircular core of non-magnetic material, such as vulcanised rubber, is uniformly wound from end to end with a known number of turns of insulated wire. If this wound core is now arranged on a bar magnet as shown in Fig. 31, then the total magneto-motive-force round the broken-line path will be zero, since no magnetising ampere-turns are anywhere linked with this path (see also pages 179 and 219). That is to say,

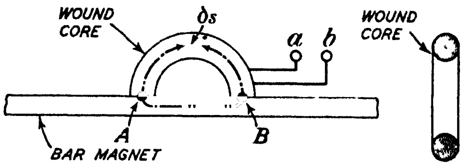


Fig. 31.

$$\int_A^B Hds \text{ along the path in the core}$$

$$= -\int_B^A Hds \text{ along the path in the bar magnet} \quad . \quad . \quad (27)$$

If now the ends ab of the vulcanised rubber core winding of Fig. 31 are connected to a galvanometer, the mean value of H may be measured by the procedure which has been described already in connection with equation (27).

In Fig. 32 is shown diagrammatically a system of connections by means of which this method may be applied to the measurement of the null magnetisation curve and the hysteresis loop of a strip of sheet material. This strip bridges the two arms of a U-shaped yoke, and a magnetising coil C is wound on the transverse limb, as shown in the diagram. The sample is provided with a search coil D of a known number of turns. The current in the exciting coil C may be adjusted by means of the resistance R and the current in this coil can be reversed by means of the switch S. A ballistic galvanometer G is connected to a change-over switch T, so that when the switch T is on the contacts ab the magnetic

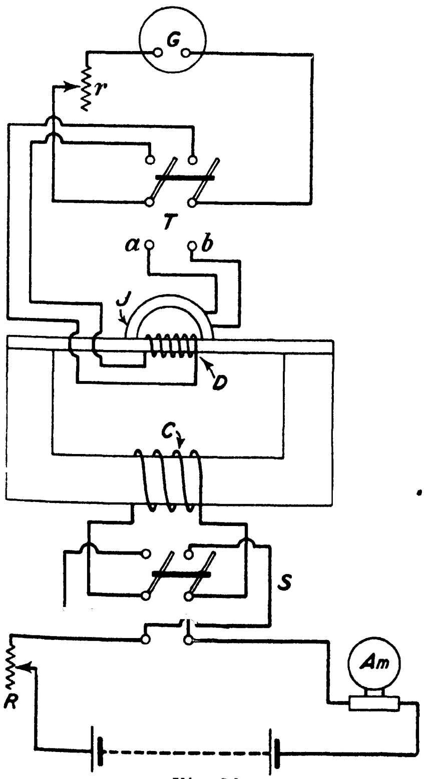


Fig. 32.

potential coil J is connected to the galvanometer and the galvanometer deflection θ, which is obtained when the magnetising current in the coil C is reversed, is then a measure of the magnetising force H. When the switch T is moved over to the upper contacts, the search coil D is connected to the galvanometer and the deflection obtained when the current in coil C is reversed is a measure of the magnetic induction B in the sample. The hysteresis loop may also be determined by means of the same switching procedure as has been described already with reference to Fig. 26.

Calibration of the Ballistic Galvanometer

In Fig. 26 is shown a long straight solenoid C having an air core and uniformly wound with w_1 turns *per centimetre length*, and by means of the switch S_3 this solenoid winding may be placed in series with the d.c. supply and the reversing switch S_R. A search coil v is wound with a total number of w_s turns, the winding being of mean cross-section A sq. cm. and is arranged coaxially with the solenoid and near the central point of the axis. This search coil will be seen to be permanently in series with the search coil w_2 of the test sample.

Now it is known from other considerations (see Chapter VIII, p. 218) that the magnetic intensity in the neighbourhood of the central part of the axis of a long straight solenoid is defined by the equation

$$H = \frac{4\pi}{10}\left\{\begin{matrix}\text{ampere-turns per centimetre}\\ \text{length of the solenoid}\end{matrix}\right\}$$

oersted, that is

$$H = \frac{4\pi}{10} I w_1 \text{ oersted} \qquad . \quad . \quad . \quad . \quad . \quad (28)$$

when the current flowing in the solenoid winding is I amperes.

Suppose that when this current is reversed, the galvanometer deflection is θ, which, when corrected for damping, becomes Θ; the method of making this correction will be explained in what follows. Then it can be shown that (see page 227) the quantity of electricity discharged through the galvanometer is,

$$Q = \frac{2\Phi w_s}{r \,.\, 10^8} = \frac{b}{k}\frac{\tau}{2\pi}\Theta \text{ coulomb} \qquad . \quad . \quad . \quad (29)$$

where r ohms is the total resistance in the galvanometer circuit including the search coils v and w_2, Φ maxwells is the magnetic flux linked with the search coil v, $\frac{b}{k}$ is a constant of the galvanometer, and τ seconds is the time of one complete oscillation of the galvanometer (see also Chapter X, page 341). Now when the solenoid current is I amperes, the corresponding value of the magnetic flux which is linked with the search coil v will be

$$\Phi = H.A = \frac{4\pi}{10} I w_1 A \text{ maxwells} \qquad . \quad . \quad . \quad (30)$$

where A sq. cm. is the cross-section of the search coil v so that from equation (29)

$$\frac{b}{k}\frac{\tau}{2\pi} = G = \frac{2\Phi . w_s}{r . 10^8 . \Theta} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and this gives the ballistic galvanometer constant G.

Hence, when a magnetic test is made on a sample as shown in Fig. 26 and the galvanometer deflection (when corrected for damping) is Θ, then the corresponding value of the induction will be given by means of equation (31), viz.,

$$B = \frac{\Phi}{F} = \left[\frac{r . 10^8}{2w_s F}\right] G\Theta \text{ gauss} \quad . \quad . \quad . \quad (32)$$

where F sq. cm. is the cross-sectional area of the sample. Since all the quantities on the right-hand side of the equation (31) are known, the corresponding value of the magnetic induction B can be found at once.

As regards the relationship between the observed deflection θ_1 of the galvanometer and the corrected deflection (Θ), it is to be observed that if the deflection during a series of successive swings of such a galvanometer is plotted as a function of the time, a damped sine wave will be obtained such as is shown in Fig. 33, in which $\theta_1 : \theta_3 : . . .$ indicate the respective amplitudes of the successive swings in the same direction of the galvanometer scale. It can then be shown that the true deflection corrected for damping is given by

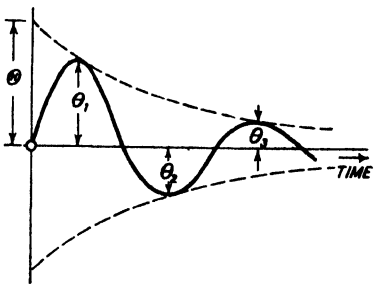


Fig. 33.

$$\Theta = \theta_1\left(1 + \frac{\lambda}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where

$$\lambda = \frac{1}{n - 1} \log_e \frac{\theta_1}{\theta_n} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where n is the number of galvanometer swings as, for example, in Fig. 33, where $n = 3$.

The quantity λ is termed the "logarithmic decrement" (see also Chapter X, page 341).

Chapter VIII

ELECTRO-MAGNETISM

Oersted's Experiment

IN 1819 the Danish physicist Oersted discovered ~~that a wire~~ carrying an electric current is surrounded by a magnetic field due to the current. Suppose in Fig. 1 a wire is held in the magnetic meridian so that it is parallel to, and a little distance above, a compass needle. If an electric current is caused to flow through the wire in the direction from south to north the needle will be deflected so that the N-seeking (i.e. red) pole moves towards the west, whilst if the current in the wire flows from north to south, the red pole of the needle will be deflected

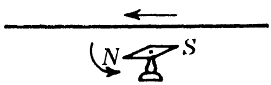


Fig 1.

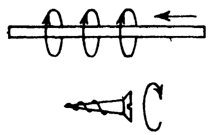


Fig. 2.

towards the east. If, however, the current-carrying wire is placed below the compass needle, the corresponding directions of movement of the needle will be opposite to those which are obtained when the wire is above the needle.

The following rule gives the relationship between the direction of the current in the conductor and the direction of the magnetic force due to the current (see also Fig. 2):

Imagine a right-handed screw with its axis in the conductor and pointing in the direction in which the current is flowing. The direction of the lines of force is the same as that in which the screw must be turned to cause it to advance in the direction of the current.

Now suppose that a long wire is bent to a coil of rectangular shape and placed with its plane in the magnetic meridian (Fig. 3), a compass

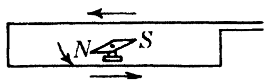


Fig. 3.

needle being supported in the plane of the rectangle. If a current flows round the coil in the direction shown, it follows from what has been said in the foregoing, that the action of the current in both of the horizontal sides of the rectangle as well as the two vertical sides

will deflect the red pole of the needle to the west. If the coil is wound with a number of turns and the current is maintained at its original value, each additional turn of the coil will increase the magnetic action, that is to say, the deflecting effect on the compass needle will be proportional to the product of the magnitude of the current and the number of turns in the coil. If the current is measured in amperes, the magnetic force exerted by the coil on the compass needle will be proportional to the number of "ampere-turns" in the coil.

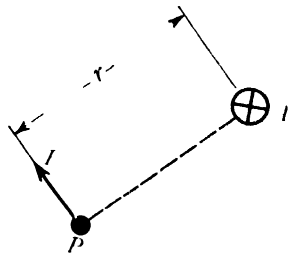


Fig. 4.

Suppose in Fig. 4 the circle A represents the trace in the plane of the paper of a long straight conductor which is perpendicular to the plane of the paper and is carrying a current of i amperes flowing in the direction away from the observer, as indicated by the symbol ⊗. Then it can be shown (see also expression (6), page 213) that the magnetic force on a unit positive magnetic pole placed at any point P which is distant r cm. from the axis of the wire is

$$F = 2\frac{i}{10r} \text{ dynes} \quad . \quad . \quad . \quad . \quad (1)$$

and this force will act in a direction at right angles to the line AP as shown in Fig. 4. The expression (1) should be compared with the corresponding expression (9) in Chapter III for the electrostatic force due to a charged long straight wire, it being observed that in the case of the magnetic force defined by the expression (1) the system is placed in a medium of which the magnetic permeability is $\mu = 1$, that is, the conditions which hold for air.

Since the force has the same magnitude at all points at a given distance r cm. from the centre of the circle A, it follows that the lines of magnetic force are circles which are concentric with the conductor A. In Fig. 5 is shown a series of such lines of force, the intensity of the magnetic forces associated with the respective circles being given by the values $H : 2H : 4H : 8H$ oersted.

An experimental proof that the magnetic force due to a long straight

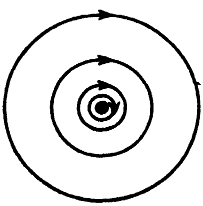


Fig. 5.

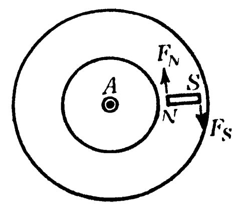


Fig. 6.

conductor in which a current is flowing is inversely proportional to the distance r from the axis of the conductor, was devised by Maxwell as follows. The conductor is arranged in a vertical position as indicated by A in Fig. 6, which shows a plan view of the arrangement. A light carriage of non-magnetic material is suspended so as to be free to turn round the conductor as an axis. It is found that when a magnet is secured to this carriage and a current is flowing in the wire, there will be no couple tending to turn the carriage round the conductor, and this is true whatever the position of the magnet on the carriage may be. Assume, for example, that a thin uniform bar magnet, NS in Fig. 6, is used. This magnet may be regarded as composed of two equal and opposite magnetic poles fixed at its extremities. Let F_n and F_s be the forces on the respective poles, each force acting at right angles to the plane through the conductor and the poles. If r_1 and r_2 are the distances of the N and S poles respectively, from the conductor, the couple tending to turn the magnet (and hence also the carriage) round the conductor will be $F_n r_1 - F_s r_2$. The actual experiment, however, shows that this resultant couple is zero, so that

$$\frac{F_n}{F_s} = \frac{r_2}{r_1},$$

or the forces are inversely proportional to the distances of the pole from the conductor.

The Magnetic Force at the Centre of a Plane Circular Coil and Due to a Current in the Coil

Fig. 7 represents a circular coil of one turn arranged perpendicular to the plane of the paper, the radius of the coil being r cm. A current of i amperes is assumed to be flowing in the coil in the direction away from the observer, as denoted by the symbol ⊗, and towards the observer as denoted by the symbol ⊙ (see also page 229). It can be shown (see page 4) that the force at the centre P is given by the expression,

$$F = \frac{2\pi . i}{10r} \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

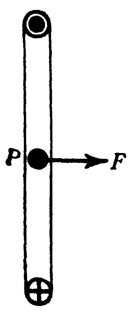


Fig. 7.

and is directed as shown in the diagram.

The intensity of the force at any point other than the centre of the coil can be determined by means of the solid angle which is subtended at the point by the coil as explained on page 215. Fig. 8 shows the lines of magnetic force in a diametrical plane which is perpendicular to the plane of the coil AB.

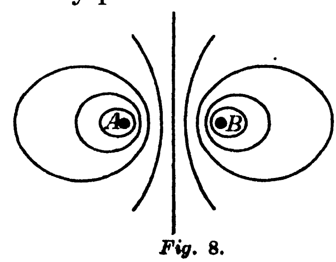


Fig. 8.

Laplace Formula (see also Chapter I, page 3)

Laplace showed that the experimental results obtained by Oersted, Biot and Savart, and Maxwell, could be summarised by the formula,

$$F = \frac{i}{10}\,\delta s . H \sin \theta \text{ dynes} . \qquad . \qquad . \qquad . \qquad (3)$$

where i amperes is the current strength in an element of conductor of length. δs cm. which is placed in a magnetic field of strength H oersted, and θ is the angle between the conductor element and the direction of the field intensity H as shown in Fig. 9 (*a*). The direction of the force will be at right-angles to the plane which contains the conductor element δs and the direction of H as shown in Fig. 9 (*a*). The force will be a maximum when $\theta = 90°$ in which case the direction of H will be at right-

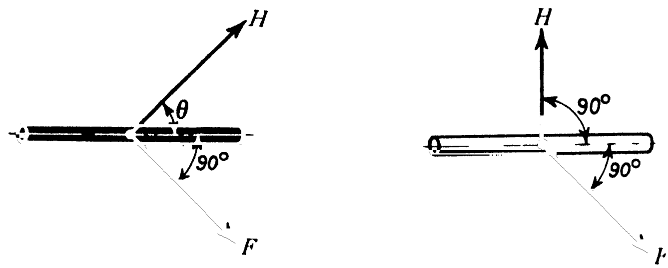


Fig. 9 (*a*). *Fig.* 9 (*b*).

angle to the conductor element δs as shown in Fig. 9 (*b*). Now suppose in Fig. 10 an element of conductor δs is carrying a current of $\frac{i}{10}$ electromagnetic units and that a unit magnetic N pole is placed at P, then the force on the unit pole will be,

$$F = \frac{m}{r^2}\frac{i}{10} . \delta s . \sin \theta = \frac{1}{r^2}\frac{i}{10} . \delta s . \sin \theta \text{ dynes} . \qquad . \qquad (4)$$

where r is the distance CP and θ is the angle between the conductor and the line CP, it being observed that the lines of force due to the unit pole are radiating straight lines. The intensity of the magnetic force at C will therefore be $\frac{1}{r^2}$ dynes. Since the force which is exerted on the unit pole by the current element is equal and opposite to the force exerted on the current element by the unit pole, the expression (4) will also define the force on the current element. Now consider a circular conductor of radius r cm. and assume a unit magnetic pole to be placed at the centre P of the circle. The lines of force will be uniformly distributed straight lines, as shown in Fig. 11, and will be everywhere at right angles to the current-carrying conductor, that is, the angle θ

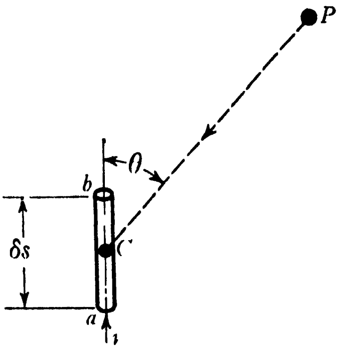


Fig. 10.

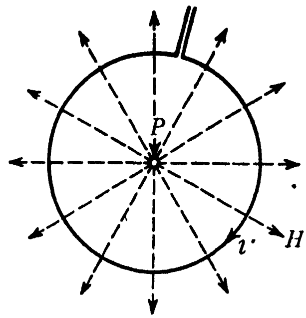


Fig. 11.

of the expression (4) will be 90°. The total force on the unit pole will then be .

$$F = \frac{i}{10}\int_0^{2\pi}\frac{1}{r^2}\,ds = \frac{2\pi}{r}\frac{i}{10} \text{ dynes} \quad . \quad . \quad . \quad (5)$$

when i is the current in amperes.

If, now $r = 1$ cm. and $F = 2\pi$ dynes, then $\frac{i}{10} = 1$ and this relationship defines the absolute unit of electric current in the electromagnetic c.g.s system That is to say, *the electromagnetic c g s. unit of current is that current which, when flowing in a plane circular conductor of 1 cm. radius will act with a force of 2π dynes on a unit magnetic pole placed at the centre of the circle.* This is equivalent to saying that each centimetre length of the conductor will exert a force of 1 dyne on the unit magnetic pole at the centre of the circular conductor (see also page 4).

For technical purposes, the unit of current is the *ampere*, the relationship between the electromagnetic absolute unit and the technical unit being

10 amperes = 1 electromagnetic c.g.s. unit.

A general consideration of units will be found in Chapter I.

The Electromagnetic Force at a point P distant a cm. from an Infinitely Long Wire which carries a Current of i Amperes

In Fig. 12 is shown a straight wire AB which is assumed to extend to an infinite distance in each direction, and it is also assumed that the wire is carrying a current of i amperes. Consider a unit magnetic pole at P which is distant a cm. from the wire so that the wire is situated in the magnetic field of this unit pole. The force exerted on a current element $i\delta s$ of the wire, which is r cm. from the unit pole, will be [by expression (4)]

$$F = \frac{i}{10}\,\delta s H \sin\alpha = \frac{i}{10}\,\delta s H \cos\theta \text{ dynes},$$

where $H = \frac{1}{r^2}$ oersted and is the intensity of the magnetic field at the place occupied by the current element $i\delta s$.

The total force on the wire will then be

$$F = 2\int_{s=0}^{s=\infty} \frac{i}{10} H \cos\theta ds = \frac{2i}{10}\int_{s=0}^{s=\infty} \frac{1}{r^2} \cos\theta ds,$$

but $\qquad \frac{a}{r} = \cos\theta$:

so that $\qquad \frac{1}{a} = \sec^2\theta \frac{d\theta}{ds} : ds = \frac{r^2}{a} d\theta,$

from which it is seen that the total force on the wire will be

$$F = \frac{2i}{10}\int_0^{\pi/2} \frac{1}{a} \cos\theta\, d\theta = \frac{2i}{10a} \text{ dynes} \quad . \quad . \quad (6)$$

and this is also the force magnetic at P due to the current in the wire. This result should be compared with the corresponding expression for the intensity of the electric field at a point in the neighbourhood of a uniformly charged straight wire of infinite length [Chapter III, page 83].

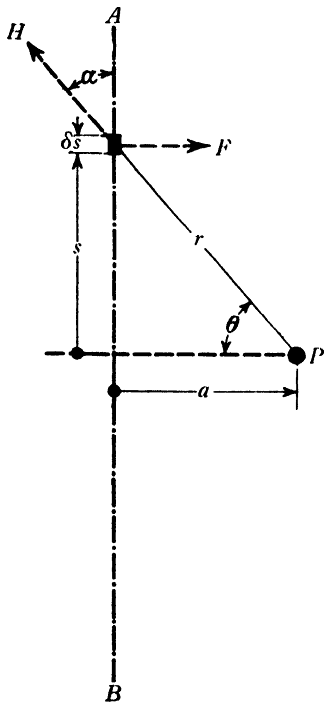


Fig. 12.

The Electromagnetic Force and the Electromagnetic Potential at a Point *P* Due to the Current in a Coil in the Neighbourhood

Fig. 13 shows a cross-section of a plane circular coil NM of radius a cm. and which is assumed to be set with its

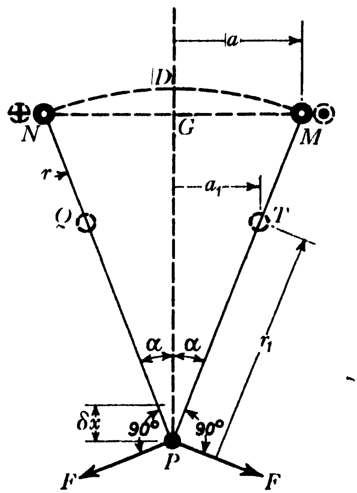


Fig. 13.

plane at right angles to the plane of the paper. The current of magnitude C electromagnetic units is assumed to flow in the direction shown by the signs at N and M respectively. If a unit magnetic pole is placed on the axis of symmetry at P, where PN is r cm., then the force on the pole due to an elementary length δs cm. of the coil at, say, N will be given by the Laplace formula of page 3, viz.

$$F = \frac{1}{r^2} C \, \delta s \text{ dynes,}$$

and the component of this force in the direction of the axis of symmetry will be

$$F' = \frac{1}{r^2} C \, \delta s \sin \alpha \text{ dynes,}$$

so that the total resultant force at P due to the whole length of the coil will be

$$F_P = \int_0^{2\pi a} \frac{C}{r^2} \sin \alpha \, ds = 2\pi C \sin \alpha \frac{a}{r^2} = 2\pi C \frac{\sin^2 \alpha}{r} . \qquad (7)$$

The work done in moving the unit magnetic pole at P through a small distance δx cm. along PG will be

$$\delta U = 2\pi C \frac{\sin^2 \alpha}{r} \delta x \text{ ergs,}$$

where $\qquad \delta x = \delta(GP) = \delta(a \cot \alpha) = -\frac{a}{\sin^2 \alpha} \delta \alpha$

and $\qquad \frac{a}{r} = \sin \alpha,$

so that $\qquad \delta U = 2\pi C \sin \alpha \, \delta \alpha \text{ ergs} \qquad (8)$

If the unit magnetic pole is brought along the direction of the axis of symmetry PG from an infinite distance to the point P, the position of P being defined by the angle α, then the work done will be

$$U = \int_0^{\alpha} 2\pi C \sin \alpha \, d\alpha = 2\pi C (1 - \cos \alpha) . \qquad (9)$$

Again, for a coil QT, Fig. 13, which carries the same current C and subtends the same angle α at P, but for which the distance $PT = r_1$ cm. and the radius is a_1 cm., so that $\frac{a_1}{r_1} = \frac{a}{r} = \sin \alpha$, then the force at P due to this coil will be

$$F_1 = C \int_0^{2\pi a_1} \frac{1}{r_1^2} \sin \alpha \, ds_1 = \frac{2\pi C \sin^2 \alpha}{r_1} \text{dynes,}$$

so that

$$\frac{\text{Force at } P \text{ due to the current } C \text{ in coil } NM}{\text{Force at } P \text{ due to the current } C \text{ in coil } QT} = \frac{r_1}{r} . \qquad (10)$$

The work done in moving the unit magnetic pole at P through a small distance

$$\delta x = \delta(a_1 \cot \alpha) = \frac{a_1}{\sin^2 \alpha}\delta\alpha$$

is

$$\delta U = 2\pi C \frac{\sin^2 \alpha}{r_1} \frac{a_1}{\sin^2 \alpha}\delta\alpha = 2\pi C \sin \alpha \, \delta\alpha$$

that is

$$U = 2\pi C(1 - \cos \alpha),$$

and this is the same value as has been found already [see equation (9)] for the electro-magnetic potential at P due to the current C in the coil NM.

If the unit magnetic pole is brought from an infinite distance to the centre G of the coil NM, that is, to the position for which the angle $\alpha = 90°$, the work done will be

$$U = 2\pi C \text{ ergs} \quad . \quad . \quad . \quad . \quad (11)$$

Now the solid angle Ω which is subtended at the point P by the coil NM is given by the relationship

$$\frac{\Omega}{4\pi} = \frac{\text{Area of the spherical segment } NDM}{\text{Area of the sphere of radius } r},$$

that is

$$\frac{\Omega}{4\pi} = \frac{2\pi r^2(1 - \cos \alpha)}{4\pi r^2},$$

or

$$\Omega = 2\pi(1 - \cos \alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From a comparison of the expressions (9) and (12) it will be seen that the work done in moving the unit magnetic pole along the direction PG from an infinite distance to the point P is given by

$$U = C \times \left\{\begin{matrix}\text{Change of the solid angle which is subtended} \\ \text{at the magnetic pole by the circular coil } NM\end{matrix}\right\} \text{ergs} \quad . \quad (13)$$

If the unit magnetic pole is moved in the direction PG from infinity and passes through the coil to infinity in the opposite direction, the total change in the solid angle which is subtended at the magnetic pole by the coil will be given by the expression (13), viz.

$$U = 4\pi C$$

If the coil has w turns then,

$$U = 4\pi C w = \frac{4\pi}{10} i w \quad . \quad . \quad . \quad . \quad (14)$$

where C is the current in electromagnetic c.g.s. units and i is the current in amperes. This is the expression for the magneto-motive force round any closed path which is linked once with a coil of w turns and carrying

a current of C electromagnetic units. This same expression is derived in a different way on page 219.

The component of the magnetic force in any direction x at a point P and due to a current-carrying coil in the neighbourhood is given by the expression

$$H_x = -\frac{dU}{dx} \text{ oersted} \quad . \quad . \quad . \quad . \quad (15)$$

that is to say, the magnitude of the component of the force in any direction x is equal to the rate of fall of the electromagnetic potential in that direction. The total magnetic force at any point P, therefore, is

$$H = -\left[\frac{dU}{dx}\right]_{max} \text{ oersted} \quad . \quad . \quad . \quad (16)$$

and acts in that direction in which the potential gradient has a maximum negative value. The foregoing results are identical in form with those which have been obtained with respect to an electric field of force (see pages 98 and 100).

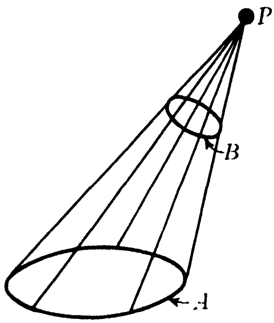


Fig. 14.

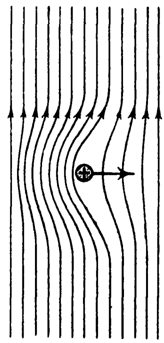


Fig. 15.

As a corollary to the foregoing results it will be seen that the electromagnetic potential at a point P due to a current-carrying coil in the neighbourhood will be the same for all coils such as A and B, as shown in Fig. 14, which carry the same current and which subtend the same solid angle at P.

The Work Done in Moving a Current-carrying Conductor in a Magnetic Field

Suppose the magnetic field is of uniform intensity H oersted and the conductor is placed in a plane which is perpendicular to the magnetic field as is shown in Figs. 15 and 16. The mechanical force acting on the conductor will then be

$$F = H.l.C = H.l\frac{i}{10} \text{ dynes} \quad . \quad . \quad . \quad . \quad (17)$$

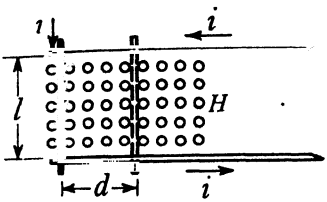


Fig. 16.

where l cm. is the length of the conductor, C is the strength of the current in electro-magnetic units, and i the strength of the current in amperes. If the conductor is allowed to move under the action of this force through a distance d cm., then the work done on the conductor will be

$$Hl\frac{i}{10}d \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Reference to Fig. 16 will show that the quantity Hld represents the total number of magnetic lines cut by the conductor during the movement so that the work done during the movement will be given by the expression

$$\frac{(\text{Total number of lines cut}) \times (\text{Current in amperes})}{10} \text{ ergs} \quad . \quad (19)$$

As will be seen later, the energy necessary to perform this work is derived from the source which supplies the current to the conductor, since the movement of the conductor in the field induces an e.m.f. in the conductor which opposes the flow and hence, in order to keep the current strength constant during the movement, the p.d. which is applied to the conductor must be increased correspondingly.

If the field intensity is not uniform the same result is arrived at by considering elementary portions of the conductor moving through small fractions of the distance, so that the general statement of the result can be expressed as follows:

If a conductor carrying a current of i amperes cuts Φ unit magnetic lines of magnetic force, the work done during the movement is

$$\Phi\frac{i}{10} \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Mutual Force between Two Long Parallel Conductors in Each of which a Current is Flowing

Suppose in Fig. 17 the two conductors A B are each carrying a current of i amperes and are placed at a distance of r cm. apart. The intensity of the field at B due to the current in A is, (see page 213)

$$H = \frac{2i}{10r} \text{ lines per square centimetre.}$$

The force on the conductor B carrying a current of i amperes and situated in a field of intensity $\frac{2i}{10r}$ is, (see expression (17), page 216)

$$2\left(\frac{i}{10}\right)^2 \frac{1}{r} \text{ dynes per centimetre length of conductor.}$$

It will be clear that there must also be an equal force on conductor A, so that the total force on each conductor will be $2\left(\frac{i}{10}\right)^2 \times \frac{l}{r}$ dynes, where l cm. is the length of each conductor. If the current flows in the same direction in each conductor the force will be one of mutual attraction, whilst if the currents flow in opposite directions the force will be one of repulsion.

EXAMPLE.—*If a current of* 100 *amperes flows in each conductor and the conductors are* 3 *inches apart and* 100 *yards long, then the mutual force between them will be*

$$\frac{2 \times 10^4 \times 3{,}600 \times 2 \cdot 54}{10^2 \times 7 \cdot 62} = 2 \cdot 4 \times 10^5 \textit{ dynes} = 0 \cdot 54 \textit{ lb. weight.}$$

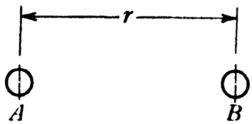


Fig. 17.

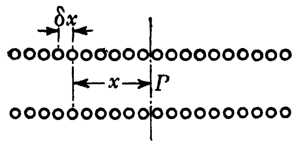


Fig. 18.

The Magnetic Force at the Middle of a Long Solenoid

Suppose the solenoid is wound with w turns *per centimetre length*, the total length of the solenoid winding being l cm. The magnetic force at the point P due to the current in the element δx of the solenoid winding (Fig. 18) distant x cm. from the mid-point P, is

$$2\pi \frac{i}{10} \frac{a^2}{(a^2 + x^2)^{3/2}} w \, \delta x \text{ dynes,}$$

where i amperes is the current strength and a cm. is the radius of the solenoid winding. The total force at P due to the whole solenoid will therefore be

$$\int_{-l/2}^{+l/2} 2\pi \frac{i}{10} \frac{a^2}{(a^2 + x^2)^{3/2}} w \, dx = 2\pi \frac{i}{10} \frac{l}{\sqrt{a^2 + \left(\frac{l}{2}\right)^2}} \text{ dynes.}$$

If a is small in comparison with l, the force acting on a unit magnetic pole at P will be

$$4\pi \frac{i}{10} w \text{ dynes.}$$

That is to say, the intensity of the magnetic field at P is

H (oersted)

$$= \frac{4\pi}{10}\{\text{ampere-turns per centimetre length of the solenoid}\} \quad . \quad (21)$$

This result is of great importance in practice, as, for example, in the calibration of a ballistic galvanometer (see Chapter VII, page 206).

Magneto-motive Force Round a Closed Circuit which is Linked with a Current-carrying Conductor

It has been seen from expression (20), page 217, that when a conductor in which a current of i amperes is flowing is cut by Φ unit magnetic lines of force, the work done in the process is $\Phi\frac{i}{10}$ ergs.

It is a matter of indifference whether the conductor cuts the lines by reason of the magnetic flux moving or the conductor moving, or both conductor and flux moving.

Now consider the electric circuit shown in Fig. 19 in which a current of i amps. is flowing, and assume that a unit magnetic pole P is carried once round the closed circuit linked with the electric circuit and shown by the broken line curve in Fig. 19. *Since a unit magnetic pole gives rise to 4π unit magnetic lines of force, then if the pole is carried once round the closed loop which is linked with the electric circuit, each of the 4π lines will cut the electric circuit once. Hence the work done in carrying the unit pole once round the closed loop is $4\pi\frac{i}{10}$ ergs.* If the electric circuit has w turns in series and the closed loop links the whole of the w turns, the work done is

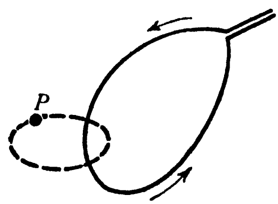


Fig 19.

$$\frac{4\pi}{10}iw \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The work done is independent of the actual path taken in making the journey round the closed loop so long as the path only links once with the circuit.

The work done in traversing the closed loop with unit magnetic pole is the *magneto-motive force* round the loop and is usually written, m.m.f. Hence—

The magneto-motive force round the path of a single closed loop is $\frac{4\pi}{10}$ times ampere-turns linked with theloop [see also page 215, expression (14)].

This is an extremely important result and, for example, forms the basis for the design of the magnetic circuit of electrical machines and apparatus.

Now the work done in carrying a unit pole round the magnetic circuit may also be stated as the sum $\Sigma(H.l)$, where H is the intensity

at any part of the circuit and l cm. the length of the part of the circuit over which the force has the value H, so that

$$\frac{4\pi}{10} i . w = \Sigma(H . l) \text{ or } iw = 0{\cdot}8\ \Sigma(H . l) \quad . \quad . \quad . \quad (23)$$

In order to find the value of the magnetic flux produced by any given value of the ampere-turns iw, it is necessary to plot a curve connecting flux and ampere-turns by assuming different values for the flux and calculating the corresponding values of $0{\cdot}8\ \Sigma H . l$. If the path of the closed loop is not wholly in air but passes partly or wholly through magnetic materials, the results stated in the foregoing still hold because the work done in carrying a unit magnetic pole round a closed path in a field due to magnetised bodies is zero, just as the work done in carrying a unit electric charge round a closed path in an electrostatic field is zero. Hence: whatever materials the closed loop may pass through, *the magneto-motive force is* $\frac{4\pi}{10} \times$ *(ampere-turns) linked with the loop.*

EXAMPLE.—As an example of the application of this result the magnetic circuit of a two-pole dynamo will be considered, as shown in Fig. 20. The poles are marked N and S, and on each pole is wound an exciting coil. The magnetic flux passes from the N pole, across the air-gap, through the armature, across the second air-gap, through the S pole, and divides into the two halves of the yoke, thus completing the magnetic circuit to the N pole.

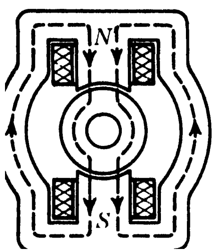


Fig. 20.

The total flux of magnetic induction is constant at any cross-section of the circuit (neglecting any leakage which passes from the sides of the poles into the sides of the yoke).

The induction density at any cross-section of the circuit is obtained by dividing the magnitude of the flux by the cross-section. Having thus determined the value of the induction B at different parts of the circuit, the corresponding values of H are read off from the B-H curves such as are given, for example, in Chapter VI.

Then, if i is the current in amperes in the exciting coils and w the total number of turns in series, the m.m.f. round the magnetic circuit is—

$$\frac{4\pi}{10} wi = 1{\cdot}257 wi.$$

For this example it is assumed that the cross-section of the magnetic circuit in poles and yoke is the same.

The cross-section of the yoke is the sum of the sections of the two limbs through which the total pole flux passes.

The following numerical values have been assumed, viz.:

Area of air-gap, $A_g = 100$ sq. cm.
Length of double air-gap, $l_g = 0{\cdot}4$ cm.
Total cross-sectional area of (iron) armature, $A_a = 75$ sq. cm.
Length of the mean path of the flux in the armature, $l_a = 17$ cm.
Cross-sectional area of (steel) poles and yoke, $A_y = 50$ sq. cm.
Total length of the mean path of the flux in poles and yoke, $l_y = 88$ cm.

The area of the gap is taken as being twice the area of the poles, it being assumed that pole shoes are fitted but are not shown in Fig. 20.

In the following table, a series of values of total flux N is taken and the corresponding values of the ampere-turns calculated.

Total Flux N	$B_g = H_g = \frac{N}{A_g}$	$B_a = \frac{N}{A}$	H_a	$B_y = \frac{N}{A_y}$	H_y	$H_g l_g$	$H_a l_a$	$H_y l_y$	ΣHl	$wi = 0{\cdot}8\Sigma Hl$
100,000	1,000	1,330	0·6	2,000	2·2	400	10	200	610	488
200,000	2,000	2,660	1	4,000	2·9	800	17	255	1,072	858
300,000	3,000	3,990	1·2	6,000	3·6	1,200	20	320	1,540	1,232
400,000	4,000	5,320	1·5	8,000	4·9	1,600	25	430	2,055	1,644
500,000	5,000	6,650	1·7	10,000	6·6	2,000	29	580	2,610	2,088
600,000	6,000	7,980	2	12,000	9·5	2,400	35	840	3,275	2,620
700,000	7,000	9,310	2·5	14,000	15	2,800	42	1,320	4,162	3,332
800,000	8,000	10,640	3	16,000	27·2	3,200	51	2,400	5,651	4,521

The results are plotted as a curve in Fig. 21, from which it is possible to read off the value of the total flux for any given value of the exciting ampere-turns.

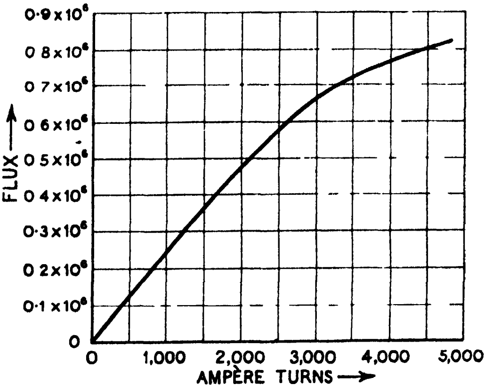


Fig. 21.

Magnetic Reluctance

If Φ is the flux of induction in a magnetic circuit which consists of different media it has been seen that the magneto-motive force is given by the expression—

$$\text{m.m.f.} = \frac{4\pi}{10}wi = (H_1l_1 + H_2l_2 + H_3l_3 \ldots),$$

where H_1 is the intensity of the magnetic force along a path of length l_1 cms. and similarly for H_2;H_3. . . .

But $$H_1 = \frac{B_1}{\mu_1} = \frac{\Phi}{A_1\mu_1},$$

where A_1 sq. cm. is the crsss-sectional area of path of l_1.

Similarly $$H_2 = \frac{\Phi}{A_2\mu_2}. \ldots$$

Hence $$\text{m.m.f.} = \frac{4\pi}{10}wi = \Phi\left[\frac{l_1}{A_1\mu_1} + \frac{l_2}{A_2\mu_2} + \frac{l_3}{A_3\mu_3} + \ldots\right]. \quad (24)$$

From a comparison of this equation with the analogous one for the electric circuit, viz.:

$$\text{e.m f.} = \text{current} \times \text{resistance},$$

the quantity $$\left[\frac{l_1}{A_1\mu_1} + \frac{l_2}{A_2\mu_2} + \frac{l_3}{A_3\mu_3} + \ldots\right] \quad (25)$$

has been termed the *reluctance* of the magnetic circuit, and consequently the following relationship holds, viz.:

$$\text{m.m f.} = \text{flux} \times \text{reluctance}.$$

The reluctance of *any part* of a magnetic circuit of which the length is l cm., the cross-section A sq. cm., and the permeability is μ, is—

$$\frac{l}{A\mu}.$$

For example, consider an electromagnet, as shown in Fig. 22.

If l_1 cm. is the length of the magnetic circuit in the iron and l_2 cm the length of the gap, and if A sq. cm. is the area of the circuit—

$$\frac{4\pi}{10}wi = \Phi\left[\frac{l_1}{A\mu_1} + \frac{l_2}{A}\right] \quad (26)$$

the permeability of the air-gap being unity.

For values of flux density in the iron below, say $B = 15{,}000$ (see Chapter VII) the value of μ_1 will be large and $\frac{l_1}{A\mu_1}$ will be relatively negligibly small as compared with $\frac{l_2}{A}$ and hence the m.m.f. is almost entirely used in driving the flux across the air-gap.

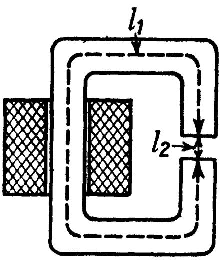


Fig. 22.

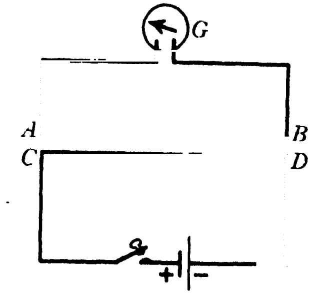


Fig. 23.

Electro-magnetic Induction

In Fig. 23 is shown a stretched wire, AB, connected to a galvanometer, G, so that a closed circuit is obtained. A wire, CD, is arranged parallel to AB and connected through a switch to an accumulator cell. When a steady current flows in the wire CD no deflection of the galvanometer needle is to be observed. When the switch is being closed, however, the galvanometer shows a momentary deflection and then comes to rest. When the switch is being opened again, the needle again shows a momentary deflection and then comes to rest, the deflection when opening the switch being in the opposite direction to that which is obtained on closing the switch. A current is therefore induced in the circuit of AB only when the electrical condition of the neighbouring circuit CD is disturbed, that is, when the current in CD is changed.

If the current in CD flows in the direction from C to D when the switch is closed, the deflection of the galvanometer will correspond to a current flowing in AB in the direction from A to B. When the switch is being opened (that is, when the current in CD is being stopped) the deflection shown by the galvanometer will correspond to a current flowing in AB in the direction from A to B. This electromagnetic induction effect between the two circuits is due to the variation of the magnetic flux in the neighbourhood of the wires when the current is being altered. It has already been seen on page 208 that when a current flows in a wire such as CD in Fig. 23, a magnetic field is established in the neighbouring space, the lines of magnetic force being closed concentric circles. When the current is being started in CD the magnetic field may be considered as becoming established by the lines of force expanding outwards from the wire CD and linking the circuit of which AB is a part, thus generating an e.m.f. in the circuit of AB. When the current in CD has become steady, the magnetic field associated with this current also becomes steady and the condition is then reached that a magnetic flux is linked with the circuit of AB, and so long as this flux is steady, no e.m.f. will be induced in the circuit of AB.

When the current in CD is being stopped, the magnetic field due

to the current simultaneously disappears and this may be considered as taking place by the lines of force shrinking and eventually collapsing on to the conductor CD. In thus shrinking, the flux linked with the circuit of AB becomes withdrawn, and *during the withdrawal of the flux an e.m.f. is induced in the circuit of AB*, the direction of this e.m.f. being opposite to that which is induced when the magnetic field is being established, that is, when the lines of force were expanding.

By revolving a given circuit at different speeds in a magnetic field Faraday arrived at the following result:

In a circuit, any part of which moves across lines of magnetic induction, an e.m.f. is induced which is proportional to the rate at which the number of lines linked with the circuit changes.

Lenz's Law

Almost immediately after Faraday's discovery of the principle of electromagnetic induction, Lenz gave the rule by which the direction

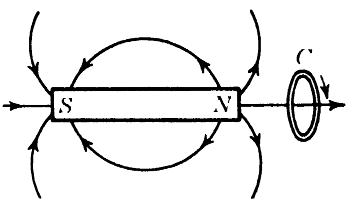


Fig 24

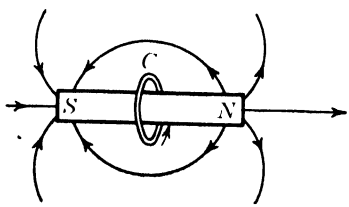


Fig 25.

of the induced e.m.f. could be determined, and this rule is generally known as Lenz's Law, viz.:

The direction of the induced e.m.f. is always such that, by its electro magnetic action, it tends to oppose the effect which produces it.

Thus, in Fig. 23, when the switch is being closed and the current started in CD, an e.m.f. is induced in AB in a direction such that it tends to prevent the magnetic field due to the current in CD from linking with the circuit AB, hence the current induced in AB will be in the opposite direction to that started in CD. When the switch is being opened the stoppage of the current in CD will induce an e.m.f. in the circuit of AB in a direction such that it will tend to oppose the withdrawal of the flux linked with AB, hence the current induced in AB will be in the same direction as the current which flowed in CD before opening the switch.

Further Examples of Electromagnetic Induction

In Figs. 24 and 25 NS is a permanent magnet and C is a short-circuited coil of insulated wire. If this coil is threaded quickly over the N-pole of the magnet (see Fig. 24), a current will be induced in the coil during

the movement, the direction being such that it will oppose the flux of the magnet from becoming linked with the coil. If the coil is now quickly withdrawn from the position shown in Fig. 25, a current will be induced in the reverse direction to that shown in Fig. 24, and Lenz's law shows that the direction of the current induced in the coil during the withdrawal will be as shown in Fig. 25, whether the coil be removed by drawing it off the N end or the S end of the magnet.

In Fig. 26 a coil is shown, wound on an iron core and connected through a switch to an accumulator battery of low e.m.f., say 30 volts, and across the terminals of the coil is connected a filament electric lamp, say, for example, rated at 100 volts. When the switch is closed only a very small current will flow in the lamp filament, which will therefore remain dark. When the switch is opened, however, the electromagnetic induction is such that it tends to maintain the flux linking the coil, and an e.m.f. is thereby induced which may be sufficiently large to cause the lamp to glow. If the lamp were removed, the electro-

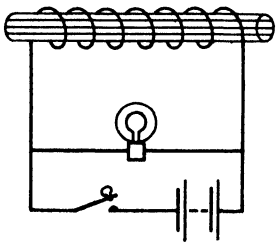


Fig. 26.

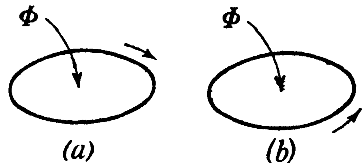


Fig. 27.

magnetic action would be such as to cause a substantial spark to pass across the switch contacts when breaking the circuit.

The Magnitude of the E.M.F. which is Induced by Electromagnetic Action

In 1831 Faraday established the conditions under which electromagnetic induction takes place, and in 1845 Neumann gave an expression for the magnitude of the induced e.m.f. The law of electromagnetic induction may be stated as follows.

Whenever the flux of magnetic induction which links a circuit is changed, an e.m.f. is induced in the circuit of magnitude equal to the rate of change of the magnetic flux and acts in such a direction as to tend to prevent that change of flux.

It is to be noted that, in so far as the induced e.m.f. is concerned, it is only the *relative movement* of the circuit and the flux linked which matters, and it makes no difference whether this relative movement corresponds to a stationary circuit or to a stationary field.

Suppose in Fig. 27*a* the magnetic flux linked with the coil at any moment t is Φ. If Φ varies, then an e.m.f. e will be induced in the coil such that $e = -\frac{d\Phi}{dt}$ electromagnetic units. If there are w turns in the coil closely wound so that each turn embraces the whole of the Φ lines, then,

$$e = w\frac{d\Phi}{dt} \text{ electromagnetic units} \quad . \quad . \quad . \quad (27)$$

The negative sign in this expression means that the direction of the induced e.m.f. is such that it tends to prevent the change in the number of lines which are linked with the coil. Thus, in Fig. 27*a* the flux Φ is assumed to be diminishing so that the induced e.m.f. produces a current which tends to prevent Φ from diminishing. In Fig. 27*b* the flux Φ is increasing so that the induced e.m.f. produces a current which tends to prevent Φ from increasing. The expression (27) for the induced e.m.f. leads to the definition of the electromagnetic unit of e.m.f. as follows

The electromagnetic unit of e.m.f. (or p.d.) is that e.m.f. induced in a coil of one turn by a flux of magnetic induction linked with the coil and changing at the rate of 1 *c.g.s. unit magnetic line per second.*

For practical purposes, however, this unit is far too small and consequently, a unit one hundred million times (10^8) as large as the electromagnetic unit is used and is termed the *volt*, that is (see also Chapter I, Table II, page 8)

$$1 \text{ volt} = 10^8 \text{ (electromagnetic units of e.m.f. or p.d.)} \quad . \quad (28)$$

It is seen, therefore, that an e.m.f. of 1 volt will be induced in a conducting coil of one turn through which the flux of magnetic induction is varying at the rate of 10^8 c.g.s. unit lines per second. Using the same symbols as before,

$$e = -\frac{d\Phi}{dt}10^{-8} \text{ volts}$$

for a coil of one turn. If the coil has w turns, each of which is linked with the whole flux Φ, then,

$$e = -w\frac{d\Phi}{dt}10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad (29)$$

Quantity of Electricity Set in Motion by Electromagnetic Induction

Suppose a closed circuit such as that shown by the coils in Figs. 27*a* and 27*b* has a resistance of R ohms and is linked with a steady magnetic flux of Φ c.g.s. lines. If this flux is then withdrawn from the coil or alternatively, if the coil is removed from the neighbourhood of the flux, an e.m.f. e will be induced in the coil such that

$$e = -\frac{1}{10^8}\frac{d\Phi}{dt} \text{ volts}$$

for a coil of one turn, and consequently, a current i will flow round the coil where

$$i = \frac{e}{R} \text{ amperes.}$$

A quantity of q coulombs of electricity will therefore pass round the coil circuit such that,

$$q = \int idt = \frac{1}{R}\int edt$$

that is,

$$q = -\frac{1}{R.10^8}\int_{\phi}^{0} d\Phi \text{ coulomb}$$

or

$$q = \frac{\Phi}{R.10^8} \text{ coulombs}$$

where Φ is the total change of flux which is linked with the coil. If the coil has w turns each of which is linked with the flux Φ then the quantity of electricity which will pass round the circuit of the coil will be

$$q = \frac{\Phi . w}{R.10^8} \text{ coulomb} \qquad . \quad . \quad . \quad . \quad (30)$$

Electric Generators

In Fig. 28 is shown a conductor placed in the magnetic field between two poles N and S, the axis of the conductor being perpendicular to the direction of the field as shown in the diagram. The conductor is connected to a wire of resistance R ohms so as to form a closed circuit and the length of the pole face perpendicular to the plane of the paper is l cm. The magnetic intensity of the field between the poles is H oersted, and the conductor is moving parallel to itself with a velocity of v cm. per second. At any time t the distance of the conductor from the upper edge of the pole is x cm. Then the flux linking the circuit of the moving conductor at the time t will be

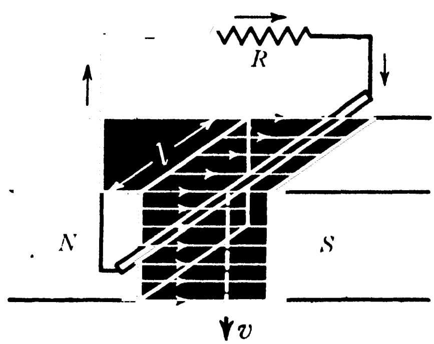


Fig. 28.

$$\Phi = H.l.x \text{ c.g.s. lines} \qquad . \quad . \quad . \quad . \quad (31)$$

The *magnitude* of the e.m.f. which will be induced in the circuit AB will then be

$$e = -\frac{d\Phi}{dt}10^{-8} = -\frac{Hl}{10^8}\frac{dx}{dt} = -\frac{Hlv}{10^8} \text{ volts} \qquad . \quad . \quad (32)$$

The direction of the induced e.m.f. is such as to tend to oppose the movement which produces it, hence the current which will flow in the

circuit of the moving conductor and the resistance R will be (Fig. 28) directed towards the observer because such a current will produce a flux which, when superposed on the magnetic field due to NS will give a resultant intensity which will be greater in front of the conductor and less behind the conductor than the intensity of the original field due to NS as shown in Fig. 29 (see also Fleming's Rule, Fig. 31). Since, however, a characteristic feature of the magnetic lines of force is that they tend to straigthen out, it will be clear from Fig. 29 that the magnetic field opposes the downward movement of the conductor. The same result is arrived at if it be considered that the e.m.f. induced in the conductor when cutting across the lines of force and may be stated as follows :

The e.m.f. which is induced in a conductor when cutting across lines of magnetic induction at the rate of Φ unit lines per second, has the magnitude $\Phi . 10^{-8}$ volts.

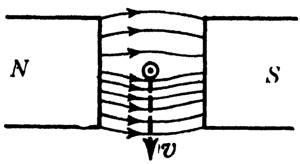


Fig. 29.

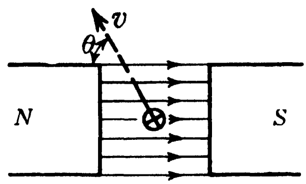


Fig. 30.

The conception that the induced e.m.f. is due to the conductor *cutting across* the lines of force is most convenient for application when considering the case of a generator or a motor. When dealing with transformers, however, the change of flux *linked with* the circuit is the more convenient method of applying the formula for the induced e.m.f.

If a conductor moves in any way in a magnetic field, the induced e.m.f. is given by the rate at which it cuts the lines of force. Thus, in Fig. 30, if the lines of force are parallel to the plane of the paper and the conductor is perpendicular to the plane of the paper, and if the conductor is moved parallel to itself with a velocity of centimetres per second, in the direction shown in Fig. 30, then the induced e.m.f. will be

$$e = (Hlv \sin \theta)10^{-8} \text{ volts} \quad . \quad . \quad . \quad . \quad (33)$$

where l cm. is the length of the conductor and H is the intensity of the magnetic field. Generally stated, the magnitude of the e.m.f. induced in the element δl cm. of a conductor which is moving across a magnetic field is, $\frac{\delta l}{10^8}\frac{d\Phi}{dt}$ where $\frac{d\Phi}{dt}$ is the rate at which the elementary length δl cuts across the lines of magnetic induction.

A very convenient rule for determining the direction of the induced e.m.f. in a conductor which is moving across a magnetic field is Fleming's "Right-hand Rule";

Let the thumb, forefinger, and the middle finger of the right hand be placed mutually at right angles as shown in Fig. 31. Let the thumb point in the direction of movement of the conductor relatively to the magnetic field, the forefinger in the direction of the field, then the middle finger will point in the direction of the induced e.m.f.

This rule is easily remembered by noting that the word "thumb" containing the letter "m" indicate the direction of *motion*, the fore-finger indicates by the letter "f" the *direction of the magnetic field*, and the middle finger indicates by the letter "i" the *direction of the induced current* (observing that the technical symbol for current is the letter "*i*").

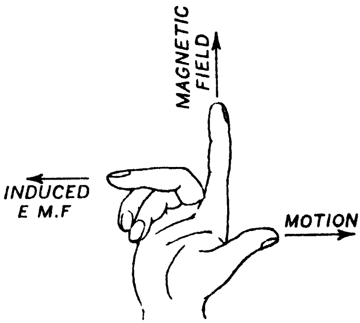


Fig. 31.

It is to be carefully noted that the "direction of motion" in this rule is the *direction* of motion of the *conductor relatively to the field*. Thus, for example, if the field is stationary and the conductor moves, say, from left to right, the induced e.m.f. is in the same direction, as is the case when the conductor is stationary and the field moves from right to left. In both cases the direction of movement of the conductor relatively to the field is from left to right.

A very useful convention for diagrammatically representing the direction of the current and e.m.f. in a conductor is as follows: If the conductor is assumed to be perpendicular to the plane of the paper, a current directed towards the observed is denoted by the sign ⊙, which is intended to represent the point of an arrow moving towards the observer. If the current is directed away from the observer, this is denoted by the symbol ⊗, which is intended to represent the featherhead of an arrow moving away from the observer (see, for example, Figs. 29, 30, and 32*a*).

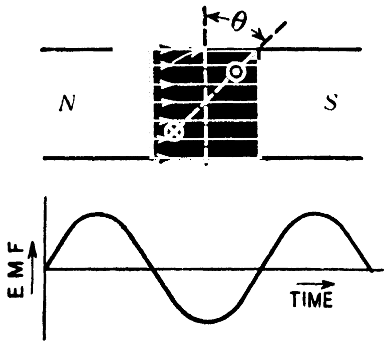
EXAMPLE.—As a numerical example of the magnitudes of the different factors which are involved in the formula (32) for the magnitude of the generated e.m.f. in a conductor, suppose the conductor is 50 cm. long, the flux density is H = 10,000 oersted, and the velocity of the conductor is 30 metres per second, that is, 3,000 cm. per second. Assuming that the direction of motion is at right angles to both field and conductor, as shown in Fig. 29, the induced e.m.f. will then be

$$e = \frac{H.l.v}{10^8} = \frac{10{,}000 \times 50 \times 3{,}000}{10^8} = 15 \text{ volts} \quad . \quad . \quad (34)$$

Now suppose in Fig. 32*a* that a coil of rectangular shape is arranged so that it can rotate about an axis perpendicular to the plane of the

pole sides, NS, the arrangement being shown in perspective in Fig. 32*b*. Let the coil be driven at a speed of n revs. per second and let $2r$ cm. be the length of the side of the coil which is in the plane of the paper, the area of the coil thus being $2l.r$ sq. cm. If H oersted is the intensity of the magnet field, the flux through the coil when inclined at the angle θ to the vertical (Fig. 32*a*), will be

$$\Phi = 2l.r.H \cos\theta$$



(a)

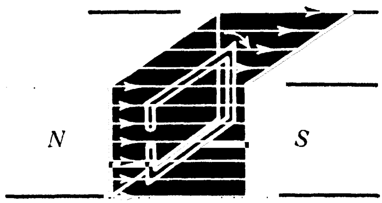


Fig. 32. (b)

If the time t is measured from the moment at which the coil is in the vertical position, then $\theta = 2\pi nt$, so that

$$\Phi = 2l.r.H \cos 2\pi nt.$$

The magnitude of the induced e.m f. will therefore be,

$$e = \frac{1}{10^8}\frac{d\Phi}{dt} = \frac{1}{10^8}4\pi H.n.l.r \sin 2\pi n.t$$

that is $$e = 2\pi n 10^{-8}\Phi_{max} \sin\theta \text{ volts} \qquad . \quad . \quad . \quad (35)$$

where Φ_{max} is the maximum flux which is embraced by the coil- that is, the flux through the coil when its plane is at right angles to the field, in which case $\theta = 0$. It is seen, therefore, that the e.m.f. varies as a sine function of θ, under the assumption that the angular velocity of the coil $w = 2\pi n$ radians per second remains constant. The e.m.f. wave as obtained is shown in Fig. 32*a* as a function of the time t. The same result will of course be obtained if the e.m.f. is calculated from the rate at which each conductor of the loop cuts the lines of force.

In practice, in order to obtain in an efficient manner, a strong magnetic field, the coil is wound on an iron core which revolves between the pole pieces, a small gap being thus formed between the core and pole faces as shown in Fig. 33. If the length of one of the active coil sides is l cm. the velocity v cm. per second, and the intensity of the magnetic field is H oersted, then the e.m.f. which will be induced in the coil side will be $Hlv10^{-8}$ volts. If the core is driven at a constant speed—that is, if v is constant—it will be seen that the e.m.f. will be proportional to H, the intensity of the magnetic field at that point in the gap at which

the coil side is placed at the instant considered. The wave of e.m.f. induced in the coil shown in Fig. 33 will then be as shown in that diagram.

Instead of connecting the ends of the coil each to a separate slip-ring,

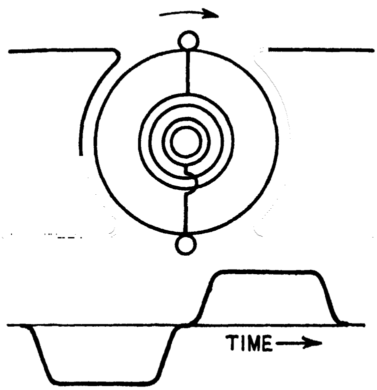


Fig. 33.

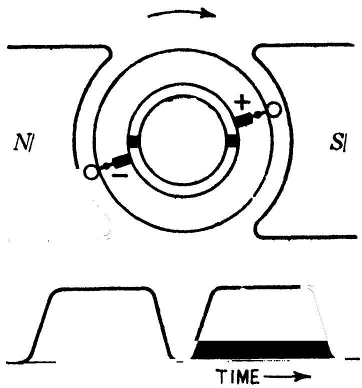


Fig. 34.

suppose a single slip-ring is cut into two equal parts, these two parts being mounted on the axle of the core, so that they are insulated from each other. Let the two segments be connected respectively to the ends of the coil as shown in Fig. 34. If two brushes be arranged diametrically opposite to each other and on an approximately horizontal axis as in Fig. 34, it will be seen that when a coil side is under the N pole it is connected to the brush marked −, and when the coil side passes to a position under the S pole it is connected to the brush marked +. The negative brush is thus maintained in contact with that coil side which is under the N pole and the positive brush with whichever coil side is under the S pole. The polarity of each brush thus remains the same as the coil revolves. The e.m.f. developed between the brushes will then be as shown in Fig. 34 and is therefore *uni-directional.*

This split-ring collector device thus rectifies the alternating e.m.f. wave shown in Fig. 33, so that the uni-directional wave of Fig. 34 is obtained, and the arrangement of Fig. 34 is thus capable of giving a current which will always flow in the same direction. The split-ring collector is thus a very simple form of *commutator.* A single coil and a two-segment commutator arrangement as shown in Fig. 34 is not, however, a practicable arrangement since the e.m.f. varies from a maximum to zero twice during every revolution of the coil.

A practical form of direct-current generator is shown diagrammatically in Fig. 35, in which a number of coils wound on a laminated iron ring and displaced relatively to each other are connected in series so as to form a closed circuit, the assembly of iron ring and copper coils being then known as the "armature" of the machine. From each coil a

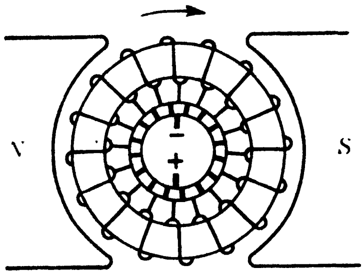


Fig. 35.

tapping is brought to an insulated segment of the commutator and the two brushes are placed diametrically opposite to each other and on that axis which is perpendicular to the magnetic axis of the poles. For the direction of rotation of the armature and the direction of the magnetic field as shown, the right-hand rule of Fig. 31 shows that the upper brush will form the negative terminal and the lower brush the positive terminal. The e.m.f. which is induced between the brushes in such a machine may be calculated as follows: Let Φ be the total flux measured in c.g.s. lines which passes from the N pole to the S pole of the machine field system. Let Z be the total number of active conductors on the surface of the armature, that is, for the case illustrated in Fig. 35, $Z = 16$, and let n rvs. per second be the speed at which the armature is driven. It will be clear that each conductor will cut Φ lines in half a revolution of the armature, that is, in $\frac{1}{2n}$ second. The mean rate of cutting of the lines of force by each conductor will therefore be

$$\frac{\Phi}{\frac{1}{2n}} = 2\Phi n \text{ lines per second,}$$

and the mean e.m.f. induced in each conductor will then be $\frac{2\Phi n}{10^8}$ volts. Since there are $\frac{Z}{2}$ conductors in series between the brushes, the mean e.m.f. induced between the brushes will be

$$E = \frac{2\Phi . n . Z}{10^8 \times 2} = \frac{\Phi . n . Z}{10^8} \text{ volts} \quad . \quad . \quad . \quad (36)$$

If the armature is wound with a relatively small number of coils in series and a correspondingly small number of commutator segments, the e.m.f. induced between the brushes will pulsate considerably above and below the mean value. If, however, a large number of coils are arranged in series between the brushes and a correspondingly large number of commutator segments provided, the mean e.m.f. will approach more closely to the actual e.m.f. at any instant: in other words, the pulsation of the e.m.f. at the brushes becomes relatively insignificant.

Energy Involved in Generating an Electric Current

It has been shown on page 217, expression (20), that in the case of a circuit in which a current of i amperes is flowing, then if the magnetic flux which is linked with the circuit is caused to change by an amount

Φ c.g.s. unit lines, the mechanical work expended in producing the change of flux linked with the circuit will be

$$\Phi \frac{i}{10} \text{ ergs} = \Phi \frac{i}{10^8} \text{ joules.}$$

If the flux linked changes at the rate of Φ lines per second, the mechanical power expended will be,

$$\Phi i . 10^{-8} \text{ joules per second, i.e. watts.}$$

It has been shown, however, that the e.m.f. induced in the circuit when the flux linked with the circuit changes at the rate of Φ lines per second is

$$e = \Phi 10^{-8} \text{ volts.}$$

Hence the mechanical work expended in producing the change of flux through the circuit in which a current of i amperes is flowing is $e.i$ watts.

If the resistance of the circuit is R ohms (see also Fig. 28) the electric power dissipated in heating this resistance is as shown on page 59,

$$i^2 R = e.i \text{ watts,}$$

so that for the arrangement of Fig. 28 the mechanical power expended in driving the conductor across the field is completely accounted for by the electrical energy which is dissipated in the form of heat in the circuit resistance.

The Electric Motor

In Fig. 36 the conductor AB is placed in a uniform magnetic field of intensity H oersted and a current of i amperes is passed through the conductor from a battery of which the e.m.f. is E volts. If the con-

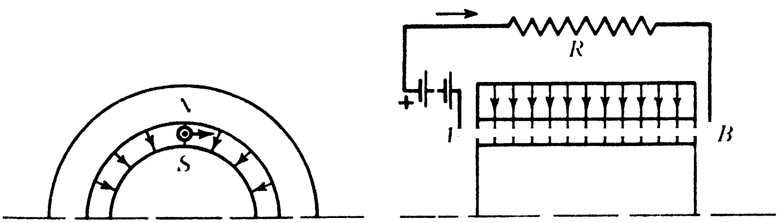


Fig. 36.

ductor is stationary and the resistance is R ohms, then by Ohm's Law $E = i.R$ and from the results given on page 216 it will be clear that there will be a mechanical force F acting on the conductor such that

$$F = H.l.\frac{i}{10} \text{ dynes} \quad . \quad . \quad . \quad . \quad (37)$$

where l cm. is the length of the conductor, which is actually situated in the magnetic field. Suppose now that the conductor is free to move and

that it develops a velocity of v cm. per second under the action of the force F. An e.m.f. will be induced in the conductor since it is cutting across magnetic lines and the magnitude of this e.m.f. will be

$$e = \frac{H.l.v}{10^8} \text{ volts} \quad . \quad . \quad . \quad . \quad (38)$$

The direction of this induced e.m.f. will, by Lenz's Law, be such that it will oppose the effect which produces it, and in this case it will therefore reduce the current i and hence reduce the force to which the movement is due. Hence the resultant e.m.f. in the circuit will be

$$(E - e) \text{ volts} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and the current will be i amperes, where

$$(E - e) = i.R \quad . \quad . \quad . \quad . \quad . \quad (40)$$

For the relative direction of current and field shown in Fig. 36, the direction of F will be from left to right. The direction of the force E may be ascertained by an application of the "right-hand rule" of Fig. 31, it being observed that the induced e.m.f. will oppose the current. Generally, however, the more convenient rule for relating the factors involved in the case of an electric motor is Fleming's "*left-hand rule*" as follows:

Place the thumb, forefinger, and middle finger of the left hand mutually at right angles as shown in Fig. 37. If the forefinger points in the direction of the field and the middle finger in the direction of the current, then the thumb will point in the direction of the force which will act on the conductor—that is to say, the direction in which the conductor will tend to move.

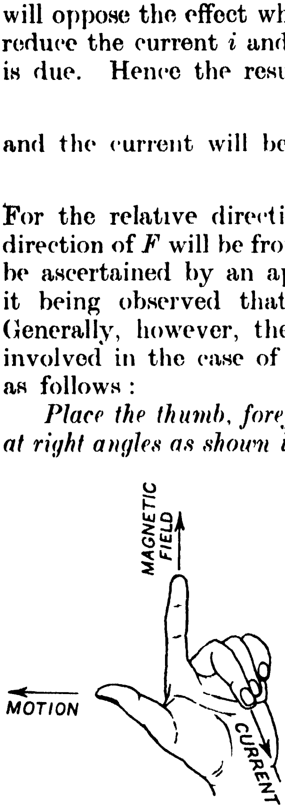


Fig. 37.

It will be seen, therefore, that the "*left-hand rule*" applies for the conditions of a *motor*, whilst the "*right-hand rule*" applies for the conditions of a *generator*.

Now if the mechanical force on the conductor, which is electromagnetically developed for the conditions shown in Fig. 36, is F dynes, the mechanical power which will be developed for a current of i_1 amperes will be

$$Fv \text{ ergs per second, i.e. } \frac{F.v}{10^7} \text{ watts,}$$

where v cm. per second is the speed at which the conductor is cutting across the magnetic field. Substituting for F from equation (37) gives the mechanical power developed by the motor as

$$\frac{H.l.i.v}{10^8} \text{ watts,}$$

and substituting for e from equation (38) it is seen that the mechanical power developed will be

$$e.i \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The induced e.m.f. e of the motor is termed the "*back e.m.f.*" of the motor, and the difference between the e.m.f. E of the battery and the back e.m.f. e of the motor is the pressure which is absorbed in driving the current of i amperes through the resistance of the whole circuit of battery and conductor of total value R ohms.

Any direct-current generator will run as a motor if the field system is excited and current is supplied to the armature winding. Thus, in the case of the machine shown in Fig. 35, if the lower brush is connected to the positive pole of a battery and the upper brush to the negative pole, current will flow in the armature winding in the reverse direction to that which is shown in Fig. 35, but the direction of rotation of the armature which is now due to its action as a motor will remain the same as that shown in Fig. 35, as will easily be seen by applying the left hand rule. The mechanical power developed by this motor will then be

$$e.i = (V - i.R_a)i \text{ watts} \quad . \quad . \quad . \quad . \quad (42)$$

where V volts is the p.d. applied to the brushes and R_a ohms the resistance of the armature winding. If the armature revolves at a speed of n revolutions per second, the torque τ will be given by the equation

$$\tau.2\pi n = e.i.10^7 \text{ ergs per sec.} \quad . \quad . \quad (43)$$

where τ is measured in dyne-centimetres so that

$$\tau = \frac{e.i.10^7}{2\pi n} \text{ dyne-centimetres} = \frac{e.i}{2\pi n \times 9{\cdot}81} \text{ kg.m.}$$

If τ is measured in lbs.-feet, then

$$\tau = 0{\cdot}118\frac{e.i}{n} \text{ lbs.-ft.} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

As a numerical example, suppose the length of the conductor which is in the magnetic field as shown in Fig. 36 is 50 cm. and that a current of 100 amperes is flowing in the conductor, the intensity of the magnetic field being $H = 10{,}000$ oersted. The force on the conductor will then be

$$F = 10{,}000 \times \frac{100}{10} \times 50 = 5 \times 10^6 \text{ dynes} = 11{\cdot}2 \text{ lbs.-weight.}$$

If the conductor is fixed to the surface of a cylindrical core of 2 feet diameter the torque on the core will be $\tau = 11{\cdot}2$ ft.-lbs. $= 1.55$ kg.m.

Inductance or Coefficient of Self-Induction

Suppose a coil of w turns is carrying a current of i amperes and that the magnetic flux Φ c.g.s. lines due to this current is linked with the whole of the w turns, then the product $\Phi.w$ is termed the "flux-linkages" of the coil. The number of flux-linkages per ampere is then

$$\frac{\Phi w}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and this quantity is a measure of the *inductance* of the coil. The practical unit of inductance is the "henry" and is defined as the flux-linkages per ampere divided by 10^8. That is to say, the inductance of the coil of w turns which is carrying a current of i amperes, and each turn of the coil being linked with the flux Φ c.g.s. lines is

$$L = \frac{\Phi w}{i \times 10^8} \text{ henry} \quad . \quad . \quad . \quad . \quad (46)$$

If the whole of the flux Φ is not linked with all the turns of the coil (as will usually be the case in practice), then the coefficient of self-induction is given by the expression

$$L = \frac{\Sigma(\Phi_x . w_x)}{10^8 . i} \text{ henry} \quad . \quad . \quad . \quad . \quad (47)$$

where the flux Φ_x is linked with w_x turns and the sum of the products such as $\Phi_x . w_x$ is taken to include all the turns of the coil.

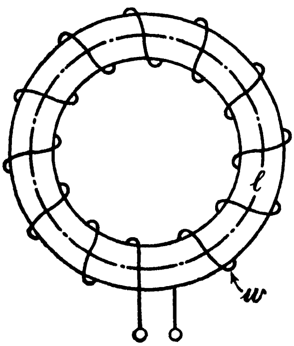


Fig. 38.

EXAMPLE - Suppose a ring of iron is uniformly wound with an exciting coil as shown in Fig. 38 and on pages 19 and 178. The cross sectional area of the iron is $A = 6$ sq cm., the mean length of the path of the magnetic flux in the ring is $l = 20$ cm. and the number of turns in the exciting coil is $w = 190$. The magnetisation curve for the iron is that shown in Fig. 4, page 181.

The relationship between the magnetising current i amperes and the corresponding magnetising force of H oersted is given by the expression (191), viz.

$$\frac{4\pi}{10} i . w = H . l$$

so that $$H = \frac{1 \cdot 26 \times 190}{20} i = 12\, i \text{ oersted.}$$

If the current in the coil is $i = 0 \cdot 15$ ampere, then $H = 1 \cdot 8$ oersted and the corresponding value of the induction density as obtained from the curve of Fig. 4, page 181 is $B = 9{,}200$ gauss. Hence the inductance for this particular value of the magnetising current is

$$L = \frac{\Phi . w}{10^8 i} = \frac{B . A . w}{10^8 i} = \frac{9{,}200 \times 6 \times 190}{0 \cdot 15 \times 10^8} = 0 \cdot 7 \text{ henry.}$$

It is to be observed that if there is no magnetic material in the neighbourhood of the coil, the coefficient of self-induction will be constant and

independent of the magnitude of the current. If, however, there is magnetisable material in the neighbourhood of the coil, the inductance will depend upon the strength of the current i and the magnetic characteristics of the material (see also page 181).

It will be seen from the foregoing, therefore, that *the inductance of a coil measured in henry is equal to the number of flux-linkages of the coil per ampere divided by* 10^8.

If the current is varying, there will be a back e.m.f. induced in the coil of magnitude (see page 226)

$$e = -\frac{d}{dt}\left(\frac{\Phi w}{10^8}\right) \text{ volts} \quad . \quad . \quad . \quad . \quad (48)$$

and since from expression (214) $\frac{\Phi w}{10^8} = Li$, it follows that

$$e = -\frac{d}{dt}(L.i) = -L\frac{di}{dt} \text{ volts} \quad . \quad . \quad . \quad (49)$$

Hence an alternative to the foregoing definition is the following: *The inductance of a coil in henry may be defined as being numerically equal to the induced e.m.f. in volts when the rate of change of the current is one ampere per second.*

Mutual Inductance or Coefficient of Mutual Induction

Suppose there are two coils, 1 and 2, as shown in Fig. 38, each having a concentrated winding, viz. coil 1 has w_1 turns and coil 2 has w_2 turns. Assume the coils are arranged in close association as diagrammatically indicated in Fig. 39 and such that, when a current flows in either coil, the magnetic flux so produced will be linked with all the turns of the other coil. For example, if a current of i_1 amperes in coil 1 produces a flux Φ_1 which is linked with all the w_1 turns of coil 1 as well as with all the w_2 turns of coil 2, then the flux-linkages per ampere of coil 1 will be $\frac{\Phi_1 w_1}{i_1}$, so that the inductance of this coil will be

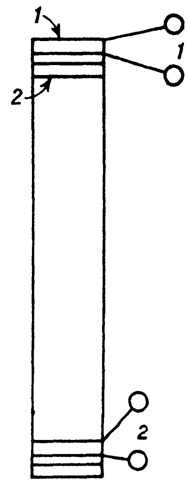


Fig. 39.

$$L_1 = \frac{\Phi_1 . w_1}{10^8 i_1} \text{ henry} \quad . \quad . \quad . \quad . \quad (50)$$

The corresponding flux-linkages of coil 2 will then be $\frac{\Phi_1 w_2}{i_1}$ and this quantity is a measure of the "mutual inductance" of the two coils, that is to say, the coefficient of mutual induction of the two coils will be given by

$$M = \frac{\Phi_1 w_2}{i_1 10^8} \text{ henry} \quad . \quad . \quad . \quad . \quad (51)$$

Similarly, if a current of i_2 amperes in coil 2 produces a flux Φ_2 which links with the whole of the w_2 turns of this coil as well as with the whole of the w_1 turns of coil 1, then the mutual inductance of the two coils will be

$$M = \frac{\Phi_2 w_1}{i_2 10^8} \text{ henry} \quad . \quad . \quad . \quad . \quad (52)$$

If now, R_M is the reluctance of the path of the magnetic flux due to the current in either of the coils 1 and 2, of Fig. 38, then from expression (26), on page 179,

$$\left.\begin{aligned} \frac{4\pi}{10} i_1 w_1 &= R_M . \Phi_1 \\ \frac{4\pi}{10} i_2 w_2 &= R_M . \Phi_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (53)$$

so that from cross-multiplication of these two expressions

$$\frac{\Phi_1 w_2}{i_1} = \frac{\Phi_2 w_1}{i_2} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

It is seen, therefore, that the two expressions (51) and (52) for the mutual inductance are identical. Stated in words, therefore, *the "mutual inductance" or "coefficient of mutual induction" of two coils measured in henry is equal to the (flux-linkages) × 10^{-8} of either coil when a current of 1 ampere is flowing in the other coil.*

Further, since

$$L_1 = \frac{\Phi_1 . w_1}{i_1 . 10^8} ; \; L_2 = \frac{\Phi_2 . w_2}{i_2 . 10^8}$$

$$M = \frac{\Phi_1 . w_2}{i_1 . 10^8} = \frac{\Phi_2 . w_1}{i_2 . 10^8}$$

it is seen that, for the conditions of Fig. 38, in which all the flux due to a current in one coil is assumed to be linked with all the turns of the other coil,

$$\left.\begin{aligned} L_1 &= M \frac{w_1}{w_2} \\ L_2 &= M \frac{w_2}{w_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and consequently

$$M^2 = L_1 . L_2 \quad \text{or} \quad M = \sqrt{L_1 . L_2} \quad . \quad . \quad . \quad (56)$$

and this is the highest value of the mutual inductance which two coils can have. When this condition holds, the "coupling coefficient" of the two coils is said to be unity or, otherwise stated, the "coupling factor" is

$$k = \frac{M}{\sqrt{L_1 L_2}} = 1 \quad . \quad . \quad . \quad . \quad (57)$$

Now suppose, as is generally the case in practice, that the two coils are separated so that when a current is flowing, for example, in coil 1, the whole of the flux so produced does not link with all the turns of coil 2. It will be clear that whilst in such a case the coefficient of self-induction of each coil remains unaffected by the separation of the two coils, the coefficient of mutual induction will be reduced—that is to say, the coupling factor will now be less than unity. In this case, therefore, the expression (55) may be written (see Fig. 39):

$$(L_1 - S_1) = M\frac{w_1}{w_2}$$

$$(L_2 - S_2) = M\frac{w_2}{w_1}$$

that is, $$(L_1 - S_1)(L_2 - S_2) = M^2$$

where S_1 henry is the "coefficient of leakage induction" of coil 1 and S_2 henry is the "coefficient of leakage induction" of coil 2, so that

$$M = \sqrt{(L_1 - S_1)(L_2 - S_2)} \quad . \quad . \quad . \quad (58)$$

and the coupling factor is now

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

and is less than unity. When the coupling factor k is not much less than unity the condition is said to be one of "tight coupling", and when the factor k is much less than unity the coupling is said to be "loose". The expression (58) shows that two coils having mutual induction relationship may be represented diagrammatically as shown in Fig. 40.

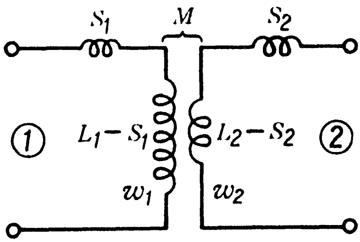


Fig. 40.

Since in the case for which there is no leakage inductance, the quantity $L_1L_2 - M^2$ is equal to zero, the magnitude of this quantity in the general case may be taken to be a measure of the leakage inductance, and various leakage factors of practical importance may then be defined as follows. The "total leakage factor", sometimes known as the "Blondel leakage factor", is

$$\tau = \frac{L_1L_2 - M^2}{L_1L_2} \quad . \quad . \quad . \quad . \quad (59)$$

so that $$(1 - \tau) = \frac{M^2}{L_1L_2} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and the coupling factor is

$$k = \frac{M}{\sqrt{L_1L_2}} = \sqrt{(1 - \tau)} \quad . \quad . \quad . \quad . \quad (61)$$

Writing $\frac{w_1}{w_2} = u$, then

$$(L_1 - S_1) = M.u$$

$$(L_2 - S_2) = M.\frac{1}{u}$$

writing also $$\tau_1 = \frac{S_1}{L_1 - S_1} : \tau_2 = \frac{S_2}{L_2 - S_2}$$

then $$\frac{1}{(1 + \tau_1)} \cdot \frac{1}{(1 + \tau_2)} = \frac{M^2}{L_1.L_2} = 1 - \tau$$

so that $$\tau = \frac{L_1.L_2 - M^2}{L_1.L_2} = 1 - \frac{1}{(1 + \tau_1)(1 + \tau_2)}. \quad . \quad (62)$$

If τ_1 and τ_2 are each small in comparison with unity, then

$$\tau \triangleq 1 - (1 - \tau_1)(1 - \tau_2) \triangleq \tau_1 + \tau_2 . \quad . \quad . \quad (63)$$

It has already been pointed out that if there is any magnetisable material in the path of the magnetic flux, the inductance will not be a constant quantity but will depend upon the flux density in that material, that is, upon the magnitude of the current in the coil. If a current of i amperes produces a flux-linkage Q, then the coefficient of self-induction for the particular value of the flux density in the magnetic material which corresponds to that value of the current will be $\frac{Q}{10^8.i}$ henry. In order to find the value of the inductance for any given value of the magnetising current, it is necessary to refer to the magnetisation characteristic of the magnetic material concerned.

The Energy of the Magnetic Field

It has been seen in expression (50) that when a current of i amperes is flowing in a coil of self-induction L henry, the flux-linkages of the coil will be $L.i.10^8$. It has also been seen on page 217, expression (20), that the work done in causing a conductor which carries a current of i amperes to link with a flux of Φ maxwells is $\frac{\Phi.i}{10}$ ergs. In the case of a coil of self-induction L henry the work done in increasing the current from i to $i + \delta i$ amperes is intermediate between

$$\frac{\delta Q.i}{10} \text{ and } \frac{\delta Q(i + \delta i)}{10} \text{ ergs,}$$

where Q is the total flux-linkages when the current of i amperes is flowing in the coil, that is, $Q = \Sigma\Phi_x.w_x$. In the notation of the calculus when the increments are indefinitely small, the work done is

$$i\frac{dQ}{10} \text{ ergs,}$$

where $dQ = 10^8 d(L.i)$. Hence the work done is

$$10^7 Li\,di \text{ ergs} = Li\,di \text{ joules.}$$

The total work done, therefore, in establishing the current of i amperes in a coil of self-induction of L henry is

$$\int_0^i L.i\,di \text{ joules} = \tfrac{1}{2}Li^2 \text{ joules} \quad . \quad . \quad . \quad (64)$$

Otherwise expressed, it may be said that the work done in establishing a current of i amperes in a coil of self-induction of L henry is

$$\tfrac{1}{2}Li^2 = \tfrac{1}{2}(Li)i - \tfrac{1}{2}\left\{\frac{\text{Flux-linkages}}{10^8}\right\}i \text{ joules} \quad . \quad . \quad (65)$$

The same result may be arrived at in a somewhat different way as follows. The back e.m.f. induced in the coil when the current increases from i to $i + \delta i$ amperes is

$$e = L\frac{di}{dt} \text{ volts.}$$

The energy which is supplied in the time δt second in order that the current of i amperes may be driven against the back e.m.f. of e volts is

$$v.i.\delta t = e.i.\delta t = Li\frac{di}{dt}\delta t \text{ joules,}$$

where v is the p.d. of the source which supplies the current of i amperes. The total energy supplied in establishing the current is therefore

$$V = \int_0^t L.i\frac{di}{dt}dt = \int_0^i Li\,di = \tfrac{1}{2}\,Li^2 \text{ joules.}$$

Consider next the case of two coils which are magnetically coupled as shown diagrammatically in Fig. 40. If the mutual inductance is M henry and the self-inductances of the individual coils are respectively L_1 and L_2 henry, then by definition of mutual inductance, the $\left\{\frac{\text{flux-linkages}}{10^8}\right\}$ of the coil 1 due to a current of i_2 amperes in coil 2 will be Mi_2, so that the energy which is expended in establishing these flux-linkages in coil 1 will be in accordance with expression (65)

$$\tfrac{1}{2}(Mi_2)i_1 \text{ joules.}$$

Similarly, the $\left\{\frac{\text{flux-linkages}}{10^8}\right\}$ of the coil 2 due to the current of i_1 amperes in coil 1 will be $M.i_1$, so that the energy expended in establishing these flux linkages in coil 2 will be

$$\tfrac{1}{2}(Mi_1)i_2 \text{ joules.}$$

The total energy expended, therefore, in establishing the electromagnetic field due to the mutual inductance between the two coils will be

$$M.i_1.i_2 \text{ joules,}$$

and consequently the total energy expended is establishing the electromagnetic field of the whole system comprising two coils in mutual induction relationship with each other will be

$$\tfrac{1}{2}L_1 i_1^2 + \tfrac{1}{2}L_2 i_2^2 \pm M i_1 . i_2 \text{ joules} \quad . \quad . \quad . \quad (66)$$

It is to be noted that the product $i_1 . i_2$ in this expression will be positive or negative according to whether the currents i_1 and i_2 have the same signs or opposite signs. The expression (66) may also be derived from considerations of the induced back e.m.f. in the respective coils as has been explained in the foregoing with regards to the expression $\tfrac{1}{2}Li^2$ for a single coil.

The foregoing results hold if the inductances are constant throughout the range of flux considered, that is to say, if the flux is proportional to the current throughout the range considered. In order to calculate the energy stored in the electromagnetic field when there is iron in the magnetic circuit, it is necessary to have reference to the magnetisation curve. For example, in the case of an iron ring magnetised by means of a closely and uniformly wound exciting coil as shown in Fig. 2, Chapter VII, page 178, suppose that the magnetisation curve for the iron is as shown in Fig. 41. The energy expended in establishing the magnetic flux in the ring is

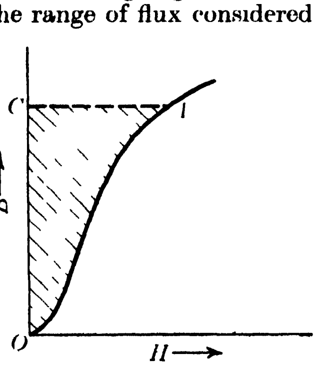


Fig. 41.

$$\int ei\, dt \text{ joules,}$$

where e volts is the back e.m.f. induced by the changing flux and i ampere is the corresponding current in the coil, the integration of this expression being taken over the time required to establish the flux in the iron ring. Hence the energy expended during the magnetisation process is

$$\frac{1}{10^8}\int \frac{d}{dt}(B.A.w)i\, dt \text{ joules}$$

since $e = -\dfrac{d}{dt}\left(\dfrac{B.A.w}{10^8}\right)$, where w is the number of turns in the exciting coil. The energy expended in magnetising the iron ring is therefore

$$U = \frac{1}{4\pi \times 10^7}\int_0^B H.l.A.dB \text{ joules,}$$

since $H.l = \dfrac{4\pi}{10} i.w$, and $l \times A$ c.cm. is the volume of the iron so that

$$U = \frac{1}{4\pi . 10^7}\int_0^B HdB \text{ joules per c.cm.} \quad . \quad . \quad . \quad (67)$$

The energy expended, in *ergs* per cubic centimetre of the iron, in order to produce a flux density of B gauss is therefore equal to the shaded area shown in Fig. 41 divided by 4π.

Energy Loss in Iron Due to Magnetic Hysteresis

Suppose in Fig. 42 *a* the initial magnetic condition of a piece of iron is defined by the point P and that the iron is then subjected to an increasing magnetising force H, so that the magnetising process is defined by the curve PQR in Fig. 42 *a*, the ultimate state of magnetisation being defined by the point R. Then, from the results which have been derived on page 242, it will be seen that the energy *expended* during this magnetising process will be

$$U_1 = \frac{\{\text{Shaded Area } pPRr \text{ in } B : H \text{ units}\}}{4\pi} \text{ ergs per c. cm.}$$

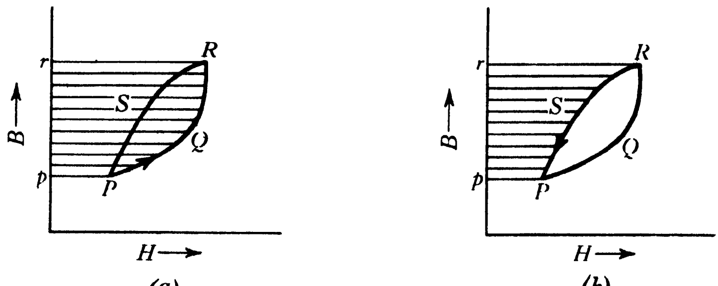


Fig. 42.

If, now, the magnetising force be gradually reduced so that the iron is subjected to a demagnetising process as defined by the curve RSP in Fig. 42 *b*, the magnetic condition will eventually reach the initial condition as defined by the point P. By means of reasoning similar to that considered on page 242 it can be shown that during this demagnetising process an amount of energy U_2 will be *recovered*,

$$U_2 = \frac{\{\text{Shaded Area } RrpP \text{ (Fig. 42 } b\text{) in } B : H \text{ Units}\}}{4\pi} \text{ ergs per c. cm.}$$

It follows therefore that the net *expenditure* of energy in passing through the magnetic cycle $PQRSP$ will be given by

$$U_h = U_1 - U_2 = \frac{\{\text{Area of the Loop } PQRSP\}}{4\pi} \text{ ergs per c. cm.}$$

For the particular case in which the iron is magnetised in accordance with the symmetrical cycle $PQRSP$ in Fig. 43 the " hysteresis loss " will be given by

$$U_h = \frac{\{\text{Area of the Hysteresis Loop } PQRP \text{ in } B : H \text{ units}\}}{4\pi} \text{ ergs per c. cm.}$$

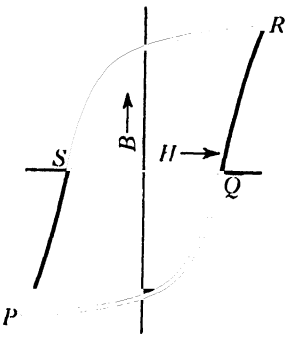


Fig. 43.

Growth of the Current in a Coil of Inductance L Henry and Resistance R Ohms

Suppose the coil in Fig. 44 is connected to a source of steady p.d. of

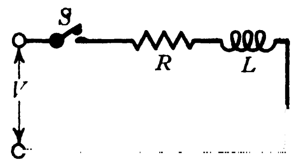


Fig. 44.

V volts. A current will then begin to flow in the coil and a corresponding magnetic flux will be developed linking the coil winding. By Lenz's Law the increasing magnetic flux will induce a back e.m.f. of magnitude $e = L\frac{di}{dt}$ volts, and acting in the direction such that it will oppose the growth of the current. The actual effective e.m.f. in the circuit at any instant t will then be

$$V - e = V - L\frac{di}{dt} \text{ volts,}$$

and this is the e.m.f. which is available for driving the current through the resistance of the coil of R ohms. Hence

$$V - L\frac{di}{dt} = Ri, \quad \text{or} \quad V = L\frac{di}{dt} + Ri. \qquad . \quad . \quad (68)$$

and from this equation the value of the current may be deduced at any instant after closing the switch S, Fig. 44. Since at the moment of closing the switch the current is zero, it follows that

$$\left.\frac{di}{dt}\right]_{t=0} = \frac{V}{L},$$

so that a line drawn through the origin in Fig. 45 and at such an angle θ_1 that $\tan \theta_1 = \frac{V}{L}$ will give the initial slope of the curve of rise of current. After a small time interval δt the current will be $i_1 = \delta t \tan \theta_1$ and hence at this instant, $t = t_1$:

$$V - L\left.\frac{di}{dt}\right]_{t=t_1} + Ri_1$$

and
$$\left.\frac{di}{dt}\right]_{t=t_1} = \frac{V - Ri_1}{L} = \tan \theta_2.$$

This expression, therefore, defines the slope of the curve at the instant

$t = t_1$. In this way the whole curve showing the rise of current as a function of the time may be plotted and, by taking the time intervals δt between the successive points sufficiently small, the curve of current growth may be plotted with as high a degree of accuracy as may be desired. This graphical method for determining the shape of the curve of current growth in an inductive circuit is particularly useful in those cases in which the magnetic circuit includes magnetisable material. Under these conditions the value of the inductance L will not be constant, but will be a function of the current.

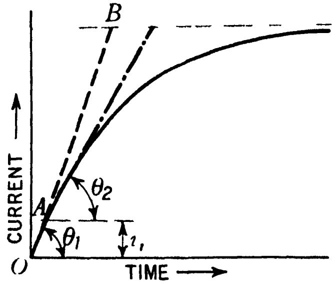


Fig. 45.

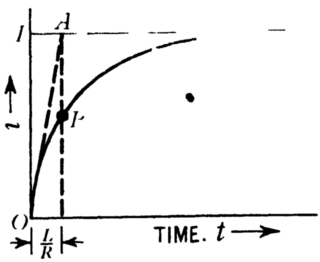


Fig. 46.

When the inductance L can be assumed to be constant or approximately constant, the most convenient way for obtaining the curve of current rise is to write down the solution of the differential equation (68), viz.

$$i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) = I(1 - e^{-t/T_0}) . \qquad . \qquad . \quad (69)$$

where $I = \frac{V}{R}$ and is the final steady value to which the current will rise. This relationship is shown graphically in Fig. 46 and the quantity $\frac{L}{R}$ is termed the "electro-magnetic time constant" T_0 of the circuit, that is to say, it is the time which would be required for the current to reach its final steady value I if the initial rate of rise were to remain constant. It is of interest to note that the actual value of the current at the time $t = \left(\text{time constant } \frac{L}{R}\right)$ is given by the point P in Fig. 45. That is to say, inserting $t = \frac{L}{R} = T_0$ in the equation (69) gives

$$i_P = I(1 - e^{-1}) = I(1 - 0{\cdot}368) = 0{\cdot}632I,$$

since $e = 2{\cdot}73$ . . . , that is, the base of natural logarithms. Hence the time constant of the circuit may also be defined as the time required for the current to attain 63·2 per cent. of its final steady value.

Chapter IX

ALTERNATING CURRENTS—THE USE OF COMPLEX QUANTITIES

Alternating Currents of Sine Wave Form

SUPPOSE the current varies with the time according to a sine law—viz. :

$$i = I_m \sin \omega t \text{ amperes,}$$

where t is the time in seconds, ω is a constant, and I_m the maximum value which the current attains during a cycle. Such a wave form is taken as the standard in electrical engineering. Of all possible wave forms it is the simplest to deal with mathematically, and it can be proved that any continuous periodic function can be analysed into a series of sine waves of different frequencies (see Chapter XIII).

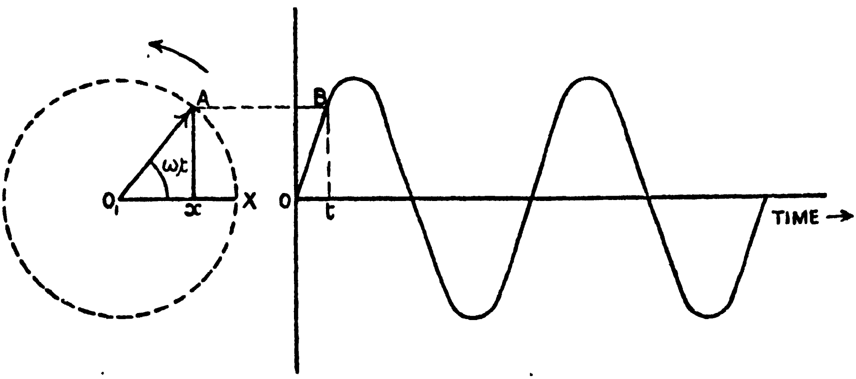


Fig. 1*a*.

The e.m.f. wave forms of modern alternators are very approximately sine curves, and, as will be seen later, the sine wave form of e.m.f. has the advantage that there is no component of high frequency which might give rise to accidental resonance effects in circuits containing inductance and capacity.

Suppose a circle be drawn (Fig. 1*a*) of radius O_1A representing to scale the maximum value I_m of the current, and suppose O_1A revolves about O with a uniform angular velocity ω radians per second in a counter-clockwise direction.

Let O_1X be the starting position of O_1A (i.e. let the zero of time

be reckoned from the moment when O_1A coincides with O_1X). At the time t, the angle which O_1A makes with O_1X will therefore be—

$$\omega t \text{ radians.}$$

From A draw Ax perpendicular to O_1X.

Then
$$\begin{aligned} Ax &= O_1A \sin \omega t \\ &= I_m \sin \omega t\,; \end{aligned}$$

or Ax represents the value at the time t of the alternating current of sine wave form, of which the maximum value is I_m.

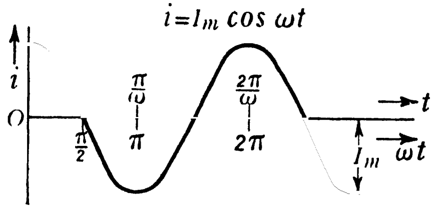


Fig. 1*b*.

Hence, if the values of t be plotted as abscissæ, the corresponding projections of O_1A on the vertical will give the sine wave form of current.

Since the angular velocity of O_1A is ω radians per second it follows that O_1A makes $\frac{\omega}{2\pi}$ revolutions per second—i.e. the current passes through $\frac{\omega}{2\pi}$ cycles in one second.

Hence the relationship between the frequency f and the angular velocity ω is—

$$f = \frac{\omega}{2\pi}.$$

The time of one complete cycle is—

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega} \text{ second.}$$

The quantity ω is also termed the " circular frequency ".

In the following the symbols I_m, V_m, E_m, will be used to represent the maximum values (or amplitudes) of sine waves of current, potential difference, and e.m.f. respectively.

In Fig. 1*b* is shown the current wave form as defined by the expression,

$$i = I_m \cos \omega t.$$

Vector Method of representing Sine Wave Forms of Alternating Current

In Fig. 2 let the length of the vector OA represent the maximum value I_m of a sine wave form of current, the direction of the vector being from O to A. If the vector rotate with uniform angular velocity ω in a counter-clockwise direction it has been shown in the preceding paragraph that the projection of the vector on the vertical represents the instantaneous value of the current $i = I_m \sin \omega t$ at the time t, the zero of time being taken as the moment at which OA coincides with OX.

When the vector OA is in quadrant I or quadrant II the current is positive, and when OA is in quadrant III or quadrant IV the current is negative. From the numerous examples given in subsequent pages it will be seen that the vector method forms a very powerful and simple means of dealing with alternating current problems.

The remarks made in the foregoing paragraphs with reference to alternating currents apply also to alternating e.m.f.s and alternating p.d.s of sine wave forms.

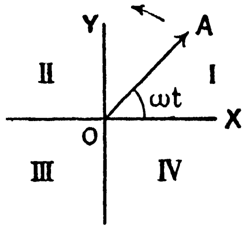


Fig. 2.

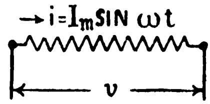


Fig. 3.

P.D. necessary to send an Alternating Current of Sine Wave Form through a Non-inductive Resistance

Let R ohms be the value of the resistance and let the current be $i = I_m \sin \omega t$ amperes (Fig. 3)

The instantaneous value of the p.d. across the resistance will then be

$$v = iR = I_m R \sin \omega t \text{ volts};$$

or $$v = V_m \sin \omega t \text{ volts},$$

where $$V_m = I_m R \text{ volts}.$$

The p.d. across the resistance is of sine wave form, and is proportional at every instant to the current at that instant.

EXAMPLE.—Let

$$R = 5 \text{ ohms} : f = 50 : \omega = 2\pi \times 50 = 314$$

$$T = \frac{1}{50} = 0{\cdot}02 \text{ second}$$

$$i = 20 \sin \omega t = 20 \sin 314t \text{ amperes.}$$

Then $$v = 20 \times 5 \sin 314t$$
$$= 100 \sin 314t \text{ volts.}$$

The waves of current and p.d. are shown in Fig. 4. The two waves pass through their zero values simultaneously, and also reach their maximum values simultaneously. The two waves are therefore said to be *in phase*, and it is a characteristic feature of a non-inductive resistance that the p.d. across the resistance is in phase with the current flowing in the resistance.

The vector diagram for the current and p d. in this case is given in Fig. 5. The vector OA represents the current and the vector OB the p.d., the two vectors coinciding in direction.

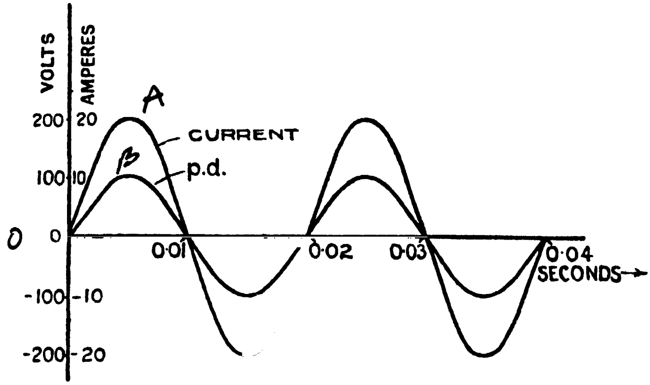


Fig. 4.

P.D. necessary to send an Alternating Current of Sine Wave Form through an Inductance Coil of Negligibly Small Resistance

Let the inductance of the coil be L henrys (Fig. 6). It was shown on p. 237, Chapter VIII, that if a current of i amperes flowing in a coil of inductance L henrys varies at the rate $\frac{di}{dt}$, a back e.m.f. of $L\frac{di}{dt}$ volts is developed which tends to oppose the variation of the current.

If the current is

$$i = I_m \sin \omega t \text{ amperes,}$$

the back e.m.f. developed in the coil is

$$L\frac{di}{dt} = LI_m\omega \cos \omega t \text{ volts,}$$

the direction of this back e.m.f. at any instant being such as to oppose the change of current at that instant. Hence in order to force the

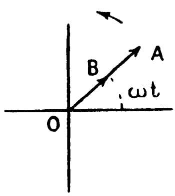


Fig. 5.

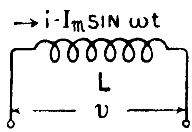


Fig. 6.

current to vary as defined by the expression $I_m \sin \omega t$, a p.d. must be applied to the terminals of the coil, such as to be exactly *equal and opposite* to the back e.m.f. $-LI_m\omega \cos \omega t$. The applied p.d. is therefore—

$$v = I_m L\omega \cos \omega t \text{ volts}$$
$$= V_m \cos \omega t \text{ volts ;}$$

where $$V_m = I_m L\omega \text{ volts.}$$

The quantity $L\omega$ is termed the **inductive reactance** of the coil, and is measured in ohms.

EXAMPLE.—Let

$$L = 0{\cdot}01 \text{ henry} : f = 50 : \omega = 2\pi f = 314.$$
$$L\omega = 3{\cdot}14 \text{ ohms} : i = 30 \sin \omega t = 30 \sin 314t \text{ amperes.}$$

Then $\quad v = V_m \cos \omega t = I_m L\omega \cos wt$ volts,

or $\quad v = 94 \cos 314t$ volts.

In Fig. 7 the waves of current and applied p.d. are shown, and it will be seen that the current reaches its maximum positive value one-quarter of a period $\left(\text{i.e. } \frac{\pi}{2} \text{ radians or } 90° \text{ in the vector diagram}\right)$ later than the applied p.d. The current is therefore said to *lag* on the applied p.d. by one-quarter of a period or 90° in the vector diagram.

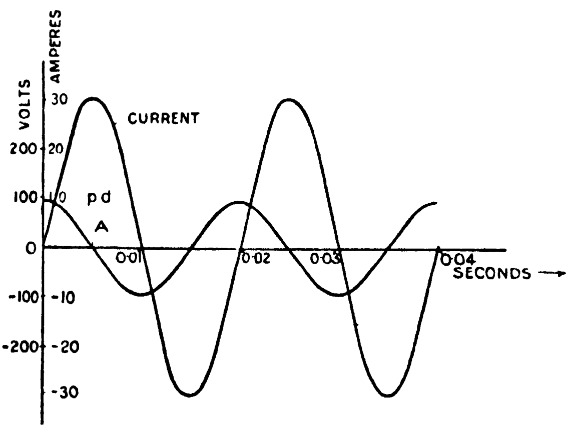


Fig. 7.

It is characteristic of a purely inductive coil (i.e. a coil of negligibly small resistance) that the sine wave of alternating current *lags* by 90° on the applied sine wave of p.d.

The vector diagram of this case is shown in Fig. 8, in which OA is the vector of current and OB the applied p.d.

It follows at once that if the applied p.d. is

$$v = V_m \sin \omega t,$$

the current will be—

$$i = \frac{V_m}{L\omega} \sin (\omega t - 90°) = -\frac{V_m}{L\omega} \cos \omega t\,;$$

that is, lagging by a quarter period on the applied p.d.

It is of interest to consider briefly the physical meaning of the fact that the alternating current in an inductive coil lags by 90° on the

applied p.d. Take the instant at which the current is a positive maximum—e.g. the time moment A (Fig. 7). At this moment the current, having reached a maximum value, is momentarily steady, and since the resistance of the coil is zero no p.d. is necessary to keep the current flowing, and the wave of applied p.d. therefore passes through its zero value at the time A. The current then begins to diminish and consequently the magnetic flux threading the coil diminishes.

According to Lenz's Law, page 224, however, a back e.m.f. is induced by the shrinking flux, and tends to keep the current flowing undiminished in strength. Hence the applied p.d., which must oppose this back e.m.f., is then negative. By similar reasoning the direction of the applied p.d. at every instant may be deduced.

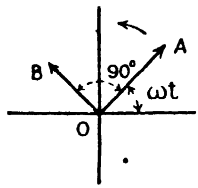


Fig. 8.

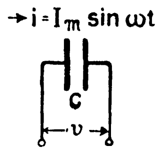


Fig. 9.

The Alternating Current through a Condenser due to a Sine Wave Form of Applied P.D.

Let C be the capacity of the condenser (Fig. 9) in farads, and let the applied p.d. be given by

$$v = -V_m \cos \omega t \text{ volts (see Fig. 10).}$$

If at any time t the quantity (in coulombs) of the charge of the condenser is Q, the p.d. across the condenser plates is—

$$v = \frac{Q}{C} \text{ volts.}$$

The current through the condenser is (see also page 41, Chapter II.)

$$i = \frac{dQ}{dt} \text{ amperes,}$$

$$= C\frac{dv}{dt} \text{ amperes,}$$

the condenser being assumed to be uncharged when $t = o$.

That is $\quad i = \omega C V_m \sin \omega t$ amperes,

or $\quad i = I_m \sin \omega t$ amperes,

where $\quad I_m = \omega C V_m$,

or $\quad V_m = \dfrac{I_m}{\omega C}$.

The quantity $\frac{1}{\omega C}$ is termed the **capacitance reactance,** and is measured in ohms.

EXAMPLE.—Let $C = 200 \times 10^{-6}$ farads $= 200$ micro-farads : $f = 50$: $\omega = 2\pi f = 314$: $\frac{1}{\omega C} = 16$ ohms : $v = -V_m \cos \omega t = -128 \cos 314t$.

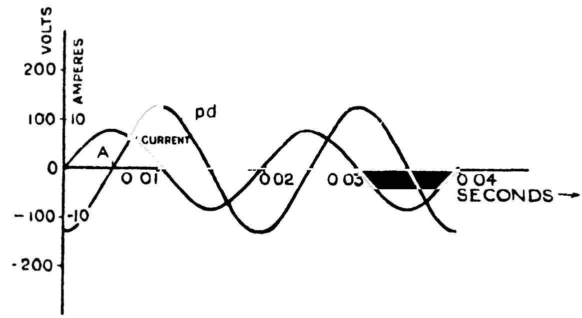


Fig. 10.

Then

$$i = \omega C V_m \sin \omega t = \frac{314 \times 200 \times 128}{10^6} \sin \omega t \text{ amperes,}$$

that is $$i = 8 \sin 314t \text{ amperes.}$$

The waves of current and applied p.d. for this case are shown in Fig. 10. It will be observed that the current reaches its maximum value one quarter period before the applied p.d.—i.e. the current *leads* on the applied p.d. by 90°. It is a characteristic feature of a condenser in an alternating current circuit that the sine wave of current leads by 90° on the applied p.d., and this is a valuable feature which has important practical applications.

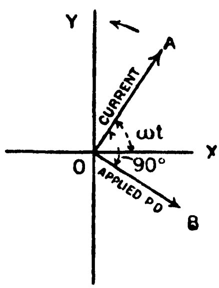


Fig. 11.

The vector diagram for the case of an alternating current through a condenser is given in Fig. 11. The current vector is OA and the vector of applied p.d. is OB.

Since $V_m = \frac{I_m}{\omega C}$ or $I_m = V_m \omega C$ it follows that if ωC is large a small value of the applied p.d. will produce a large current in the condenser. The higher the frequency the larger the current which a given applied p.d. will produce through a given condenser. Important practical consequences of this result arise when the wave of applied p.d. comprises harmonics (see Chapter XIII).

If the applied p.d. at the condenser terminals is given by the expression—

$$v = V_m \sin \omega t \text{ volts,}$$

it will be clear from the foregoing that the current will be given by the expression—

$$\begin{aligned} i &= I_m \sin (\omega t + 90°) \\ &= V_m \omega C \sin (\omega t + 90°) \\ &= V_m \omega C \cos \omega t \text{ amperes.} \end{aligned}$$

The physical meaning of the fact that the current through the condenser leads by 90° on the applied p.d. may be seen from the following considerations :

In Fig. 12 let the positive direction of the applied p.d. be such as to charge the plate A positively and the plate B negatively. The positive direction of the current will then be from A to B through the condenser.

Consider the moment immediately after time 0·01 second (Fig. 10). The current in the circuit begins to flow in the negative direction—i.e. from plate A through the generator G to the plate B Fig. 12 This means that at the moment 0·01 second the condenser has its maximum positive

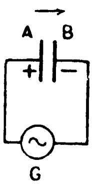


Fig. 12.

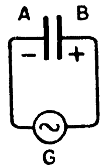


Fig. 13.

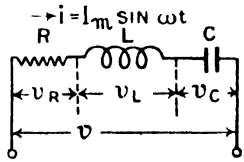


Fig. 14.

charge, and the next moment the charge has diminished. Since the applied p.d. is directly proportional to the charge, the applied p.d. at time 0·01 second (Fig. 10) has its maximum positive value. Similarly at the time 0·02 second (Fig. 10) the current is zero and increasing in a positive direction, which means that the condenser charge at this moment has its maximum negative value, and the current begins to flow from B through the generator G to the plate A (Fig. 13)—i.e. in the positive direction.

Applied P.D. necessary to send an Alternating Current of Sine Wave Form through a Resistance, Inductance, and Condenser in Series

In Fig. 14 let v_R be the p.d. across the resistance R ; v_L the p.d. across the inductance L ; v_C the p.d. across the condenser C ; and let $i = I_m \sin \omega t$ amperes be the current through the series arrangement at any moment t. Then, from the results already obtained in the foregoing,

$$\begin{aligned} v_R &= I_m R \sin \omega t \text{ volts,} \\ v_L &= I_m L\omega \cos \omega t \text{ volts,} \\ v_C &= -\frac{I_m}{\omega C} \cos \omega t \text{ volts.} \end{aligned}$$

The applied p.d. across the series arrangement at the moment considered is —

$$v = v_R + v_L + v_C = I_m\left[R \sin \omega t + \left(L\omega - \frac{1}{\omega C}\right) \cos \omega t\right] \text{ volts};$$

that is
$$v = I_m Z\left\{\frac{R}{Z} \sin \omega t + \frac{L\omega - \dfrac{1}{\omega C}}{Z} \cos \omega t\right\} \text{volts},$$

where
$$Z = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}.$$

That is —
$$v = V_m \sin (\omega t + \phi) \text{ volts},$$

where
$$V_m = I_m\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}$$

and
$$\tan \phi = \frac{L\omega - \dfrac{1}{\omega C}}{R}.$$

The quantity $\sqrt{R^2 + \left(L\omega - \dfrac{1}{\omega C}\right)^2}$, denoted by the letter Z, is termed the **impedance** of the circuit. The quantity $\left(L\omega - \dfrac{1}{\omega C}\right)$ is termed the **reactance** of the circuit, and is denoted by the letter X.

The above is the most general case of an alternating current series circuit, and each of the individual cases considered in the foregoing may be deduced from this general result.

It is to be observed that if the condenser is out of the circuit (or. what is the same thing, is short-circuited by a piece of thick copper wire), the condition is expressed by putting C equal to infinity in the formula. Thus, for a circuit containing an inductance L and resistance R in series—

$$C = \text{infinity and } \tan \phi = \frac{L\omega}{R};$$

$$v = I_m\sqrt{R^2 + (L\omega)^2} \sin (\omega t + \phi) \text{ volts}.$$

The results may be expressed generally as follows:

If a sine wave of alternating current flows in a series connected circuit, the applied p.d. must be of sine wave form and displaced in phase relatively to the current wave by an angle ϕ where

$$\tan \phi = \frac{L\omega - \dfrac{1}{\omega C}}{R}.$$

If $L\omega$ is $> \frac{1}{\omega C}$ the current *lags* on the applied p.d., and ϕ in the above formula is positive.

If $L\omega$ is $< \frac{1}{\omega C}$ the current *leads* on the applied p.d., and ϕ in the above formula is negative.

If $L\omega = \frac{1}{\omega C}$ the current is *in phase* with the applied p.d. This is a special and important case which will be considered more fully in what follows.

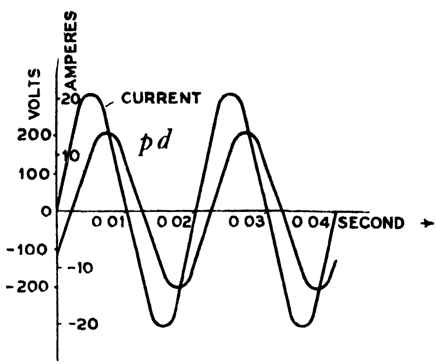


Fig. 15.

Since for the current $i = I_m \sin \omega t$ amperes, the necessary applied p.d. is $v = V_m \sin (\omega t + \phi)$ volts, it follows at once that if the applied p.d. is $v = V_m \sin \omega t$ volts the current will be $i = I_m (\sin \omega t - \phi)$ amperes.

A more general treatment of the case of an alternating current series circuit is given later (see also Chapter X).

Example.—Let $L = 0{\cdot}03$ henry : $C = 200$ micro-farads : $f = 50$: $\omega = 2\pi f = 314$: $R = 8$ ohms : $L\omega = 9{\cdot}42$ ohms : $\frac{1}{\omega C} = 16$ ohms :

$L\omega - \frac{1}{\omega C} = -\ 6{\cdot}58$ ohms : $\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} = 10{\cdot}3$:

$\tan \phi = \frac{-\ 6{\cdot}58}{8} = -\ 0{\cdot}825$: $\phi = -\ 39° \ 30'$. Hence the current *leads* on the applied p.d. by 39° 30′.

If
$$i = 20 \sin \omega t = 20 \sin 314t \text{ amperes}$$
$$v = 206 \sin (314t - 39° \ 30') \text{ volts.}$$

In Fig. 15 the waves of current and applied p.d. for this case are shown.

The same result is, of course, obtained if the component p.d. waves v_R, v_L, and v_C are drawn and the corresponding ordinates added.

Thus
$$v_R = 160 \sin 314t :$$
$$v_L = 188{\cdot}4 \cos 314t :$$
$$v_C = -\ 320 \cos \omega t.$$

These component waves of p.d. are shown in Fig. 16.

The vector diagram for such a series circuit is shown in Fig. 17.

The current vector is OA, the vector of p.d. V_{mR} is OB, the vector

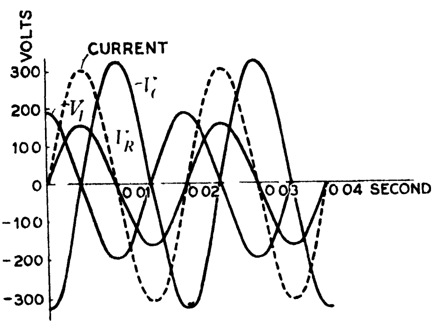


Fig. 16.

of p.d. V_{mL} is OG, and the vector of p.d. V_{mC} is OD. OK is the vector sum of OB, OG, and OD, and is the vector of applied p.d. V_m. The current vector OA leads by the angle ϕ on the vector OK of applied p.d.

Resonance.

As was pointed out in the previous paragraph, a special and important case of a series circuit is one in which $L\omega = \dfrac{1}{\omega C}$. In this case the angle ϕ is zero -that is, the current is in phase with the applied p.d.

If
$$L\omega = \frac{1}{\omega C},$$

then
$$\omega^2 = \frac{1}{LC};$$

or
$$\omega = \frac{1}{\sqrt{LC}}$$

and $$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}.$$

As will be seen later (page 270) $\frac{1}{2\pi\sqrt{LC}}$ is the frequency at which a current, started in a closed circuit of negligibly small resistance, and having an inductance of L henry and capacitance C farads (Fig. 18), would continue to oscillate; or, in other words, $\frac{1}{2\pi\sqrt{LC}}$ is the *natural frequency* of the circuit. If the applied p.d. has the frequency $f = \frac{1}{2\pi\sqrt{LC}}$ the circuit is said to be in *resonance* with the supply frequency. It will

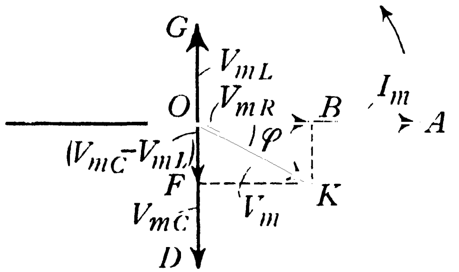


Fig. 17.

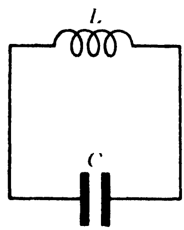


Fig. 18.

also be seen later (p. 270) that for relatively moderate values of the resistance R in the circuit the natural frequency is practically independent of the resistance of the circuit.

The condition of resonance is—

$$L\omega = \frac{1}{\omega C},$$

and since
$$v_L = L\omega I_m \cos \omega t$$
$$v_C = \frac{I_m}{-\omega C} \cos \omega t,$$

it follows that
$$v_L = -v_C$$

and
$$v = v_R + v_L + v_C = v_R;$$

that is
$$v = I_m R \sin \omega t \text{ volts.}$$

Hence, if R is small the current I_m may attain a very high value for a moderate value of the applied p.d. The consequence of this is that V_{mL} and V_{mC} may reach values which are enormously higher than the value of the applied p.d. V_m.

EXAMPLE.—Let L = 0·03 henry : $C = 200$ micro-farads :
$f = \dfrac{1}{2\pi\sqrt{LC}} = 65$: $\omega = 2\pi f = 408$: $L\omega = \dfrac{1}{\omega C}$: $V_m = 100$ volts : $R = 1$ ohm.
Then $I_m = 100$ amperes : $V_{mL} = L\omega I_m = 1{,}224$ volts $= V_{mC}$.

The maximum value of the pressure across the inductance and across the capacity is thus about ten times greater than the maximum value of the applied p.d. across the whole circuit, and this high value of the pressure might be sufficient to rupture the insulation of the inductance and condenser.

The vector diagram for this case is shown in Fig. 19.

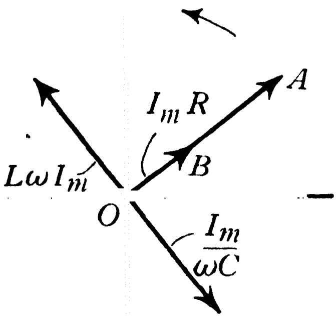


Fig. 19.

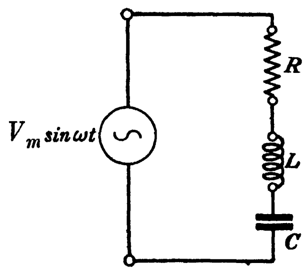


Fig. 20.

Resonance Conditions for an Oscillatory Circuit

If a p.d. $V_m \sin \omega t$ is applied to the terminals of a series-connected oscillatory circuit as shown in Fig. 20, the applied p.d. at any moment t must be equal to the sum of the back e.m.f. components so that

$$L\frac{di}{dt} + Ri + \frac{q}{C} = V_m \sin \omega t$$

or, since $i = \dfrac{dq}{dt}$, this equation may be written in the equivalent form

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_m \sin \omega t \qquad . \quad . \quad . \quad (1)$$

where q coulombs is the quantity of electricity which is stored in the capacitance at any moment t. The solution of this equation is,

$$q = \frac{V_m}{\omega Z} \sin (\omega t - \beta) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $Z = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2} = \dfrac{1}{\omega C}\sqrt{(R\omega C)^2 + \left(\dfrac{\omega^2}{\omega_0^2} - 1\right)}$

and $\omega_0 = \frac{1}{\sqrt{LC}}$ is the natural frequency of oscillation of the circuit for the case in which the resistance $R = O$. In general, the complete solution of the equation (1) will comprise a transient term and such transients will be considered on pages 270 and 284. For the present purpose, however, it is assumed that the transient current has died away and that the steady a.c. current of the circuit has become established. The current in the circuit will then be,

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad . \quad . \quad . \quad . \quad (3)$$

and I will be the r.m.s. value of the current if V is the r.m.s. value of the applied p.d. (see page 266). This equation shows that, when $\omega L = \frac{1}{\omega C}$, that is, when $\omega^2 = \frac{1}{LC} = \omega_0{}^2$, the impedance Z of the circuit becomes a minimum and the current will then have its maximum value

$$I_{res} = \frac{V}{R} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the supply frequency is $\omega = \omega_0$, a condition of *resonance* will exist and for a given value of the supply pressure, the current will be determined solely by the resistance of the circuit and the impedance is then $Z = R$. The pressure which will develop across the capacitance under these conditions of resonance will be

$$(E_C)_{res} = \frac{I_{res}}{\omega_0 C} = \frac{V}{\omega_0 RC} = \frac{\sqrt{\frac{L}{C}}}{R} V \quad . \quad . \quad . \quad (5)$$

The pressure across the capacitance for any value of the supply frequency ω will be

$$E_C = \frac{q}{C} = \frac{I}{\omega C} = \frac{V}{\omega CZ} = \frac{V}{\sqrt{(R\omega C)^2 + \left(\frac{\omega^2}{\omega_0{}^2} - 1\right)}} \quad . \quad (6)$$

and at resonance,

$$\left.\begin{aligned} (E_C)_{res} &= \frac{V}{\omega_0 RC} \quad . \quad . \quad . \quad (a) \\ (I)_{res} &= \frac{V}{R} \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting in (7*a*) the expression for the electro-magnetic time constant, viz. $T = \frac{L}{R}$ sec., also, $C = \frac{1}{\omega_0{}^2 L}$, then

$$(E_C)_{res} = \frac{VL\omega_0}{R} = VT\omega_0 \quad . \quad . \quad . \quad (8)$$

from which it will be seen that the pressure which will develop across the condenser terminals at resonance *will be proportional to the time constant and to the natural frequency of the circuit.* For example, if $\omega_0 = 200$: $L = 0{\cdot}25$ henry : $R = 5\Omega$ so that $T = 0{\cdot}05$ sec., then

$$(E_C)_{res} = 10V,$$

that is, 10 times the value of the supply pressure. In a practical case such as is illustrated in Fig. 21, in which the condenser is formed by

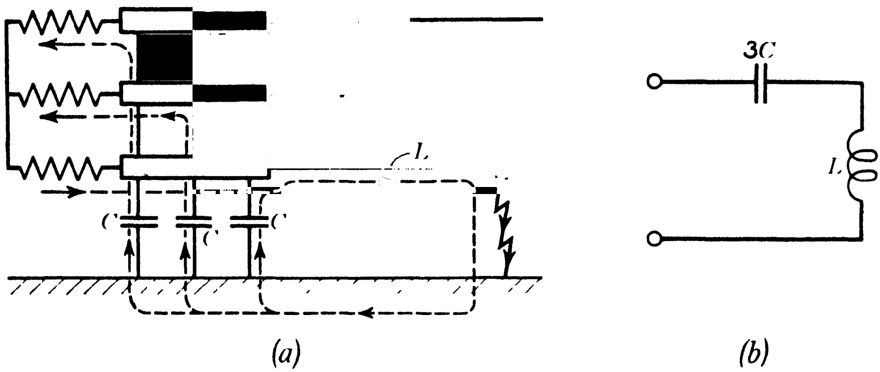


Fig. 21.

an underground cable and the inductance is that of a short-circuited overhead line, such a pressure rise would usually produce an immediate breakdown of the cable dielectric.

The way in which the current rises as the resonance frequency is approached can be seen by reference to Fig. 22, which has been drawn for the following conditions :

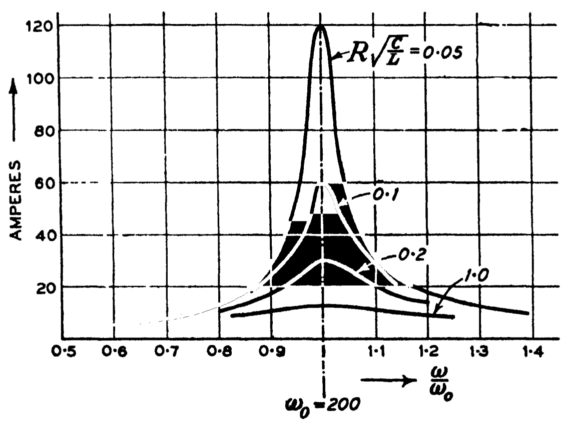


Fig. 22.

$$\omega_0 = 200 : \ C = 100\mu\text{F} : \ L = 0{\cdot}25 \text{ henry} : \ V = 300 \text{ volts} :$$

so that
$$R\omega_0 C = R\sqrt{\frac{C}{L}} = \frac{R}{50}.$$

The individual curves shown in Fig. 22 apply respectively to the following value of the characteristic constant $R\omega_0 C$,

R ohms	2·5	5	10	50
$R\omega_0 C = R\sqrt{\frac{C}{L}}$ . .	0·05	0·1	0·2	1
I_0, amperes . . .	120	60	30	6

Fig. 23 shows one phase of an open-circuited high tension cable for which the total distributed capacitance per phase is represented by the condenser of C farad. The effective inductance of the transformer referred to the high-tension winding is $L_S = \tau L_1$, as may be derived from Fig. 1 of Test Papers, Chapter XII. The pressure drop across the effective inductance when normal full-load current I_0 is flowing is then (Fig. 23)

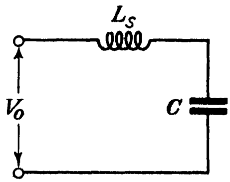


Fig. 23.

$$V_S = I_0 \omega L_S \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and this pressure drop is usually expressed as a percentage of the supply pressure, thus a reactance of 10 per cent. means that

$$V_S = I_0 \omega L_S = 0{\cdot}1 V_0.$$

Similarly, the capacitance current I_C due to the normal supply pressure V_0 is

$$I_C = \omega C V_0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and this is usually expressed as a percentage of the normal full load current I_0, so that a capacitance current of 10 per cent. means that

$$I_C = \omega C V_0 = 0{\cdot}1 I_0.$$

The natural frequency of the circuit of Fig. 23 is

$$\omega_0 = \frac{1}{\sqrt{L_S C}} = \frac{1}{\sqrt{\frac{V_S}{I_0\omega}\frac{I_C}{V_0\omega}}}$$

that is
$$\frac{\omega_0}{\omega} = \frac{1}{\sqrt{\frac{V_S}{V_0}\frac{I_C}{I_0}}}.$$

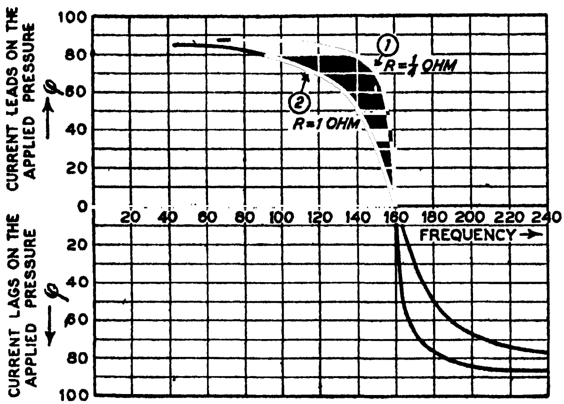


Fig. 24.

Suppose, for example, that the reactance is 33 per cent. and the capacitance current is 12 per cent., then

$$\frac{\omega_0}{\omega} = \frac{1}{\sqrt{0{\cdot}33 \times 0{\cdot}12}} \simeq 5,$$

so that the natural frequency is five times the supply frequency and consequently, if the supply pressure comprises a fifth harmonic in its wave form, the circuit will be in resonance for that harmonic.

If the reactance is 17 per cent. and the capacitance current is 12 per cent., then

$$\frac{\omega_0}{\omega} = \frac{1}{\sqrt{0{\cdot}17 \times 0{\cdot}12}} = 7,$$

and the circuit will then be in resonance with the seventh harmonic of the pressure wave form.

For a reactance of 12 per cent. and a capacitance current of 7 per cent

$$\frac{\omega_0}{\omega} = \frac{1}{\sqrt{0{\cdot}12 \times 0{\cdot}07}} = 11$$

and the circuit is then in resonance with the eleventh harmonic of the pressure wave form.

The way in which the phase angle between the current and the applied p.d. varies as the frequency passes through the resonance value, may be seen by reference to Fig. 24. The circuit conditions to which this diagram refers are :

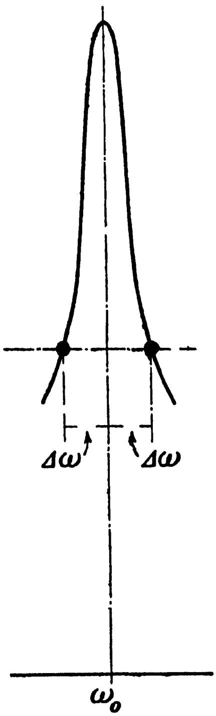


Fig 25.

$V_m = 200$ volts : $L = 0{\cdot}005$H : $C = 200\mu$F.
Graph (1) : resistance of circuit is $R = 0{\cdot}25$ ohm.
,, (2) : ,, ,, ,, $R = 1$ ohm.

When the value of the characteristic constant $R\omega_0 C$ is relatively small (see equation (6), page 259) as, for example, in the case of the curve of Fig. 22 for which the value is 0·05, the pressure across the cable will rise rapidly as the resonance frequency is approached, and this can be seen by reference to the expression (6) and Fig. 25. In the neighbourhood of resonance, $\omega = \omega_0 \pm \Delta\omega$, and it is to be noted that although when the supply frequency ω is very close to the resonance value, the factor $R\omega C$ is the controlling term of the impedance expression in (1), yet when this factor is small it very quickly becomes insignificant when $\Delta\omega$ is still small so that the pressure across the cable may be written,

$$E_C \simeq \frac{V}{\left(\dfrac{\omega \pm \Delta\omega}{\omega_0}\right)^2 - 1} \simeq \frac{V}{2\dfrac{\Delta\omega}{\omega_0}}$$

so that

$$\frac{\Delta\omega}{\omega_0} = \frac{V}{(2E_C)_p} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and this expression gives the minimum value of $\Delta\omega$ if the pressure across the cable is not to rise above the permissible safe value $(E_C)_p$, that is to say, it fixes the closest safe approach to the resonance frequency for a given limiting pressure rise $(E_C)_p$ across the cable.

Circuits in Parallel

In considering series arrangements of circuits the current is the factor which is the same throughout the circuit, and hence in determining the phase relationship of current and applied p.d. it is convenient to obtain an expression for the applied p.d. in terms of the current.

In the case of parallel circuits, such as *abc*, *adc* in Fig. 26*a*, the applied p.d. is the same for each branch, and it is therefore convenient to use the p.d. as the basis of calculation.

Let the applied p.d. be—

$$v = V_m \sin \omega t \text{ volts.}$$

The current in *abc* is—

$$i_1 = \frac{V_m}{\sqrt{R_1^2 + (Lw)^2}} \sin (\omega t - \phi_1),$$

where

$$\tan \phi_1 = \frac{L\omega}{R_1}.$$

The current in *adc* is—

$$i_2 = \frac{V_m}{\sqrt{R_2^2 + \left(\dfrac{1}{\omega C}\right)^2}} \sin (\omega t + \phi_2),$$

where $$\tan \phi_2 = \frac{1}{\omega C R_2}.$$

The current in the main lead is therefore—

$$i = i_1 + i_2.$$

Hence—

$$i = V_m\left[\frac{1}{Z_1}\sin(\omega t - \phi_1) + \frac{1}{Z_2}\sin(\omega t + \phi_2)\right] \text{ amperes,}$$

where $Z_1 = \sqrt{R_1^2 + (L\omega)^2}$ ohms ; $Z_2 = \sqrt{R_2^2 + \left[\frac{1}{\omega C}\right]^2}$ ohms.

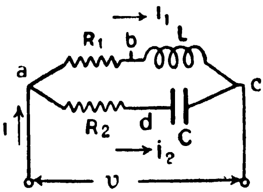


Fig. 26*a*.

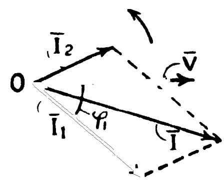


Fig. 26*b*.

That is—

$$i = V_m\left[\left(\frac{\cos\phi_1}{Z_1} + \frac{\cos\phi_2}{Z_2}\right)\sin\omega t - \left(\frac{\sin\phi_1}{Z_1} - \frac{\sin\phi_2}{Z_2}\right)\cos\omega t\right] \text{ amperes}$$

$$= V_m\left[\left[\frac{R_1}{Z_1^2} + \frac{R_2}{Z_2^2}\right]\sin\omega t - \left(\frac{L\omega}{Z_1^2} - \frac{\frac{1}{\omega C}}{Z_2^2}\right)\cos\omega t\right] \text{ amperes}$$

$$= V_m \sin(\omega t - \gamma)\sqrt{\left(\frac{R_1}{Z_1^2} + \frac{R_2}{Z_2^2}\right)^2 + \left(\frac{L\omega}{Z_1^2} - \frac{\frac{1}{\omega C}}{Z_2^2}\right)^2} \text{ amperes}$$

where $$\tan\gamma = \frac{\dfrac{L\omega}{Z_1^2} - \dfrac{\frac{1}{\omega C}}{Z_2^2}}{\dfrac{R_1}{Z_1^2} + \dfrac{R_2}{Z_2^2}}$$

that is $$\tan\gamma = \frac{\dfrac{L\omega}{R_1^2 + (L\omega)^2} - \dfrac{\frac{1}{\omega C}}{R_2^2 + \left(\frac{1}{\omega C}\right)^2}}{\dfrac{R_1}{R_1^2 + (L\omega)^2} + \dfrac{R_2}{R_2^2 + \left(\frac{1}{\omega C}\right)^2}}$$

The vector diagram for this case is given in Fig. 26*b*.

If $L\omega = \frac{1}{\omega C}$ —that is to say, the closed circuit—*abcda* is in resonance with the supply frequency—

$$\tan\gamma = \frac{(R_2 - R_1)\frac{1}{\omega C}}{R_1R_2 + \left(\frac{1}{\omega C}\right)^2} = \frac{(R_2 - R_1)L\omega}{R_1R_2 + (L\omega)^2}.$$

If $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1$, $\tan\gamma = 0$, then the current will be in phase with the applied p.d.

If $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1 = 0$, then $i = 0$, and no current flows in the mains although currents flow in each branch—viz. :

$$i = o : i_1 = \frac{V_m}{L\omega}\sin\left(\omega t - \frac{\pi}{2}\right) : i_2 = V_m\omega C\sin\left(\omega t + \frac{\pi}{2}\right).$$

The closed circuit *ABCDA* acts in this case as a *perfect insulator*, that is to say, a "rejector" circuit or "wave trap."

Similarly, in the case of a series circuit, if $L\omega = \frac{1}{\omega C}$ and $R_2 = R_1 = 0$, the circuit acts as a *perfect conductor*, that is to say, it is an "acceptor" circuit.

Root Mean Square Values

It was explained, Chapter II, page 59, that the power expended when a current flows through a resistance is proportional to the square of the current.

If the current flowing through a resistance of R ohms is i amperes the power expended is—

$$i^2R \text{ joules per second};$$

that is,
$$i^2R \text{ watts},$$

and this power is expended in heating the resistance.

Further, in any electrical apparatus the consuming device can always be considered as equivalent to a resistance, the power expended by the current being equal to the square of the current multiplied by the value of the equivalent resistance.

When the current is an alternating current the power varies from moment to moment, and the practical question arises as to what value has to be assigned to the current which, when squared and multiplied by the resistance, will give the mean value of the power expended in the resistance.

Let the current be—

$$i = I_m \sin\omega t \text{ amperes}.$$

At any instant the power expended is—

$$i^2R \text{ watts};$$

that is,

$$RI_m^2 \sin^2 \omega t \text{ watts}.$$

During one cycle the mean power is—

$$\frac{1}{T}\int_0^T RI_m^2 \sin^2 \omega t \, dt \text{ watts},$$

where T is the time of one cycle—i.e. $\frac{2\pi}{\omega}$ seconds.

Hence the mean power

$$= \frac{I_m^2 R}{T}\int_0^T \left(\frac{1 - \cos 2\omega t}{2}\right) dt$$

$$= \frac{I_m^2 R}{2} \text{ watts}.$$

The equivalent or effective value of the current which, when squared and multiplied by the resistance, gives the mean power is therefore

$$\frac{I_m}{\sqrt{2}} \text{ amperes}.$$

From the expression from which this effective value of the current is deduced it will be seen that this current is the square root of the mean square value of the current during one cycle. It is therefore called the **root mean square** (r.m.s.) value, or sometimes the **effective** value of the current, and will be denoted by I. Hence

$$I = \frac{I_m}{\sqrt{2}}.$$

The r.m.s. value of a *sine wave* of current is therefore $\frac{1}{\sqrt{2}}$ times the maximum value. Similarly the r.m.s. value of a *sine wave* of e.m.f. or p.d. is $\frac{1}{\sqrt{2}}$ times the maximum value.

The r.m.s. value of an alternating current is clearly equal to that of a direct current which gives the same heating effect in a resistance.

There is another quantity which is of practical importance, chiefly in connection with the design of alternating current machines, and that is the *mean* value of a sine wave. The mean value of the sine wave $i = I_m \sin \omega t$ is—

$$\frac{2}{T}\int_0^{T/2} I_m \sin \omega t \, dt,$$

where

$$T = \frac{2\pi}{\omega}.$$

Hence the mean value is $$= \frac{2}{\pi} I_m.$$

The ratio $\frac{\text{r.m.s. value}}{\text{mean value}}$ is termed the **form factor,** and for a sine wave is 1·11.

It will easily be seen that the form factor for a rectangular, i.e. a "flat top" wave, is unity, whilst for a sharply peaked wave the form factor is greater than 1·11 and in many practical cases reaches values of from 3 to 5 and even higher.

The Growth and Decay of a Current in a Direct Current Circuit

(*i*) AN INDUCTANCE OF L HENRY IN SERIES WITH A RESISTANCE OF R OHMS.—The solution of this problem has been obtained already in

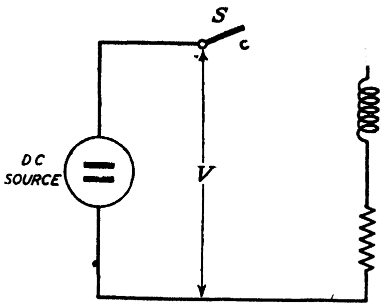


Fig. 27.

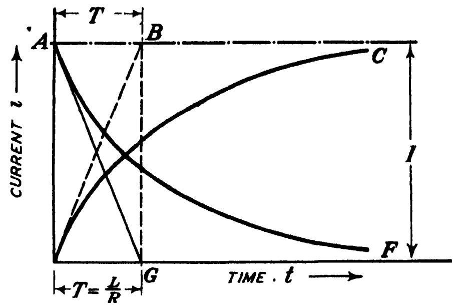


Fig. 28.

Chapter VIII, page 244, by means of the differential equation for the back e.m.f.s of the circuit. An alternative method of solution involving considerations of energy will now be given.

Let i amperes be the current in the circuit of Fig. 27 at any moment t second after the closing of the switch S. The electromagnetic energy stored in the inductance will then be $\frac{1}{2}Li^2$ joules, whilst the energy which is being dissipated in heating the resistance will be i^2R watts. The power supplied to the circuit from the d.c. source at the potential V volts must then be equal to the rate of increase of the electromagnetic energy of the inductance plus the power which is dissipated in the resistance. That is,

$$Vi = \frac{d}{dt}(\tfrac{1}{2}Li^2) + Ri^2$$

so that
$$Vi = Li\frac{di}{dt} + Ri^2$$

hence
$$V = L\frac{di}{dt} + Ri \quad . \quad . \quad . \quad . \quad . \quad (12)$$

this being the same equation which has been obtained already on page 245, and the solution of which has been found to be

$$i = \frac{V}{R}(1 - e^{-t/T}) = I(1 - e^{-t/T}) \quad . \quad . \quad . \quad (13)$$

where I amperes is the final steady value of the current and $T = \frac{L}{R}$ second is the *electro-magnetic time constant* of the circuit. The graph showing the growth of the current is given by the curve OC in Fig. 28. The time constant is given by the intercept AB on the horizontal line through A.

If, when the current has reached the final steady value of I amperes, the series circuit of resistance and inductance is short-circuited, as shown in Fig. 29, then the equation for the decay of the current is easily derived by means of considerations of the energy balance, as follows. The rate

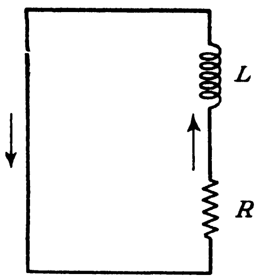


Fig. 29.

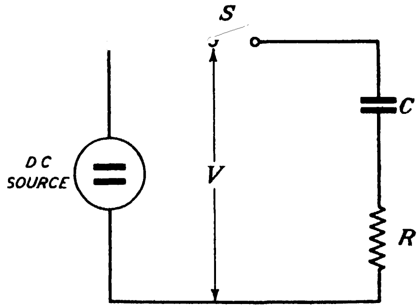


Fig. 30.

of decay of the electromagnetic energy at every moment will be equal to the rate of dissipation of energy in the resistance. That is

$$-\frac{d}{dt}(\tfrac{1}{2}Li^2) = Ri^2$$

or

$$-L\frac{di}{dt} = Ri \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The solution of this equation is

$$i = Ie^{-t/T} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

in which $i = I$ when $t = 0$. The graph of this equation is shown in Fig. 28 by the curve AF.

(*ii*) A CAPACITANCE OF C FARAD IN SERIES WITH A RESISTANCE OF R OHMS.—This circuit is shown in Fig. 30 in which the switch S connects the d.c. source of V volts to the series circuit. When the switch is closed, the energy relationships will be defined by the energy stored in the condenser at any moment t, that is, $\frac{1}{2}\frac{q^2}{C}$ joules, the power dissipated in

the resistance, and the power supplied by the d.c. source. Hence,

$$Vi = \frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C}\right) + i^2R$$

or

$$V = \frac{q}{C} + iR \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Differentiating this equation with respect to t gives

$$i = -C.R\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The solution of this equation is then

$$i = Ie^{-t/T} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

in which $T = C.R$ and $I = \frac{V}{R}$ is the initial value of the current which will flow immediately the switch is closed. The quantity $C.R$ is termed the *electrostatic time constant.*

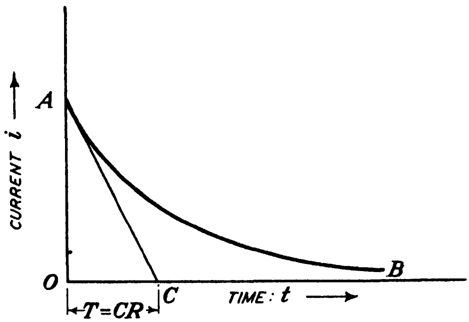


Fig. 31.

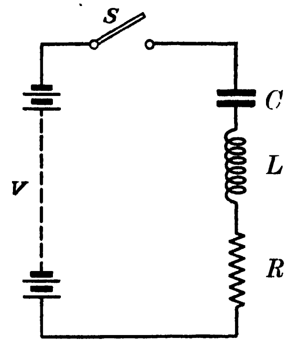


Fig. 32.

When the current has ceased to flow the condenser will have become charged with a quantity $Q = VC$ coulombs. If the series circuit of condenser and resistance is then short-circuited, the discharge current will commence with the full value $I = \frac{V}{R}$ and will decay logarithmically so that the same curve will be obtained for the discharge curve as is shown in Fig. 31 for the charge curve. It is to be observed that when the switch S of Fig. 30 is closed so that the condenser is receiving a charging current, then for the first moment of the process the condenser has no back e.m.f. and the conditions are the same as if the circuit contained only the resistance R.

A Series Oscillatory Circuit Comprising an Inductance, a Capacitance, and a Resistance is Connected Across a Steady Direct Current Pressure

The circuit arrangement is shown in Fig. 32, in which the switch S

connects the steady supply pressure of V volts across the series connection of C farad, L henry, and R ohms. The energy relationship at any moment t seconds after the switch has been closed will then be given by the equation,

$$Vi = \frac{d}{dt}\left[\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}\right] + Ri^2$$

that is,
$$Vi = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} + Ri^2$$

so that,
$$V = L\frac{di}{dt} + \frac{q}{C} + Ri. \qquad . \quad . \quad . \quad . \quad (19)$$

Since $i = \frac{dq}{dt}$, this equation may be written,

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{V}{L}. \qquad . \quad . \quad . \quad . \quad (20)$$

where q coulomb is the quantity of electricity which is stored in the condenser at any moment t. If $\left(\frac{R}{2L}\right)^2$ is $< \frac{1}{LC}$ the solution of (20) may be written

$$\left.\begin{aligned} q &= Qe^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t + \phi) - K \\ \text{where} \quad \nu &= \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{1}{2T}\right)^2} \end{aligned}\right\} \quad . \quad . \quad (21)$$

and $\omega_0 = \frac{1}{\sqrt{LC}}$ is the natural frequency for the circuit when the resistance is zero,

$T = \frac{L}{R}$ is the electromagnetic time constant as already defined on page 245,

K is an arbitrary constant which depends upon the initial conditions.

If the initial conditions are assumed to be : $t = 0$: $q = 0$, : $i = \frac{dq}{dt} = 0$,

then, after inserting the corresponding quantities in (21) it is seen that

$$K = Q\cos\phi$$

so that
$$q = Q\left[e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t + \phi) - \cos\phi\right]. \qquad . \quad . \quad . \quad (22)$$

and since, for $t = \infty$, $q = V.C$, it follows that $\cos\phi = -\frac{V.C}{Q}$

Differentiating this equation (22) with respect to t gives

$$i = \frac{dq}{dt} = -Qe^{-\frac{1}{2}\frac{t}{T}}\left[\frac{1}{2T}\cos(\nu t + \phi) + \nu\sin(\nu t + \phi)\right]. \qquad . \quad (23)$$

and inserting $\left.\frac{dq}{dt}\right]_{t=0} = 0$, it follows that

$$\tan\phi = -\frac{1}{2T\nu} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and from equation (21) and Figs. 33 (*a*) and 33 (*b*) it is seen that

$$\omega_0{}^2 = \nu^2 + \left(\frac{1}{2T}\right)^2 : \quad \cos\phi = -\frac{\nu}{\omega_0}.$$

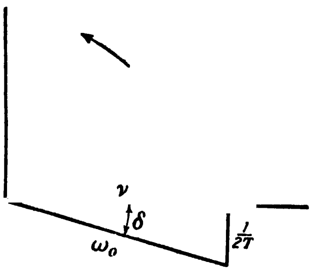


Fig. 33 (*a*).

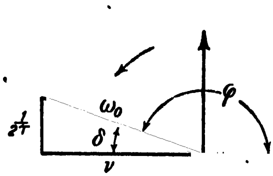


Fig. 33 (*b*).

The equation (23) for the current can then be written

$$i = -Q\omega_0 . e^{-\frac{1}{2}\frac{t}{T}}\left[\frac{1}{2T\omega_0}\cos(\nu t + \phi) + \frac{\nu}{\omega_0}\sin(\nu t + \phi)\right]$$

where (see Fig. 33),

$$\left.\begin{aligned} \frac{1}{2T\omega_0} &= \sin\delta = \sin(\pi - \phi) \\ \frac{\nu}{\omega_0} &= \cos(\pi - \phi) \end{aligned}\right\} \quad . \quad . \quad . \quad (25)$$

so that,

$$i = -Q\omega_0 . e^{-\frac{1}{2}\frac{t}{T}}\sin(\nu t + \pi) = Q\omega_0 . e^{-\frac{1}{2}\frac{t}{T}}\sin\nu t$$

that is,

$$i = Q\omega_0 . e^{-\frac{1}{2}\frac{t}{T}}\cos\left(\nu t - \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Reference to equation (20) shows that if the initial conditions are:

$$t = 0, \; q = 0, \; \frac{dq}{dt} = 0$$

then,
$$\left.\frac{d^2q}{dt^2}\right]_{t=0} = \frac{V}{L}$$

and, after differentiating (22) twice with respect to t and inserting the

above values for the initial condition, it is found that

since $$\frac{1}{LC} = \omega_0^2$$

therefore $$Q = VC\frac{\omega_0}{\nu} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Substituting for Q in equation (26), the applied pressure across the resistance R, Fig. 32, is found to be,

$$v_R = i.R = R.V.C\frac{\omega_0^2}{\nu}e^{-\frac{1}{2}\frac{t}{T}}\cos\left(\nu t - \frac{\pi}{2}\right)$$

or, substituting $$C = \frac{1}{L.\omega_0^2},$$

then $$v_R = \frac{V.R}{L.\nu}e^{-\frac{1}{2}\frac{t}{T}}\cos\left(\nu t - \frac{\pi}{2}\right) \quad . \quad . \quad . \quad (28)$$

The applied pressure across the inductance L, Fig. 32, is

$$v_L = L\frac{di}{dt}$$

and since, from equations (26) and (27),

$$\frac{di}{dt} = -\frac{V.\omega_0}{L.\nu}e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t - \phi),$$

it follows that,

$$v_L = -V\frac{\omega_0}{\nu}e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t - \phi) \quad . \quad . \quad . \quad (29)$$

The applied pressure across the condenser C, Fig. 32, is,

$$v_c = \frac{q}{C}$$

so that, after substituting for q from equations (22) and (27) it is found that,

$$v_c = \frac{V.\omega_0}{\nu}e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t + \phi) + V \quad . \quad . \quad . \quad (30)$$

The equations (28), (29), and (30), may now be assembled as follows,

$$\left.\begin{aligned} v_L &= -V_L.e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t - \phi) \\ v_R &= V_R.e^{-\frac{1}{2}\frac{t}{T}}\cos\left(\nu t - \frac{\pi}{2}\right) \\ v_C &= V_C.e^{-\frac{1}{2}\frac{t}{T}}\cos(\nu t + \phi) + V \end{aligned}\right\} \begin{aligned} &(a) \\ &(b) \quad . \quad . \quad (31) \\ &(c) \end{aligned}$$

where,

$$V_L = V_C = V . \frac{\omega_0}{\nu};$$

$$V_R = \frac{V . R}{L . \nu}$$

In Fig. 34 are shown the vector relationships for these three oscillatory back e.m.f.s, that is, for the moment $t = 0$. It can easily be shown that the sum of these three vectors is zero continuously from the moment at which the switch S of Fig. 32 is closed until the final steady state is reached at which the condenser becomes charged with the quantity VC coulombs. If these three pressure vectors are arranged as shown in Fig. 35 they will therefore form a closed triangle. Further consideration of these three vectors is given on page 277.

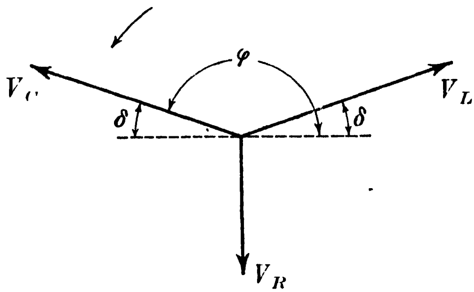


Fig. 34.

If each side of this pressure triangle is divided by the initial magnitude of the current vector, viz. $I = \dfrac{V}{L\nu}$, the resistance triangle of Fig. 36 is obtained, so that

$$\sin \delta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\tan \delta = \frac{\dfrac{R}{2}}{\sqrt{\dfrac{L}{C} - \left(\dfrac{R}{2}\right)^2}}$$

$$\cos \delta = \frac{\sqrt{\dfrac{L}{C} - \left(\dfrac{R}{2}\right)^2}}{\sqrt{\dfrac{L}{C}}} = \sqrt{1 - \left(\frac{R}{2}\right)^2 \frac{C}{L}}.$$

EXAMPLE.—A capacitance of 100 μF, an inductance of 0·25 henry and a resistance of 10 ohms are arranged in series and connected to a d.c. pressure of 300 volts. The time constant is thus $T = \dfrac{L}{R} = 0{\cdot}025$ sec. and $\dfrac{1}{2T} = 20$. The undamped circular frequency is $\omega_0 = \dfrac{1}{\sqrt{LC}} = 200$,

so that the circular frequency of the oscillating current will be [see Fig. 33 (*a*)]

$$\nu = \sqrt{\omega_0{}^2 - \left(\frac{1}{2T}\right)^2} = 199,$$

also (Fig. 33*b*) $\qquad \tan \phi = - \dfrac{1}{2T\nu} = 0{\cdot}10 : \ \phi = 174^\circ$

so that $\qquad \delta = \pi - \phi = 6^\circ.$

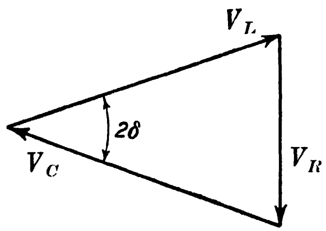


Fig. 35.

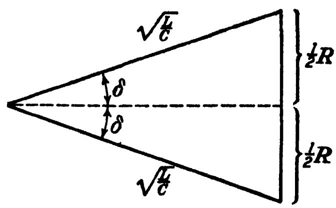


Fig. 36.

The pressure across the capacitance at any moment *t* will be given by the expression [31 (*c*), page 272], that is

$$v_C = \frac{300}{\cos \phi} e^{-\frac{1}{2}\frac{t}{T}} \cos (\nu t + \phi) + V$$

the applied pressure across the inductance will be

$$v_L = - \frac{300}{\cos \phi} e^{-\frac{1}{2}\frac{t}{T}} \cos (\nu t - \phi)$$

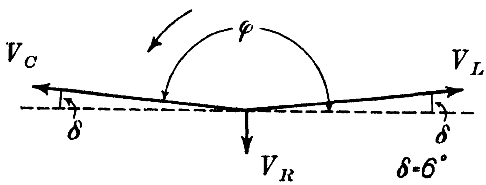


Fig. 37.

and the applied pressure across the resistance will be

$$v_R = 60 e^{-t/2T} \cos \left(\nu t - \frac{\pi}{2}\right).$$

In Fig. 37 the vector diagram for these three oscillating pressures is shown for the moment $t = 0$ and in Fig. 38 is shown the wave of the condenser pressure v_C. The foregoing expression for v_C may be

written in a somewhat more convenient form as follows. Since $\phi + \delta = \pi$,

$$v_C = V\left[1 - \frac{1}{\cos \delta}e^{-t/2T} \cos (\nu t - \delta)\right] \quad . \quad . \quad . \quad (32)$$

that is to say, the pressure across the capacitance at any moment t will be given by the arithmetical difference of the steady pressure of

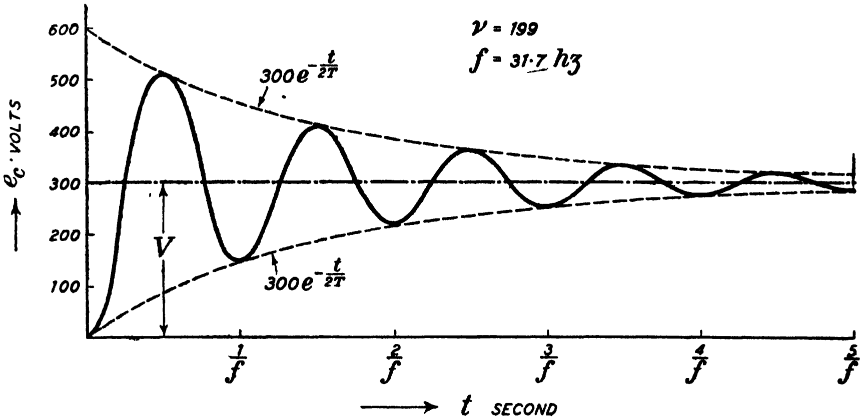


Fig. 38.

$V = 300$ volts and the damped oscillating cosine wave of frequency ν. The decay of this wave is defined by the two enveloping exponential curves $300e^{-t/2T}$, that is, $300e^{-20t}$ since $\frac{1}{2T} = 20$.

A Condenser which is Initially Charged to the Pressure V Volts Discharges through a Resistance and an Inductance in Series

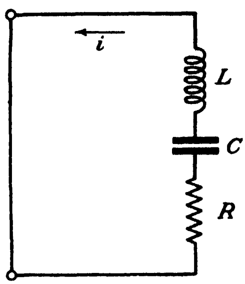


Fig. 39.

It is here assumed that the series circuit of Fig. 39 is the same as that of Fig. 32 and that the condenser, having become charged by the process which is represented by Fig. 32, is now allowed to discharge through the inductance and resistance as shown in Fig. 39. The equation defining this process of discharge is obtained from equation (20) (see page 270) by putting the right-hand side equal to zero, that is,

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{L.C} = 0 \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The solution of this equation is then

$$q = Qe^{-t/2T} \cos (\nu t - \delta) + K$$

and, inserting the initial conditions: $t = 0$: $q = VC$: it is seen that

$K = 0$, and $VC = Q \cos \delta$, so that,

$$q = \frac{VC}{\cos \delta} e^{-t/2T} \cos (\nu t - \delta) \quad . \quad . \quad . \quad (34)$$

The equation for the current at any moment t is then

$$i = \frac{dq}{dt} = -\frac{VC}{\cos \delta} e^{-t/2T} \left[\frac{1}{2T} \cos (\nu t - \delta) + \nu \sin (\nu t - \delta) \right] \quad . \quad (35)$$

and, inserting the initial conditions : $t = 0$: $i = 0$, then

$$0 = -\frac{VC}{\cos \delta} \left[\frac{1}{2T} \cos \delta - \nu \sin \delta \right]$$

so that
$$\tan \delta = \frac{1}{2T\nu} : \quad \cos \delta = \frac{\nu}{\omega_0} \quad . \quad . \quad . \quad (36)$$

The equation for the current i may now be written

$$i = -\frac{V.C}{\nu} \omega_0^2 . e^{-\frac{1}{2}\frac{t}{T}} \left\{ \frac{1}{2T\omega_0} \cos (\nu t - \delta) + \frac{\nu}{\omega_0} \sin (\nu t - \delta) \right\}$$

or
$$i = -\frac{V.C}{\nu} \omega_0^2 . e^{-\frac{1}{2}\frac{t}{T}} \left\{ \sin \delta \cos (\nu t - \delta) + \cos \delta \sin (\nu t - \delta) \right\}$$

that is,

$$i = - V.C\frac{\omega_0^2}{\nu} e^{-\frac{1}{2}\frac{t}{T}} \sin \nu t$$

or,

$$i = - V.C\frac{\omega_0^2}{\nu} e^{-\frac{1}{2}\frac{t}{T}} \cos \left(\nu t - \frac{\pi}{2} \right) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The pressure across the resistance R (Fig. 39) at any moment t is therefore

$$e_R = i.R = - V.C.R\frac{\omega_0^2}{\nu} e^{-\frac{1}{2}\frac{t}{T}} \cos \left(\nu t - \frac{\pi}{2} \right) \quad . \quad . \quad (38)$$

The pressure across the inductance L (Fig. 39) at any moment t is

$$e_L = - L\frac{di}{dt}$$

That is,

$$e_L = - V\frac{\omega_0}{\nu} e^{-\frac{1}{2}\frac{t}{T}} \cos (\nu t + \delta) \quad . \quad . \quad . \quad (39)$$

The pressure across the condenser terminals is obtained from equations (34) and (36), viz.

$$e_C = \frac{q}{C} = V\frac{\omega_0}{\nu} e^{-\frac{1}{2}\frac{t}{T}} \cos (\nu t - \delta) \quad . \quad . \quad . \quad (40)$$

The initial magnitudes of the three pressure vectors are, respectively,

$$E_C = V\frac{\omega_0}{\nu} = E_L : \; E_R = VCR\frac{\omega_0{}^2}{\nu} = \frac{VR}{L\nu} \quad . \quad . \quad . \quad (41)$$

and are shown in their appropriate vector relationships in Fig. 40.

If this figure is compared with Fig. 34 it will be seen to be identical in form with the exception that the two diagrams are displaced relatively

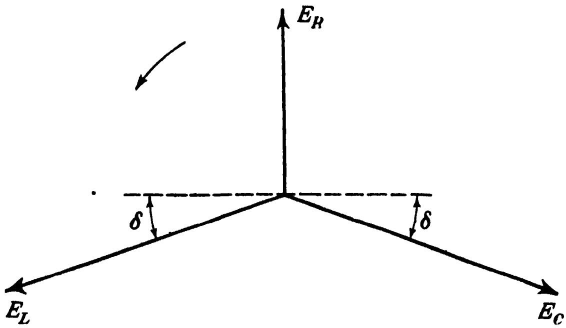


Fig. 40.

to each other by 180°. This is accounted for by the fact that the discharge current for the circuit of Fig. 39 is in the opposite direction to the charging current for the system of Fig. 32.

In Fig. 41 is shown the oscillating pressure wave as defined by the expression (40) for the same numerical values of the series circuit as have been considered on page 273. In the present case the condenser is assumed to have been initially charged to a pressure of $V = 300$ volts and then discharges through the inductance and resistance as shown in Fig. 39.

The equation (40) may now be written

$$e_C = \frac{V}{\cos\delta} e^{-t/2T} \cos(\nu t - \delta)$$

where $\nu = 199 : \; f = 31{\cdot}7 hz : \; \delta = 6^\circ : \; \dfrac{1}{2T} = 20.$

The pressure wave shown in Fig. 41 has been derived from the logarithmic spiral, which is shown on the left-hand side of the diagram. This spiral has been drawn by calculating the values of $\dfrac{300}{\cos\delta} e^{-t/2T}$ for successive values of t. Thus, for the two successive points a and b on the spiral the difference of the vector angles being 30°, that is to say, one-twelfth of the periodic time, i.e.

$$\frac{1}{12}\frac{1}{f} = \frac{1}{12}\left(\frac{1}{31{\cdot}7}\right) = 0{\cdot}00263 \text{ sec.}$$

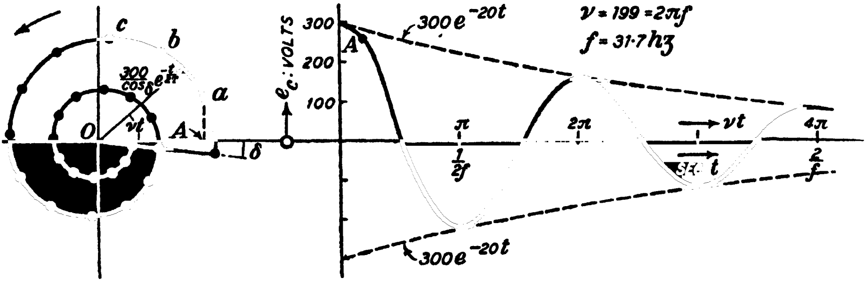


Fig. 41.

In Fig. 42 the three pressure vectors are shown in their correct relative positions.

Then, for any point such as A on the oscillatory wave of Fig. 41, the abscissa of which corresponds to $\frac{1}{12}$th of a cycle, the ordinate is obtained by projecting the vector Oa on to the horizontal axis of the logarithmic spiral.

The application of the logarithmic spiral of Fig. 41 may be explained as follows. For any moment t mark off the angle νt from the horizontal axis in the counter-clockwise direction and then reduce this angle by the amount δ. Draw the polar radius of the spiral for this resultant

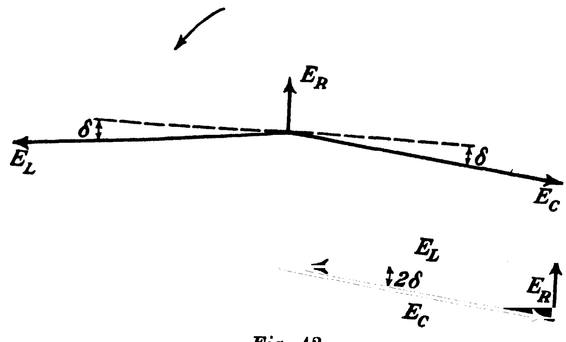


Fig. 42.

angle $(\nu t - \delta)$, then the projection on the horizontal axis of this polar radius will give the required value of the pressure at the given moment t.

The following alternative method is perhaps somewhat more convenient, although not quite so informative as the method just described. From the initial polar radius of the spiral, mark off the angle νt in the counter-clockwise direction and draw the corresponding polar radius. Then the projection of this radius on the horizontal axis will give the required value of the pressure for the given moment t.

Transient phenomena, which are closely allied to the foregoing examples, occur during the switching operations of heavy current circuits. For example, if a short-circuit develops on the consumer's side

of the protective reactance L in Fig. 43, in which C represents the capacitance of the line and S the circuit-breaker, the short-circuit current will then be approximately 90 degrees out of phase with the supply pressure, and if the circuit-breaker contacts open when the current is passing through its zero value, the stored energy of the capacitance will discharge through the closed circuit of inductance and capacitance in series, thus producing an oscillating current of frequency $\nu = \frac{1}{\sqrt{LC}}$.

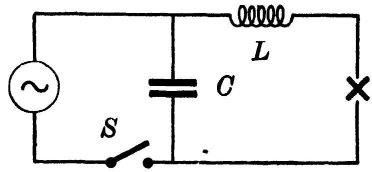


Fig. 43.

An oscillatory pressure will then develop across the capacitance and consequently also across the switch contacts, as defined by the expression,

$$e_C = Ve^{-t/2T} \cos \nu t \quad . \quad . \quad . \quad . \quad (42)$$

This transient pressure will be superposed upon the supply pressure

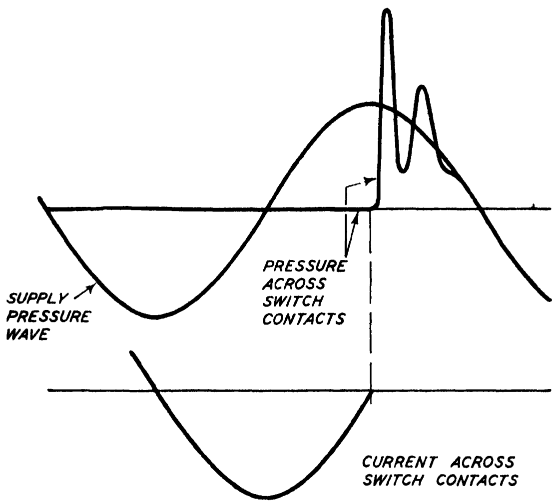


Fig. 44.

$V \cos \omega t$, so that the pressure across the switch contacts will be given by the expression

$$e = V(\cos \omega t - e^{-t/2T} \cos \nu t) \quad . \quad . \quad . \quad (43)$$

The circular frequency ν will usually be very large in comparison with the supply frequency ω, so that the transient pressure will rise to approximately double the supple pressure, as is shown in Fig. 44, and in accordance with the results obtained on page 275, Fig. 38.

A Non-Oscillatory Circuit comprising an Inductance, a Capacitance, and a Resistance in Series is connected to a D.C. Supply

The transient electrical phenomena in a series circuit of capacitance, inductance, and resistance, have an oscillatory characteristic only when the condition is fulfilled as previously stated on page 270, that is, when

$$\frac{1}{LC} \text{ is greater than } \left(\frac{R}{2L}\right)^2.$$

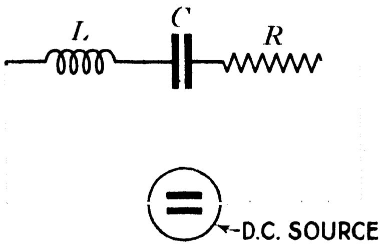


Fig. 45.

If the resistance of the circuit is relatively large so that

$$\left(\frac{R}{2L}\right)^2 \text{ is greater than } \frac{1}{LC},$$

the transient effects become non oscillatory. In this case, the solution of the general equation (19) on page 270 is

$$i = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

in which λ_1 and λ_2 are the two roots of the equation—

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{CL} = 0,$$

and A and B are constants which are determined by the initial conditions.

If these conditions are $t = 0$: $i = 0$: $\int i\,dt = 0$, as before, then $A + B = 0$, or $A = \quad B$.

Further, $\left.\frac{di}{dt}\right]_{t=0} = \frac{V}{L}$ for these initial conditions.

By differentiating the equation for i and substituting the value $\left.\frac{di}{dt}\right]_{t=0} = \frac{V}{L}$, it follows that—

$$A = \frac{V}{L}\frac{1}{(\lambda_1 - \lambda_2)},$$

and the current in the circuit will be—

$$i = \frac{V}{L}\frac{1}{(\lambda_1 - \lambda_2)}(e^{\lambda_1 t} - e^{\lambda_2 t}) \text{ amperes.}$$

Example.—As an example, suppose

$L = 0{\cdot}005$ henry : $C = \frac{200}{10^6}$ farads : $R = 20$ ohms : $V = 200$ volts.

Then $\lambda_1 = -3{,}730$, $\lambda_2 = -270$, and $A = -11{\cdot}5$,

therefore $i = 11{\cdot}5\,(e^{-270t} - e^{-3730t})$ amperes.

This current is plotted in Fig. 46.

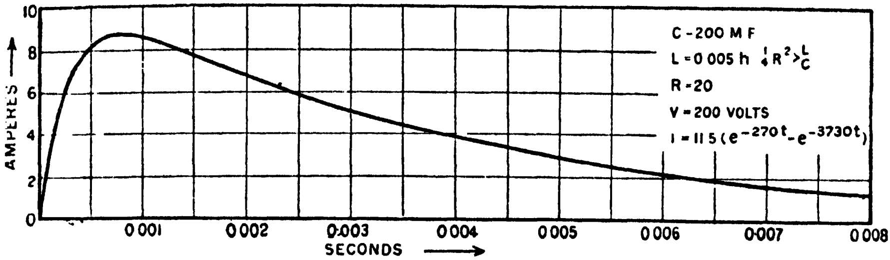


Fig. 46.

Transient Effects when an Alternating Pressure is Applied to a Circuit comprising a Resistance and an Inductance in Series

The circuit is shown in Fig. 47, and it is required to find what the current will be immediately after closing the switch—that is, before the steady cyclic conditions of current have been established as determined by the general formula given on page 254.

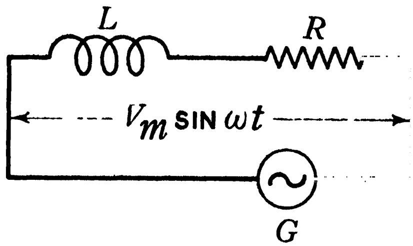


Fig. 47.

At any moment the current in the circuit will be given by the equation

$$L\frac{di}{dt} + Ri = V_m \sin \omega t,$$

where $\omega = 2\pi \times$ frequency.

The solution of this equation is—

$$i = Ae^{-\frac{R}{L}t} + \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin (\omega t - \beta),$$

in which A is a constant and $\tan \beta = \frac{\omega L}{R}$. The first term of this equation contains an exponential function of the time, and gives the transient current which flows after the switch is closed. In practice this term rapidly disappears, and the current assumes the steady cyclic state given by the equation—

$$i = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin (\omega t - \beta) \text{ amperes.}$$

If the initial conditions—i.e. at the instant at which the switch is closed—are

$$t = 0 : \; i = 0 : \; V_m \sin \omega t = 0,$$

then

$$A = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin \beta$$

and the transient current is

$$i_1 = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} e^{-\frac{R}{L}t} \sin \beta \text{ amperes.}$$

Hence $$i = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \left[e^{-\frac{R}{L}t} \sin \beta + \sin (\omega t - \beta) \right].$$

EXAMPLE.—As an example, suppose R = 2 ohms : $\omega = 2\pi$ frequency $= 2\pi \times 100 = 628$: $L = 0{\cdot}02$ henry : $V_m = 200$ volts. Then

$$\tan \beta = \frac{\omega L}{R} = 6{\cdot}28 : \ \beta = 81^\circ = 1{\cdot}41 \text{ radians} : \ \sin \beta = 0{\cdot}988.$$

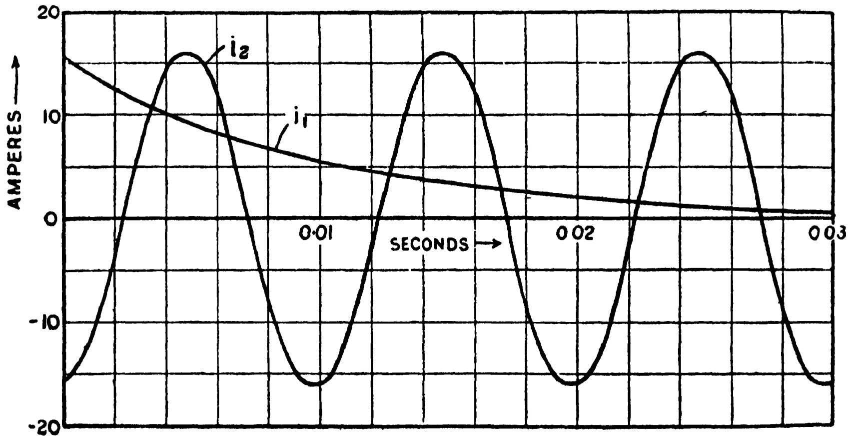


Fig. 48.

The transient term of the current is therefore

$$i_1 = \frac{200}{\sqrt{2^2 + (12{\cdot}56)^2}} 0{\cdot}988 e^{-100t} \text{ ampere ;}$$

that is, $$i_1 = 15{\cdot}5 e^{-100t} \text{ amperes.}$$

The second or permanent term of the current is

$$i_2 = 15{\cdot}8 \sin (628t - 1{\cdot}41) \text{ amperes.}$$

In Fig. 48 the two terms of the current wave are shown, and in Fig. 49 the resultant current wave is shown. It will be observed that after a time corresponding to about three complete cycles of the supply p.d. (i.e. after about 0·03 second from the time of closing the switch) the transient term has become relatively very small, and the current has reached very nearly a steady cyclic state.

As has been stated, the current waves shown in Figs. 48 and 49 correspond to the conditions that the switch is closed at the instant at

which the applied p.d. is zero. By suitably altering the value of the constant A in the foregoing example the corresponding current wave which will result if the switch is closed at any other instant of the p.d. cycle may be deduced.

An interesting special case is that of a circuit of which the resistance

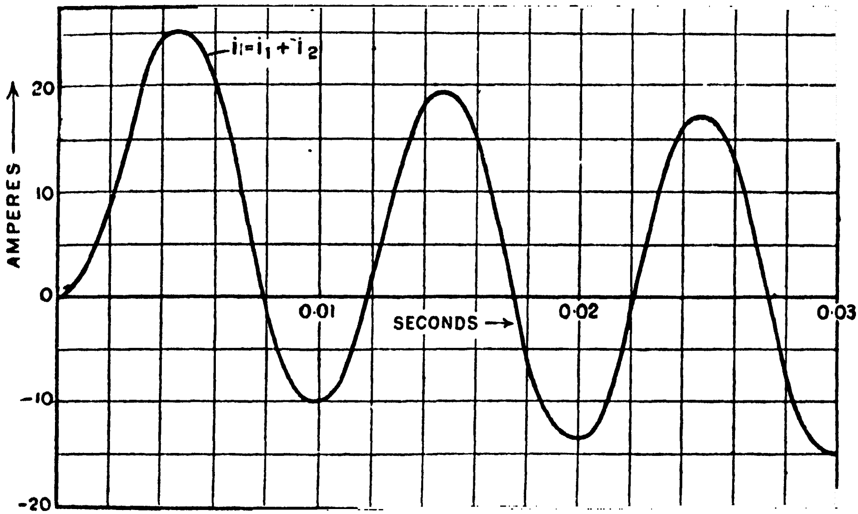


Fig. 49.

R (Fig. 47) is zero, and for which the switch is closed when the applied p.d. is zero. For this case $\tan \beta = \infty$, (i.e. infinity) and $\beta = 90°$.

Therefore $$i = A - \frac{V_m}{\omega L} \cos \omega t \text{ amperes.}$$

If the initial conditions are such that

$$t = 0: \quad i = 0: \quad V_m \sin \omega t = 0,$$

$$A = \frac{V_m}{\omega L},$$

and therefore $$i = \frac{V_m}{\omega L}(1 - \cos \omega t) \text{ amperes.}$$

This equation shows that the current never becomes negative, the minimum value being $i = 0$.

Example.—Suppose $V_m = 200$ volts : $\omega = 2\pi \times 100 = 628$: $\omega L = 12{\cdot}56$.

$$i = 16(1 - \cos \omega t) \text{ amperes.}$$

This current wave is shown in Fig. 50, the wave of the supply p.d. also being shown. It will be seen that the p.d. and current waves pass

through the zero values simultaneously and the current wave is uni-directional.

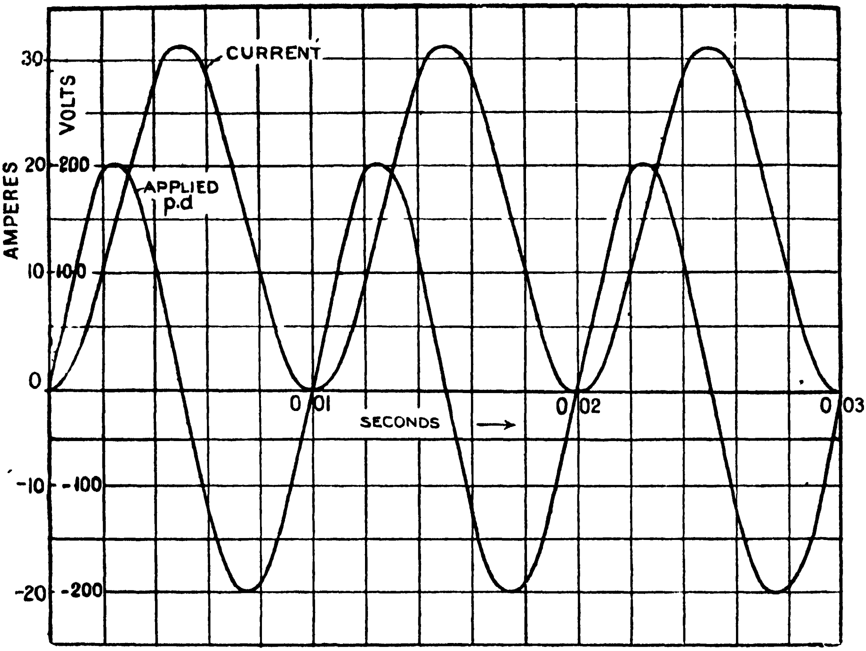


Fig. 50.

An Alternating P.D. applied to a Circuit Containing Resistance, Inductance, and Capacity in Series

The circuit is shown in Fig. 51, and it is assumed that an alternating p.d. $v = V_m \cos(\omega t - \theta)$ volts is applied to the terminals. The following equation will then hold :

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = V_m \cos(\omega t - \theta)\,;$$

that is,
$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = V_m\omega \cos\left(\omega t - \theta + \frac{\pi}{2}\right),$$

or
$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = F\cos\left(\omega t - \theta + \frac{\pi}{2}\right),$$

where
$$F = \frac{V_m\omega}{L}.$$

The solution of this equation comprises three forms dependent upon whether $\frac{1}{4}\frac{R^2}{L^2}$ is less than, equal to, or greater than $\frac{1}{LC}$. Two cases only, will be considered here.

Case I.—$\frac{1}{4}\frac{R^2}{L^2}$ *is less than* $\frac{1}{LC}$.

The solution in this case is—

$$i = Ae^{-\frac{1}{2}\frac{R}{L}t}\cos\left\{\sqrt{\frac{1}{LC} - \frac{1}{4}\frac{R^2}{L^2}}\,t + \beta\right\} + Y\cos\left(\omega t - \theta + \frac{\pi}{2} - \gamma\right)$$

where
$$Y = \frac{F}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{R^2}{L^2}\omega^2}}$$

and
$$\tan\gamma = \frac{\frac{R\omega}{L}}{\left(\frac{1}{LC} - \omega^2\right)} = \frac{R\omega C}{1 - LC\omega^2}$$

and A, β, and θ are constants determined by the initial conditions.

If, when the switch is closed, $t = 0$: $i = 0$: $\int i\,dt = 0$—that is, the condenser has no charge, and if the applied p.d. is zero at that instant—i.e. $\alpha = \frac{\pi}{2}$, then $\left.\frac{di}{dt}\right]_{t=0} = 0$, and the values of the constants A and β may be determined in terms of V_m and the constants L, R, C, and ω.

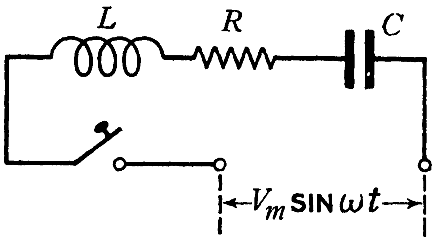


Fig. 51.

Example I.—As an example, suppose

$$L = 0{\cdot}005 \text{ henry : } C = \frac{200}{10^6} \text{ farad : } R = 0{\cdot}5 \text{ ohm,}$$

the supply frequency is $f = 100$ cycles per second—that is,

$$\omega = 2\pi \times 100 = 628, \text{ and } V_m = 200 \text{ volts.}$$

The value of F in the equation for i is—

$$F = \frac{V_m\omega}{L} = \frac{200 \times 628}{0{\cdot}005} = 25{\cdot}12 \times 10^6.$$

The natural frequency of the circuit is—

$$\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{1}{4}\frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \text{ very approximately.}$$

That is, the natural frequency is 160 cycles per second, and

$$2\pi \times \text{natural frequency} = 2\pi \times 160 = 1{,}000.$$

The constant A is therefore equal to 41·5, and the constant

$$\beta = 173\tfrac{1}{2}^\circ = 3{\cdot}03 \text{ radians.}$$

Also
$$\tan \gamma = \frac{R\omega C}{1 - LC\omega^2} = 0{\cdot}104\,;$$

that is,
$$\gamma = 6^\circ = 0{\cdot}105 \text{ radians.}$$

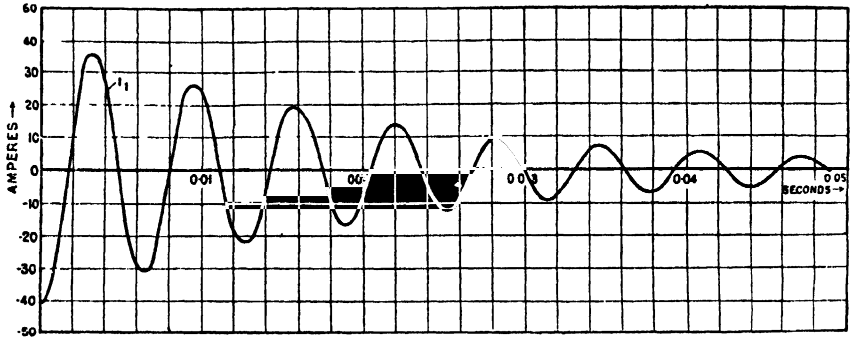


Fig. 52.

Hence

$$i = 41{\cdot}5e^{-50t}\cos(1{,}000t + 3{\cdot}03) + 41{\cdot}3\cos(628t - 0{\cdot}105) \text{ amperes.}$$

The first term is the transient term—

$$i_1 = 41{\cdot}5e^{-50t}\cos(1{,}000t + 3{\cdot}03) \text{ amperes,}$$

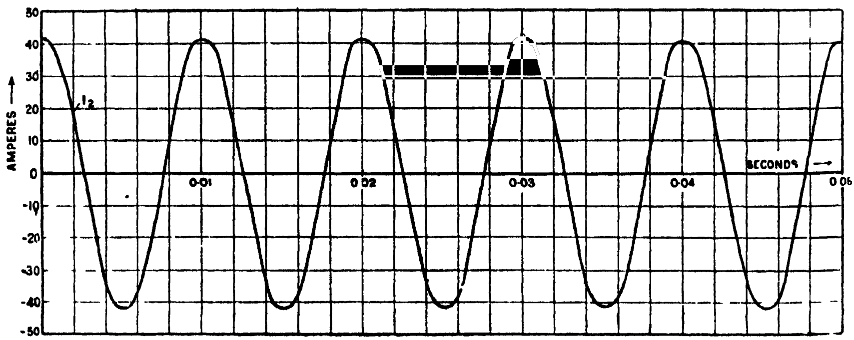


Fig. 53.

and this is plotted in Fig. 52, from which it will be seen that after about $\frac{1}{20}$th second after closing the switch the amplitude of this term is less than $\frac{1}{10}$th of its maximum amplitude 41·3. The frequency of this transient current is $\frac{1{,}000}{2\pi} = 160$ cycles per second.

The second term is the permanent term—

$$i_2 = 41{\cdot}3 \cos (628t - 0{\cdot}105) \text{ amperes},$$

and this is plotted in Fig 53. This is the current which flows in the circuit when the conditions have become steady.

The total current in the circuit is—

$$i = i_1 + i_2,$$

and this is plotted in Fig. 54. This current eventually becomes identical with the current i_2 (Fig. 53).

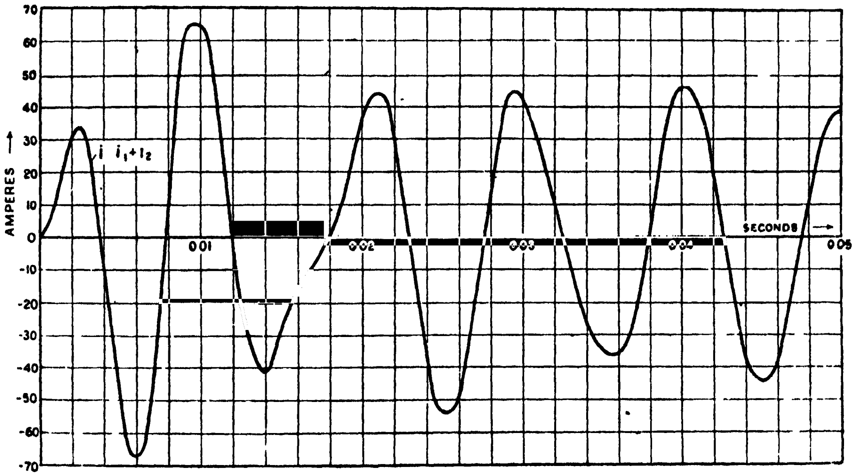


Fig. 54.

For any other given initial conditions—e.g. if the switch is closed at a moment when the applied p.d. is not zero—the transient current i_1 may be determined by finding the corresponding values of the constants A and β. The permanent term of the current—viz. i_1—is, of course, independent of the initial conditions if the values of L, C, and R are kept constant.

EXAMPLE II.—As another example of the case in which $\frac{1}{4}\frac{R^2}{L^2}$ is less than $\frac{1}{LC}$ will be chosen the condition that the natural frequency of the circuit is the same as the frequency of the applied p.d.—that is, the circuit is in *resonance* with the frequency of the supply pressure.

Let $L = 0{\cdot}005$ henry : $C = \frac{200}{10^6}$ farad : $R = 0{\cdot}25$ ohm : $V_m = 200$ volts : supply frequency $f = 160$ cycles per second :

$$\omega = 2\pi \times 160 = 1{,}000 : \; LC\omega^2 = 1.$$

Then, in the equation for the current i, given on page 284,

$$F = 40 \times 10^6 : \ \tan \gamma = \infty : \ \gamma = 90^\circ = \frac{\pi}{2} \text{ radians};$$

that is,

$$i = Ae^{-25t} \cos (1{,}000t + \beta) + 800 \sin 1{,}000t \text{ amperes.}$$

If the initial conditions are—

$$t = 0 : \ i = 0 : \ \int i\, dt = 0,$$

then $$A = 800$$

and $$B = 90^\circ = \frac{\pi}{2} \text{ radians.}$$

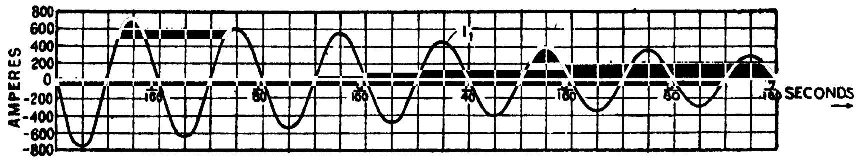


Fig. 55.

Hence $\quad i = -800e^{-25t} \sin 1{,}000t + 800 \sin 1{,}000t$ amperes.

The first term of this equation is the transient term—

$$i_1 = -800e^{-25t} \sin 1{,}000t \text{ amperes,}$$

and this is plotted in Fig. 55.

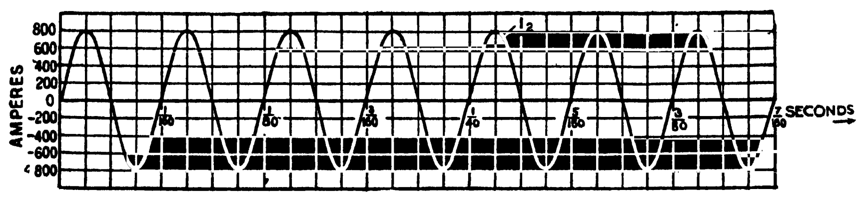


Fig. 56.

The second term is the permanent term—viz.:

$$i_2 = 800 \sin 1{,}000t \text{ amperes,}$$

and this is plotted in Fig. 56.

The resultant current is—

$$i = i_1 + i_2,$$

and this is given in Fig. 57, from which it is seen that the current is of gradually increasing amplitude until the steady condition is reached for which i_1 has become zero and $i = i_2$.

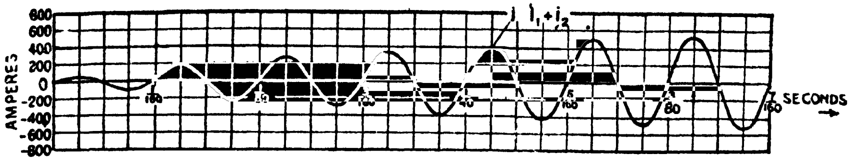


Fig. 57.

CASE II.—*Suppose* $\frac{1}{4}\frac{R^2}{L^2}$ *is greater than* $\frac{1}{LC}$.

The solution of the differential equation (p. 284) for the current in the series circuit is then—

$$i = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + \frac{F}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R\omega}{L}\right)^2}} \cos\left(\omega t - \theta + \frac{\pi}{2} - \gamma\right),$$

where A, B, and θ are constants of which the values depend upon the initial conditions.

As an example, suppose the supply frequency is 50 cycles per second, and let the constants of the circuit be as follows: $R = 10{\cdot}1$ ohms; $L = 0{\cdot}005$ henry; $C = 200 \times 10^{-6}$ farads; $V_m = 200$ volts.

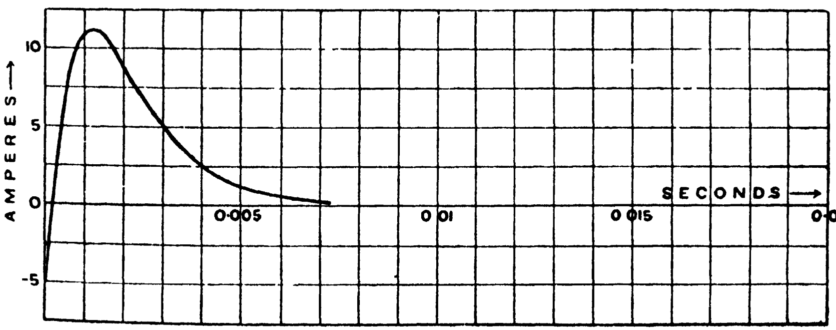


Fig. 58.

Then $\omega = 2\pi \times$ frequency $= 314$; $\tan\gamma = 0{\cdot}705$; $\gamma = 35\frac{1}{4}^\circ = 0{\cdot}616$ radian.

$$F = \frac{V_m\omega}{L} = 12{\cdot}56 \times 10^6 : \quad \frac{1}{LC} = 10^6 :$$

$$\lambda_1 = -1{,}151; \quad \lambda_2 = -868.$$

The current in the circuit is therefore—

$$i = Ae^{-1151t} + Be^{-868t} + 11{\cdot}4 \cos\left(314t - \theta + \frac{\pi}{2} - 0{\cdot}616\right),$$

where A, B, and θ are constants depending upon the initial conditions.

Suppose when the circuit switch is closed the applied p.d. is at its maximum positive value—that is, $\theta = o$ (see p. 284). Further, let the current be zero and the condenser without charge when the switch is closed. That is, the initial conditions are—

$$t = o\,; \quad i = o\,; \quad \int i\,dt = o\,; \quad \theta = o\,; \quad \frac{di}{dt} = 40{,}000.$$

By substitution in the equations for i and $\frac{di}{dt}$ it is found that the constants have the following values:

$$A = -\,131{\cdot}6\,; \quad B = 125.$$

Hence the current is—

$$i = -\,131{\cdot}6e^{-1151t} + 125e^{-868t} + 11{\cdot}4 \cos\,(314t + 0{\cdot}954) \text{ amperes.}$$

The transient component of this current is:

$$i_1 = -\,131{\cdot}6e^{-1151t} + 125e^{-868t} \text{ amperes,}$$

and this is plotted in Fig. 58.

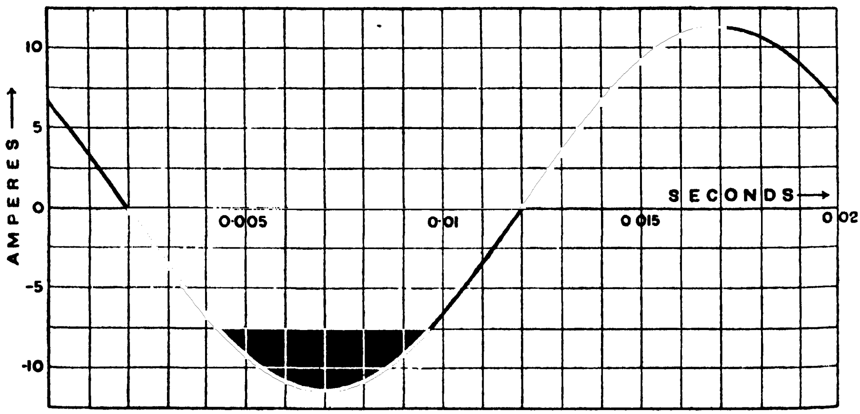


Fig. 59.

The permanent component of the current is—

$$i_2 = 11{\cdot}4 \cos\,(314t + 0{\cdot}954) \text{ amperes,}$$

and this is plotted in Fig. 59.

The actual resultant current is—

$$i = i_1 + i_2,$$

and this is shown in Fig. 60.

The case in which $\frac{1}{4}\frac{R^2}{L^2}$ is equal to $\frac{1}{LC}$ may be dealt with in a similar manner.

The calculation of transient currents may be greatly facilitated by the use of the table given in the Appendix No. I, page 529.

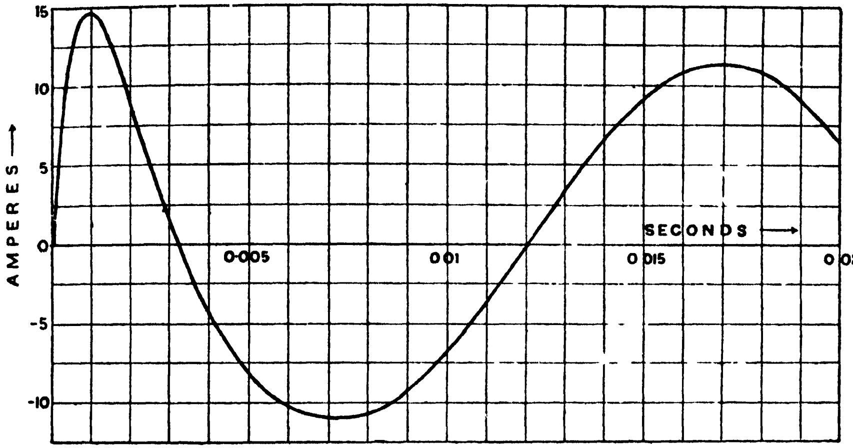


Fig. 60.

Representation of Vectors by Complex Quantities

In Fig. 61 let OA be a line drawn in the direction OX and of length a units.

That is—
$$OA = a.$$

A line OA^1 drawn in the direction OX^1 and of length a units is written—
$$OA^1 = -a;$$
that is,
$$OA^1 = -OA.$$

Now assume that a line OB (Fig. 61) drawn in the direction OY and of length a units be written—
$$OB = j(OA) = ja,$$
the prefix j denoting that OB is drawn at right angles to the direction of OA, the angle being measured in a counter-clockwise direction from OA.

According to this notation $j(OB)$ is a line drawn at right angles to OB, measuring the angle in a counter-clockwise direction from OB.

That is—
$$j(OB) = OA^1,$$
or
$$j(ja) = OA^1 = -a;$$
hence
$$j^2a = -a$$
$$j^2 = -1,$$
and
$$j = \sqrt{-1}.$$

Hence j must be assumed to be $\sqrt{-1}$ in order that the prefix j satisfies the condition that $j(OA)$ denotes a line drawn at right angles to OA in the counter-clockwise direction, and that—
$$OA^1 = -OA.$$

In Fig. 62 let a vector OB be drawn at an angle ϕ to the axis OX, and let the length of OB be r units. This vector may be resolved into two components at right angles, OA along OX, and AB in the direction OY.

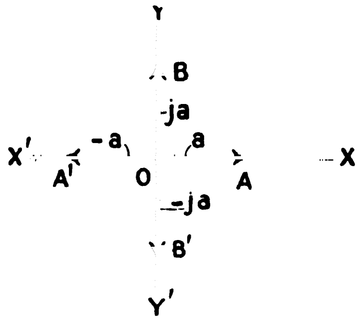


Fig. 61.

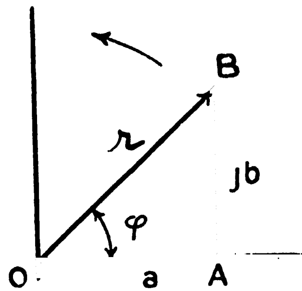


Fig. 62.

Let OA be a units and AB be b units, so that $OA = a$ and $AB = jb$. The vector OB may then be written—

$$OB = a + jb.$$

Similarly a vector OB^1 (Fig. 63) in the second quadrant is—

$$OB^1 = -a + jb.$$

A vector in the third quadrant is—

$$OB^{11} = -a - jb,$$

and a vector in the fourth quadrant is—

$$OB^{111} = a - jb.$$

Since the length of the vector OB (Fig. 62) is r units—

$$a = r \cos\phi : \ b = r \sin\phi$$

$$r = \sqrt{a^2 + b^2} : \ \tan\phi = \frac{b}{a},$$

and

$$OB = a + jb = r(\cos\phi + j\sin\phi),$$

and since it has been shown that j under this interpretation must be $\sqrt{-1}$ it follows from the previous results that—

$$OB = r(\cos\phi + j\sin\phi) = re^{j\phi}.$$

Quantities of the form $a + jb$ and the equivalent forms

$$r(\cos\phi + j\sin\phi), \text{ and } re^{j\phi}$$

are termed **complex quantities.**

The quantity r is the magnitude or the *modulus* of the vector, and ϕ is the *argument*, or $\tan\phi$ *the slope* of the vector.

In Fig. 64 let OA represent a vector in the first quadrant, so that—

$$OA = a + jb;$$

then

$$\begin{aligned} j(OA) &= j(a + jb) \\ &= ja - b \\ &= OA_1. \end{aligned}$$

Hence, if a vector OA is multiplied by j a vector OA_1 is obtained which is of the same magnitude as OA, but makes an angle of 90° with

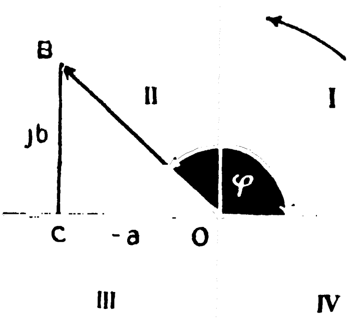


Fig. 63.

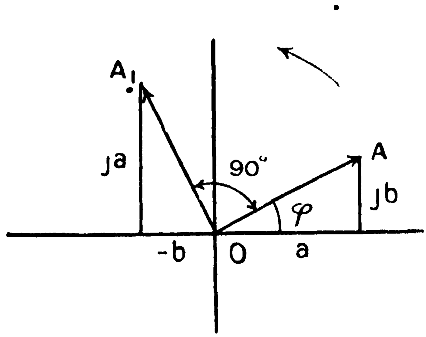


Fig. 64.

OA, the angle being measured in the counter-clockwise direction—*that is to say, multiplication of a vector by j gives a vector of equal magnitude, but 90° ahead of the original vector.*

Similarly, multiplication of a vector by $-j$ gives a vector of equal magnitude, but 90° *behind* the original vector.

Addition of Vectors.—Suppose now that two vectors OA and OB (Fig. 65) defined respectively by—

$$OA = a + jb = r_1 e^{j\phi_1},$$

and

$$OB = c + jd = r_2 e^{j\phi_2}$$

are to be added.

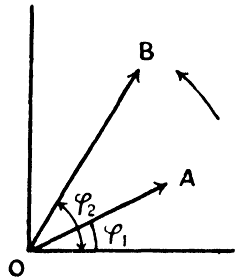


Fig. 65.

Then $\quad OA + OB = a + c + j(b + d);$

that is to say, the resultant vector is obtained by adding the real and imaginary parts respectively.

The slope of the resultant vector is—

$$\tan\phi = \frac{b + d}{a + c},$$

and the magnitude is—

$$\sqrt{(a + c)^2 + (b + d)^2}.$$

Subtraction of Vectors.—

$$OA - OB = a - c + j(b - d)$$

$$\tan \phi = \frac{b - d}{a - c},$$

and the magnitude is—

$$\sqrt{(a - c)^2 + (b - d)^2}.$$

Multiplication of Vectors.—For multiplication the vectors are most conveniently expressed in the form

$$OA = r_1 e^{j\phi_1}$$

and $$OB = r_2 e^{j\phi_2};$$

then $$OA \times OB = r_1 r_2 e^{j(\phi_1 + \phi_2)}.$$

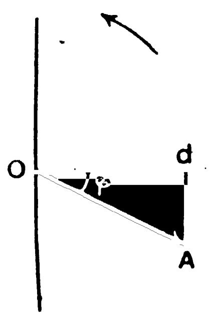


Fig. 66.

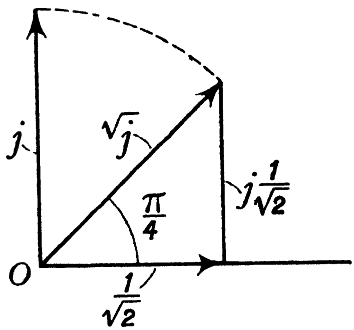


Fig. 67.

Hence the magnitude of the resultant vector is the product of the magnitudes of the respective components, and the slope of the resultant is the sum of the slopes of the components. If the vectors be written in the form—

$$OA = r_1 (\cos \phi_1 + j \sin \phi_1) : OB = r_2 (\cos \phi_2 + j \sin \phi_2);$$

and $$OA \times OB = r_1 r_2 [\cos (\phi_1 + \phi_2) + j \sin (\phi_1 + \phi_2)],$$

as is also obvious from the previous result.

Division of Vectors.—

$$\frac{OA}{OB} = \frac{r_1}{r_2} e^{(j\phi_1 - \phi_2}$$

and hence

$$\frac{OA}{OB} = \frac{r_1 (\cos \phi_1 + j \sin \phi_1)}{r_2 (\cos \phi_2 + j \sin \phi_2)} = \frac{r_1}{r_2} [\cos (\phi_1 - \phi_2) + j \sin (\phi_1 - \phi_2)].$$

To Raise a Vector to the nth Power.—

$$(OA)^n = (r_1 e^{j\phi_1})^n = r_1^n e^{j\phi_1 n},$$

and $$r^n (\cos \phi_1 + j \sin \phi_1)^n = r^n (\cos n\phi_1 + j \sin n\phi_1).$$

To Determine the nth Root of a Vector.—

$$(OA)^{1/n} = (r_1 e^{j\phi_1})^{1/n} = r_1^{1/n} e^{j\phi_1 \frac{1}{n}},$$

and
$$r_1^{1/n}(\cos\phi_1 + j\sin\phi_1)^{1/n} = r_1^{1/n}\left(\cos\frac{\phi_1}{n} + j\sin\frac{\phi_1}{n}\right).$$

If, as the result of such operations, an expression of the form $\frac{l+m}{h+jk}$ is obtained, this expression may be reduced to the standard form $a + jb$ by multiplying by $\frac{h-jk}{h-jk}$.

Thus
$$\frac{l+m}{h+jk} = \frac{(l+m)(h-jk)}{h^2+k^2} = \frac{(l+m)h - jk(l+m)}{h^2+k^2};$$

or
$$\frac{l+m}{h+jk} = \frac{(l+m)h}{h^2+k^2} - j\frac{k(l+m)}{h^2+k^2},$$

and this represents a vector OA in Fig. 66,

where
$$Od = \frac{(l+m)h}{h^2+k^2},$$

and
$$dA = \frac{k(l+m)}{h^2+k^2}.$$

It is important to notice that (Fig. 67)

$$j = e^{j\frac{\pi}{2}} : \sqrt{j} = e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \quad . \quad . \quad . \quad (44)$$

also, generally,

$$j^x = e^{j\frac{\pi}{2}x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The Exponential Function for cos x

Consider now the expression for $\cos x$, viz. (see Appendix I, p. 523.)

$$\cos x = \tfrac{1}{2}(e^{jx} + e^{-jx}) \quad . \quad . \quad . \quad . \quad (46)$$

This expression may be graphically represented as one-half the sum of the two conjugate vectors e^{jx} and e^{-jx} as shown in Fig. 68, that is, $\cos x = OB$. Now, if it be agreed that only the "real" component of the vector e^{jx} is to be taken into account, it is then permissible to write

$$\cos x = e^{jx},$$

so that, if an alternating current of circular frequency $\omega = 2\pi f$ is given by the expression,

$$\cos \omega t = e^{j\omega t} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

then the circular frequency of the current is given by the numerical value of the coefficient of t in the term on the right-hand side.

It is of interest to consider a little further at this stage the complete

expression for a cosine function of the time, for example, the current vector,

$$\mathfrak{I}_0 = I_0 \cos \omega t = I_0 . \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \quad . \quad . \quad . \quad (48)$$

and it is to be noted in what follows, that vector quantities are represented by Gothic script letters, a list of which will be found in Appendix No. III, page 531.

This expression comprises two components, (*i*) the vector $\frac{1}{2}I_0e^{j\omega t}$ and (*ii*) the vector $\frac{1}{2}I_0e^{-j\omega t}$. From what has been said in the foregoing (see also Fig. 69) it will be clear that each of these vectors will rotate with an angular velocity ω radians per second, the former component in the

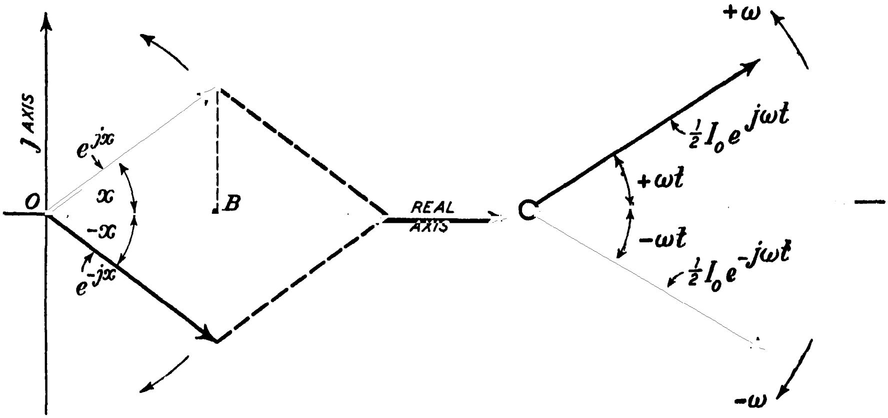


Fig. 68. *Fig.* 69.

positive (counter-clockwise) direction and the latter component in the negative (clock-wise) direction as is shown in Fig. 69. When $\omega t = 2\pi$ each of these component vectors will have made one complete revolution and returned to its initial position, that is, coincident with the positive direction of the real axis, OX, so that the time of one complete cycle of changes of the alternating current, that is, the "periodic time" τ, will be given by $\tau = \dfrac{2\pi}{\omega}$ sec., and consequently, the number of cycles which will be passed through in one second, that is, the "frequency" of the alternating current, will be given by $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$, so that $\omega = 2\pi f$, the angular velocity ω being also termed the "circular frequency."

Example of the Application of the Expression $\cos \omega t = e^{j\omega t}$

By means of the expression (47) the solution of a whole range of problems relating to oscillatory current circuits can be solved in an extremely simple manner and with the minimum amount of explanatory

statement. For example, suppose a condenser of capacitance C farad is charged with a quantity Q coulombs and is then switched across a coil of negligibly small resistance and of which the inductance is L henry (see Fig. 70). If at any moment t second after closing the switch the quantity in the condenser is q coulomb, then the current will be $i = \frac{dq}{dt}$ amperes and the p.d. across the condenser terminals will be $v_c = \frac{q}{C}$ volts. The e.m.f. of self-induction is then (see Chapter VIII, page 237)

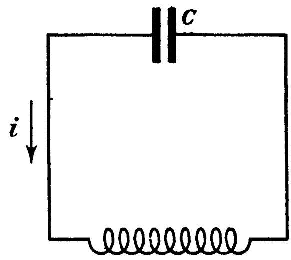


Fig. 70.

$$e = -L\frac{di}{dt},$$

so that

$$v_c = \frac{q}{C} = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$

or

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0,$$

that is, the differential equation which defines the electrical condition of the circuit at any moment t is

$$\frac{d^2q}{dt^2} + \frac{1}{L.C}q = 0 \qquad . \quad . \quad . \quad . \quad . \quad (49)$$

The solution of this equation may be written in the form $q = Ae^{\gamma t}$, and by substitution in equation (49) it is seen that—

$$\gamma^2 + \frac{1}{L.C} = 0$$

so that

$$\gamma = \pm j\frac{1}{\sqrt{L.C}},$$

and consequently the solution of the equation (49) is

$$q = A_1 e^{j\frac{1}{\sqrt{L.C}}t} + A_2 e^{-j\frac{1}{\sqrt{L.C}}t} \qquad . \quad . \quad . \quad . \quad . \quad (50)$$

where A_1 and A_2 are arbitrary constants which depend upon the initial conditions. If, when $t = 0$, $q = Q_0$ and $i = 0$, then it is easily shown that $A_1 = A_2 = \frac{1}{2}Q_0$, so that the complete solution of the equation (49) is then

$$q = Q_0\tfrac{1}{2}\left[e^{j\frac{1}{\sqrt{L.C}}t} + e^{-j\frac{1}{\sqrt{L.C}}t}\right],$$

or

$$q = Q_0 \cos \omega t \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where $\omega = \frac{1}{\sqrt{L.C}}$ and is the circular frequency of the oscillating

quantity q. The oscillating current will then be

$$i = \frac{dq}{dt} = -\omega Q_0 \sin \omega t \quad . \quad . \quad . \quad . \quad (52)$$

so that the peak value of the current is $I_0 = \omega Q_0$ and the pressure across the condenser terminals at any moment t will be (see Fig. 71),

$$v_c = \frac{q}{C} = \frac{Q_0}{C} \cos \omega t \quad . \quad . \quad . \quad . \quad (53)$$

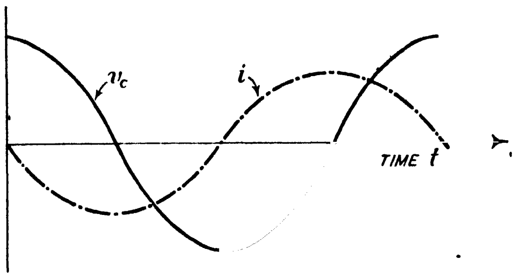


Fig. 71.

The Impedance Vector of an Alternating Current Circuit

In Fig. 72 is shown a circuit which comprises a resistance of R ohms in series with an inductance of L henry. The vector of the supply pressure is $\mathfrak{V}_0$ and the vector of the current in the circuit is defined by

$$\mathfrak{I}_0 = I_0 \cos \omega t = I_0 e^{j\omega t} \text{ amperes} \quad . \quad . \quad . \quad (54)$$

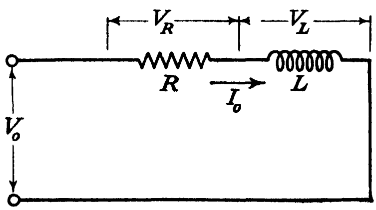


Fig. 72.

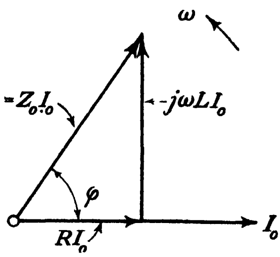


Fig. 73.

The vector of the p.d. across the inductance will be

$$\mathfrak{V}_L = -\mathfrak{E}_L = L\frac{d}{dt}(\mathfrak{I}_0) = j\omega L I_0 e^{j\omega t} = j\omega L \mathfrak{I}_0,$$

so that the vector of the supply pressure may be written

$$\mathfrak{V}_0 = \mathfrak{V}_R + \mathfrak{V}_L = \mathfrak{I}_0(R + j\omega L) = \mathfrak{I}_0 . \mathfrak{Z}_0 \text{ volts} \quad . \quad . \quad (55)$$

as shown in Fig. 73.

where $\mathfrak{Z}_0 = R + j\omega L$ ohms is the vector of the circuit impedance,
ωL ,, is the inductive reactance,
R ,, is the ohmic resistance.

$$\tan \phi = \frac{\omega L}{R}.$$

In Fig. 74 is shown a circuit comprising a resistance of R ohms, an inductance of L henry, and a condenser of capacitance C farad connected in series. The vector of the supply pressure is $\mathfrak{V}_0$ and the current vector is defined as before, viz.

$$\mathfrak{I}_0 = I_0 \cos \omega t = I_0 e^{j\omega t} \text{ amperes} \quad . \quad . \quad . \quad (56)$$

the rate of change of this current vector being given by the expression

$$\frac{d}{dt}\mathfrak{I}_0 = j\omega \mathfrak{I}_0 \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The vector of the pressure across the condenser is then $\mathfrak{V}_C = \frac{q}{C}$, the

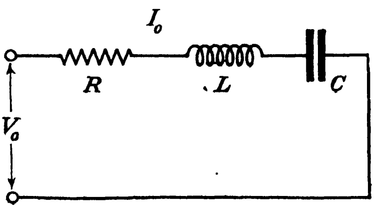


Fig. 74.

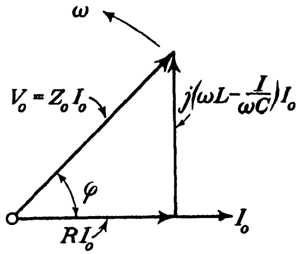


Fig. 75.

vectors of the pressures $\mathfrak{V}_R$ and $\mathfrak{V}_L$, respectively, being expressed as in the previous example, so that

$$\mathfrak{V}_0 = \mathfrak{V}_R + \mathfrak{V}_L + \mathfrak{V}_C = R.\mathfrak{I}_0 + j\omega L.\mathfrak{I}_0 + \frac{1}{C}q \quad . \quad . \quad (58)$$

and, consequently, $\mathfrak{V}_0$ will be a cosine function of the time t and may be written in the general form $\mathfrak{V}_0 = V_0 e^{j(\omega t + \phi)}$.

Differentiating both sides of the equation (58) with respect to t gives

$$\frac{d}{dt}\mathfrak{V}_0 = j\omega\mathfrak{I}_0\left(R + j\omega L + \frac{1}{j\omega C}\right),$$

and since $$\mathfrak{V}_0 = V_0 e^{j\phi} e^{j\omega t} : \frac{d}{dt}\mathfrak{V}_0 = j\omega\mathfrak{V}_0,$$

then $$\mathfrak{V}_0 = \mathfrak{I}_0\left\{R + j\left(\omega L - \frac{1}{\omega C}\right)\right\} = \mathfrak{I}_0\mathfrak{Z}_0 \quad . \quad . \quad (59)$$

as shown in Fig. 75, where

$\mathfrak{Z}_0 = R + j\left(\omega L - \frac{1}{\omega C}\right)$ ohms and is the impedance vector for the whole circuit

R ohms is the resistance,
ωL ohms is the inductive reactance,
$\frac{1}{\omega C}$ ohms is the capacitance reactance.

$$\tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

For a series circuit, therefore, the vector of the total impedance of the circuit is equal to the sum of the resistance vector and the reactance vector.

The Admittance Vector of an Alternating Current Circuit

It has been seen in the foregoing that the vectors of pressure, current and impedance are connected by the relationship

$$\mathfrak{B}_0 = \mathfrak{J}_0\mathfrak{Z} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

This expression may be re-written in the form,

$$\mathfrak{J}_0 = \mathfrak{B}_0\frac{1}{\mathfrak{Z}} = \mathfrak{B}_0\mathfrak{Y} \quad . \quad . \quad . \quad . \quad (61)$$

where $\mathfrak{Y}$ $G + jS$ is the vector of the "admittance" of the circuit expressed in "reciprocal ohms", i.e. "siemens".
G is the "conductance" in siemens,
S is the "susceptance" in siemens.

so that $$\mathfrak{Y} = G + jS = \frac{1}{\mathfrak{Z}} = \frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

or $$G + jS = \frac{R + j\left(\frac{1}{\omega C} - \omega L\right)}{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2},$$

and consequently, the "real" component is

$$G = \frac{R}{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}$$

and the "imaginary" or "j" component is—

$$jS = \frac{j\left(\frac{1}{\omega C} - \omega L\right)}{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}.$$

If the circuit contains only an inductance of L henry as shown in Fig. 76, then

$$R = 0 : \ C = \infty$$

and consequently $G = 0 : \ jS = j\left(\frac{1}{-\omega L}\right) = -j\frac{1}{\omega L} = \mathfrak{Y}.$

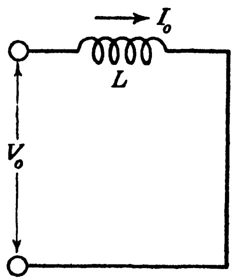


Fig. 76.

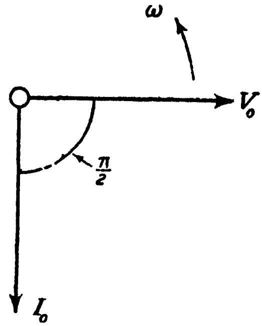


Fig. 77.

The current vector will then be

$$\mathfrak{I}_0 = \mathfrak{V}_0 . \mathfrak{Y} = j\frac{1}{\left(\frac{1}{\omega C} - \omega L\right)}\mathfrak{V}_0, = -j\frac{1}{\omega L}\mathfrak{V}_0$$

as shown in Fig. 77.

If the circuit comprises an inductance of L henry in series with a capacitance of C farad as shown in Fig. 78, then

$$R = 0 : \ G = 0 : \ jS = j\frac{1}{\left(\frac{1}{\omega C} - \omega L\right)},$$

so that the current vector will then be

$$\mathfrak{I}_0 = \mathfrak{V}_0 . \mathfrak{Y} = j\frac{1}{\left(\frac{1}{\omega C} - \omega L\right)}\mathfrak{V}_0 .$$

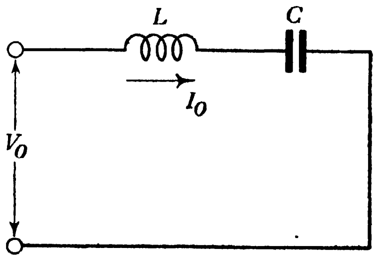


Fig. 78.

If the capacitance C is relatively small so that $\frac{1}{\omega C} > \omega L$, then the admittance will be positive and the current will *lead* by 90° on the pressure vector as shown in Fig. 79. If the capacitance reactance is equal to the inductive reactance, that is, if

$$\omega L = \frac{1}{\omega C} \text{ or } \omega = 2\pi f = \frac{1}{\sqrt{L . C}},$$

the current will be indefinitely large. The vector of pressure across the capacitance will then be equal and opposite to the vector of pressure across the inductance, and this relationship denotes a condition of *pressure resonance*, the circuit then being typical of an " acceptor circuit " and acts like a perfect *conductor*.

Next, consider a parallel arrangement of an inductance of L henry connected across a condenser of capacitance C farad, as shown in Fig. 80.

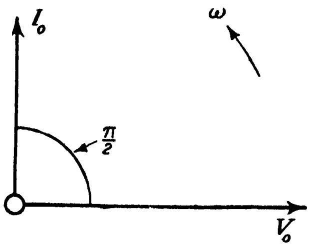


Fig. 79.

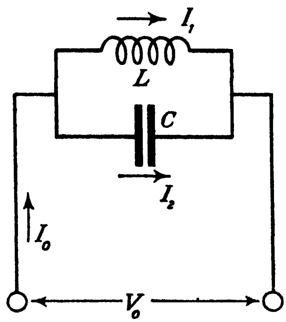


Fig. 80.

For this circuit, the vector of the current in the mains will be

$$\mathfrak{J}_0 = \mathfrak{J}_1 + \mathfrak{J}_2,$$

that is,

$$\mathfrak{J}_0 = \mathfrak{V}_0(\mathfrak{Y}_1 + \mathfrak{Y}_2) = \mathfrak{V}_0 j\left\{\omega C - \frac{1}{\omega L}\right\}$$

since

$$\mathfrak{Y}_1 = j\omega C : \ \mathfrak{Y}_2 = -j\frac{1}{\omega L}.$$

If, then, the supply frequency is such that $\omega C = \frac{1}{\omega L}$, that is,

$$\omega = 2\pi f = \frac{1}{\sqrt{L.C}}$$

the current vector will be $\mathfrak{J}_0 = 0$, and this condition defines a state of *current resonance*. There will then be a circulating current in the closed loop formed by the inductance and capacitance, the magnitude of this circulating current being

$$\mathfrak{J}_1 = -\mathfrak{J}_2 = \mathfrak{V}_0 j\omega C ;$$

but no current will flow in the mains, so that the parallel circuit then acts like a perfect *insulator* and is typical of a " rejector circuit ".

Further treatment of series circuits and parallel circuits will be found in Chapter X.

Circuits Coupled by Mutual Induction

On page 239, Chapter VIII, the relationships between the mutual inductance and the self-inductances of two coupled coils have been established, and in what follows, two practical cases of such circuits coupled by electromagnetic induction will be considered.

(*i*) THE TRANSFORMER.—The diagrammatical representation of a transformer is shown in Fig. 81. If M henry is the coefficient of mutual inductance between the two coils, then

$$X_M = \omega M : \mathfrak{E}_1 = -j\frac{w_1}{w_2}X_M\mathfrak{I}_2 : \mathfrak{E}_2 = -j\frac{w_2}{w_1}X_M\mathfrak{I}_1 \quad . \quad (62)$$

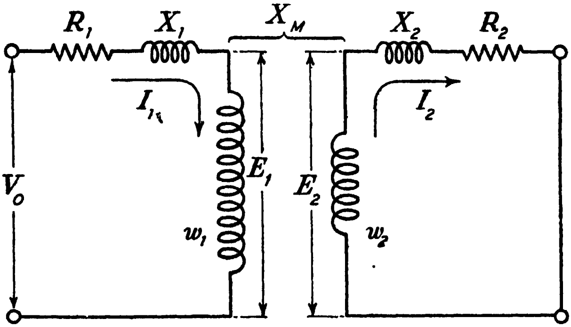


Fig. 81.

The simultaneous equations for the two circuits of the transformer windings may then be written,

$$\left.\begin{aligned} \mathfrak{V}_0 + \mathfrak{E}_1 &= \mathfrak{I}_1 . R_1 + j\mathfrak{I}_1 . X_1 \quad . \quad . \quad . \quad (a) \\ \mathfrak{E}_2 &= \mathfrak{I}_2 . R_2 + j\mathfrak{I}_2 . X_2 \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (63)$$

that is
$$\left.\begin{aligned} \mathfrak{V}_0 &= \mathfrak{I}_1 . R_1 + j\mathfrak{I}_1 . X_1 + j\mathfrak{I}_2\frac{w_1}{w_2}X_M \quad . \quad . \quad . \quad (a) \\ 0 &= \mathfrak{I}_2 . R_2 + j\mathfrak{I}_2 . X_2 + j\mathfrak{I}_1\frac{w_2}{w_1}X_M \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (64)$$

Eliminating the current vector $\mathfrak{I}_2$ between these equations gives

$$\mathfrak{V}_0 = \mathfrak{I}_1\left\{R_1 + \frac{X_M^2}{R_2 + jX_2} + jX_1\right\} \quad . \quad . \quad . \quad (65)$$

After rationalising this equation and collecting the "real" and the "imaginary" terms respectively, it is found that

$$\mathfrak{V}_0 = \mathfrak{I}_1\left\{\left(R_1 + \frac{R_2X_M^2}{R_2^2 + X_M^2}\right) + j\left(X_1 - \frac{X_2X_M^2}{R_2^2 + X_2^2}\right)\right\} \quad . \quad (66)$$

and this equation may be written in the form

$$\mathfrak{V}_0 = \mathfrak{I}_1(R_e + jX_e) \quad . \quad . \quad . \quad . \quad . \quad (67)$$

$$\left.\begin{aligned} \text{where} \quad R_e = R_1 + \frac{R_2 . X_M^2}{R_2^2 + X_2^2} &: \; X_e = X_1 - \frac{X_2 . X_M^2}{R_2^2 + X_2^2} \\ \tan \phi_1 &= \frac{X_e}{R_e} \end{aligned}\right\} \quad . \quad . \quad (68)$$

That is to say, the transformer circuit system of Fig. 81 may be replaced by the equivalent simple series circuit of Fig. 82 in which $\omega L_e = X_e$. Again, from equation (64 *b*) it is seen that

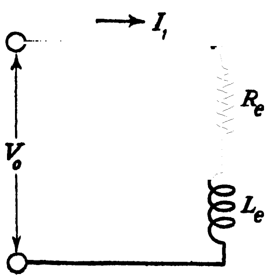


Fig. 82.

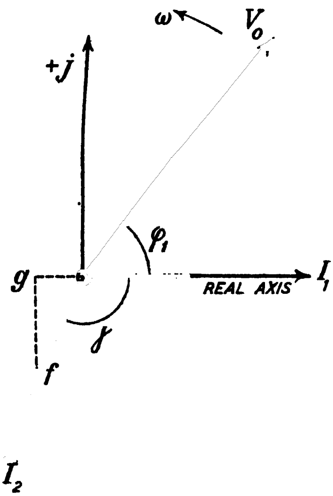


Fig. 83.

$$\mathfrak{I}_2 = -j\frac{\frac{w_2}{\omega_1}X_M}{R_2 + jX_2}\mathfrak{I}_1 = -\frac{w_2}{w_1}X_M\frac{X_2 + jR_2}{R_2^2 + X_2^2}\mathfrak{I}_1 \quad . \quad . \quad (69)$$

from which it follows that

$$\tan \gamma = \frac{-jR_2}{-X_2} = \frac{fg}{Og}$$

as shown in Fig. 83.

It will be seen from the expressions (68) that the equivalent resistance of the primary circuit of the transformer is *increased* by the amount $\frac{R_2 X_M^2}{R_2^2 + X_2^2}$ when the secondary circuit is closed, whilst the equivalent inductance of the primary circuit is *decreased* by the amount $\frac{X_2 . X_M^2}{R_2^2 + X_2^2}$ when the secondary circuit is closed. In other words, *the equivalent resistance of any circuit is increased by mutual induction effects and the equivalent reactance is decreased.*

The foregoing mathematical treatment of the transformer problem assumes that the reactances and resistances of the circuits are all constant quantities, that is to say, that they are independent of the current and frequency.

The Transformation of a Mesh-Connected System into an Equivalent Star-Connected System

The impedance vectors of the mesh-connected system shown in Fig. 84 *a* are, respectively,

$$\mathfrak{Z}_{\mathrm{I}} : \mathfrak{Z}_{\mathrm{II}} : \mathfrak{Z}_{\mathrm{III}}.$$

Let the impedances of the equivalent star-connected system be

$$\mathfrak{Z}_1 : \mathfrak{Z}_2 : \mathfrak{Z}_3.$$

If the two systems are equivalent, that is to say, if the current and power factor of the supply when connected to either system are identically the same, then the admittance between any two supply lines must be the same for each of the two equivalent systems.

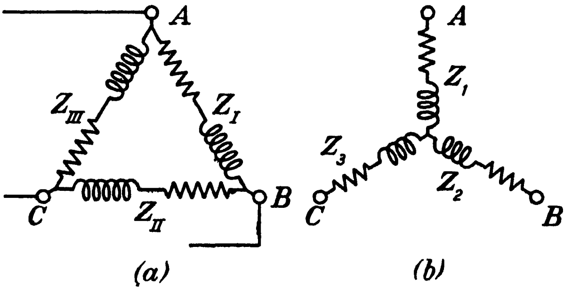


Fig. 84.

The admittance between *A* and *C* in Fig. 84 *a* is the admittance of $\mathfrak{Z}_{\mathrm{I}}$ and $\mathfrak{Z}_{\mathrm{II}}$ in series, and this series combination in parallel with $\mathfrak{Z}_{\mathrm{III}}$.

The admittance of the branch $\mathfrak{Z}_{\mathrm{III}}$ is $\mathfrak{Y}_{\mathrm{III}} = \dfrac{1}{\mathfrak{Z}_{\mathrm{III}}}$ and the admittance of $\mathfrak{Z}_{\mathrm{I}}$ and $\mathfrak{Z}_{\mathrm{II}}$ in series is $\mathfrak{Y}_{\mathrm{I}:\mathrm{II}} = \dfrac{1}{\mathfrak{Z}_{\mathrm{I}} + \mathfrak{Z}_{\mathrm{II}}}$, so that the admittance of the two branches in parallel is

$$\mathfrak{Y}_{\mathrm{III}} + \mathfrak{Y}_{\mathrm{I}\cdot\mathrm{II}} = \frac{1}{\mathfrak{Z}_{\mathrm{III}}} + \frac{1}{\mathfrak{Z}_{\mathrm{I}} + \mathfrak{Z}_{\mathrm{II}}} \quad . \quad . \quad . \quad (70)$$

The admittance between the lines *A* and *C* of the star system of Fig. 84 *b* is

$$\mathfrak{Y}_{1:3} = \frac{1}{\mathfrak{Z}_1 + \mathfrak{Z}_3} \quad . \quad . \quad . \quad . \quad (71)$$

If the two systems of Figs. 84 *a* and 84 *b* are to be equivalent, then from equations (70) and (71)

$$\left.\begin{aligned}\frac{1}{\mathfrak{z}_1+\mathfrak{z}_3}&=\frac{1}{\mathfrak{z}_{III}}+\frac{1}{\mathfrak{z}_I+\mathfrak{z}_{II}}\\ \text{and, similarly,}\quad \frac{1}{\mathfrak{z}_3+\mathfrak{z}_2}&=\frac{1}{\mathfrak{z}_{II}}+\frac{1}{\mathfrak{z}_I+\mathfrak{z}_{III}}\\ \frac{1}{\mathfrak{z}_2+\mathfrak{z}_1}&=\frac{1}{\mathfrak{z}_I}+\frac{1}{\mathfrak{z}_{II}+\mathfrak{z}_{III}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (72)$$

From which it is easily shown that

$$\left.\begin{aligned}\mathfrak{z}_1&=\frac{\mathfrak{z}_I\mathfrak{z}_{III}}{\mathfrak{z}_I+\mathfrak{z}_{II}+\mathfrak{z}_{III}}\\ \mathfrak{z}_2&=\frac{\mathfrak{z}_I\mathfrak{z}_{II}}{\mathfrak{z}_I+\mathfrak{z}_{II}+\mathfrak{z}_{III}}\\ \mathfrak{z}_3&=\frac{\mathfrak{z}_{II}\mathfrak{z}_{III}}{\mathfrak{z}_I+\mathfrak{z}_{II}+\mathfrak{z}_{III}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (73)$$

For the particular case in which each of the systems of Fig. 84 is symmetrical, that is, when

$$\mathfrak{z}_1=\mathfrak{z}_2=\mathfrak{z}_3 \text{ and } \mathfrak{z}_I=\mathfrak{z}_{II}=\mathfrak{z}_{III},$$

then $\mathfrak{z}_1' = \frac{1}{3}\mathfrak{z}_I$ and consequently $R_1 + jX_1 = \frac{1}{3}R_I + \frac{1}{3}jX_I$, so that

$$R_1 = \tfrac{1}{3}R_I : \ X_1 = \tfrac{1}{3}X_I \quad . \quad . \quad . \quad . \quad (74)$$

The Campbell Bridge for the Measurement of Capacitance and Inductance

A widely used Bridge Method for the measurement of capacitance and inductance is shown diagrammatically in Fig. 85, in which M represents a standard variable mutual inductance, whilst C is the capacitance and L is the inductance of which the respective values are to be measured.

The two branch circuits *ad* and *abd* are connected to a source of alternating-current supply, say, 500 frequency, and across the junctions *a* and *b* a telephone (or an a.c. galvanometer) is connected. Adjustments of the mutual inductance M and the resistance S are made so that no current flows in the telephone, that is to say, the telephone will be silent when the bridge is " balanced ", and the p.d. between *a* and *b* will then be zero.

Writing x_L for ωL, x_c for $\dfrac{1}{\omega C}$, and x_M for ωM, then since the branch *ad* contains only the resistance R the p.d. across *ad* will be $\mathfrak{I}_2 . R$. In the branch *abd* the resistance is $P + S$ and the reactance is $x_L - x_C$. When balance has been obtained, the mutual reactance in *ab* will be $x_M = -\omega M$ and the p.d. across *abd* will then be

$$\mathfrak{I}_1\{(P+S)+j(x_L-x_C)\} - jx_M\mathfrak{I}_2.$$

But the p.d. across *ad* has already been seen to be $\mathfrak{I}_2 . R$, so that

$$\mathfrak{I}_2 R = \mathfrak{I}_1\{(P + S) + j(x_L - x_C)\} - j\mathfrak{I} x_M$$

and since $\mathfrak{I} = \mathfrak{I}_1 + \mathfrak{I}_2$

it follows that $\mathfrak{I}_2(R + jx_M) = \mathfrak{I}_1\{(P + S) + j(x_L - x_C - x_M)\}$

that is
$$\frac{\mathfrak{I}_1}{\mathfrak{I}_2} = \frac{R + jx_M}{\{(P + S) + j(x_L - x_C - x_M)\}} \quad . \quad (75)$$

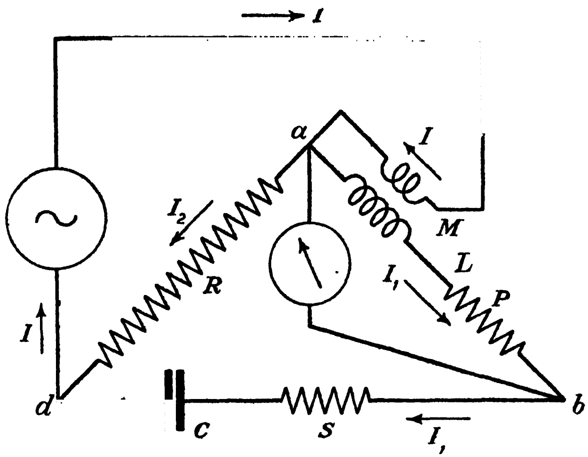


Fig. 85.

Again, since for a balance the p.d. across *ab* is zero, then

$$\mathfrak{I}_1(P + jx_L) - \mathfrak{I} jx_M = 0$$

or since $\mathfrak{I} = \mathfrak{I}_1 + \mathfrak{I}_2$

then
$$\frac{\mathfrak{I}_1}{\mathfrak{I}_2} = \frac{jx_M}{P + j(x_L - x_M)} \quad . \quad . \quad . \quad . \quad . \quad (76)$$

By equating expressions (75) and (76) it is found that

$$P.R - x_M . x_C = j\{S . x_M - R(x_L - x_M)\} \quad . \quad . \quad (77)$$

Since the real parts of the two sides of this equation must be equal and also the imaginary parts of the two sides must be equal, it follows that

$$\left.\begin{aligned} P.R - x_M . x_C &= 0 \quad . \quad . \quad . \quad (a) \\ S . x_M - R(x_L - x_M) &= 0 \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (78)$$

so that from (78 *a*)
$$C = \frac{M}{P.R} \quad . \quad . \quad . \quad . \quad . \quad (79)$$

and from (78 *b*)
$$L = M\,\frac{S + R}{R} . \quad . \quad . \quad . \quad . \quad (80)$$

The Maxwell Bridge Method for Measuring Inductance

A bridge method for measuring inductance by means of a ballistic test was given by Maxwell. The principle of this method was applied by M. Wien to the measurement of inductance using an alternating current source of supply, and in this form the bridge is widely used in practice.

The simplest arrangement of the circuit connections is shown in Fig. 86, where C is a standard capacitance of which the dielectric resistance is infinitely large and r_2 is a resistance of known value, or alternatively, the parallel connection of C and r_2 may in practice be formed by a condenser of which the effective resistance of the dielectric is r_2 and the effective capacitance is C. In the branch ad is the induct-

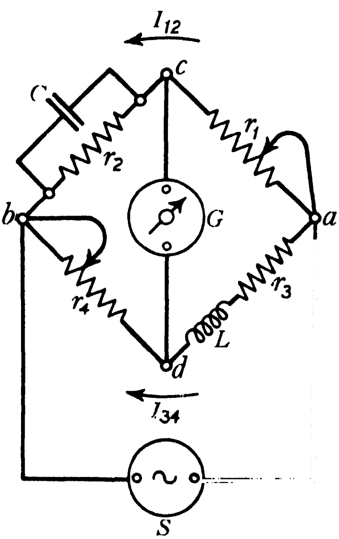


Fig. 86.

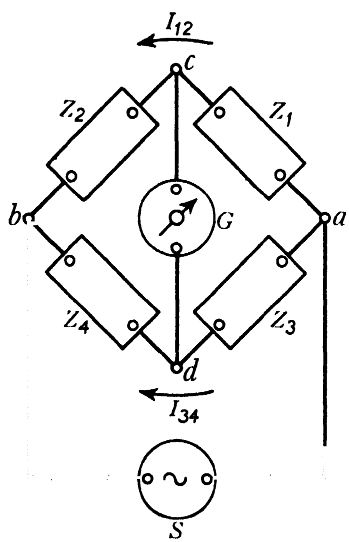


Fig. 87.

ance L which it is required to measure and a resistance r_3, and in practice the resistance r_3 may be the effective resistance of the inductance coil, L itself. The branches ac and db contain the adjustable resistances r_1 and r_4 respectively.

The equivalent simplified system of Fig. 86 is shown in Fig. 87, and when the bridge is balanced, no current will flow through the galvanometer G, so that

$$\mathfrak{I}_{12} = \mathfrak{I}_{34}$$

Further, if $\mathfrak{Z}_2$ is the impedance of the capacitance branch bc of Fig. 86, $\mathfrak{Z}_3$ is the impedance of the inductive branch ad,

then, $$\mathfrak{Z}_1 = r_1 : \mathfrak{Z}_4 = r_4$$

where r_1, r_2, r_3, and r_4 are all in ohms, L is in henry, and C is in farad. The following relationships will then hold, viz.,

pressure drop in branch ac = pressure drop in branch ad

so that $$\mathfrak{I}_{12} \cdot \mathfrak{Z}_1 = \mathfrak{I}_{34} \cdot \mathfrak{Z}_3 \quad . \quad . \quad . \quad . \quad (81)$$

and

pressure drop in branch cb = pressure drop in branch db

so that $$\mathfrak{I}_{12} \cdot \mathfrak{Z}_2 = \mathfrak{I}_{34} \cdot \mathfrak{Z}_4 \quad . \quad . \quad . \quad . \quad (82)$$

Cross-multiplying equations (81) and (82) gives

$$\mathfrak{Z}_1 \cdot \mathfrak{Z}_4 = \mathfrak{Z}_2 \cdot \mathfrak{Z}_3$$

so that $$r_2 r_3 + j\omega L r_2 = r_1 r_4 + j\omega C r_1 r_2$$

and, equating respectively the real components and the imaginary components of this equation, then

$$\frac{r_1}{r_2} = \frac{r_3}{r_4} : L = C r_1 r_4 = C r_2 r_3$$

as already stated in Chapter I, page 17.

The Wien Bridge for the Measurement of Capacitance and Dielectric Angle of Loss

The circuit diagram for this bridge is shown in Fig. 88 as arranged for example, to measure the capacitance C_X of a high-tension cable and its equivalent resistance r_X, which is also a measure of its angle of loss δ (see Chapter XI, page 363). The circuit components shown in Fig. 88 are as follows:

C_S is a standard " no loss " condenser, that is, one for which the dielectric resistance is infinitely large

C_X is the unknown capacitance

r_X is the effective dielectric resistance of the capacitance C_X

R_1 and R_2 are resistance slides

R_K and R_S are resistances the self-inductance and the self-capacitance of which are zero.

R_1 and R_2 are resistance slides.

Balance is obtained by adjusting the slide contacts s_1 and s_2, and when the bridge is balanced no current will flow through the detector G, so that the p.d. between the points a and b will then be zero. It is then easily shown that the following relationships will hold:

$$\begin{cases} \mathfrak{I}_1\left[\dfrac{r_X}{1 + j\omega C_X r_X}\right] = \mathfrak{I}_2\left[\dfrac{1 + jR_2\omega C_S}{j\omega C_S}\right] \\ \mathfrak{I}_1 R_K = \mathfrak{I}_2(R_S + R_1) \end{cases}$$

Cross-multiplying these two equations then gives

$$R_K\left[\frac{1 + jR_2\omega C_S}{j\omega C_S}\right] = (R_S + R_1)\left[\frac{r_X}{1 + j\omega C_X r_X}\right].$$

Reducing this equation to its simplest form and separating out the real and the imaginary components, it is found that

$$\left.\begin{aligned} \omega^2 R_2 r_X C_X C_S &= 1 \\ \left(\frac{r_X \omega C_S}{R_K}\right)(R_S + R_1) &= R_2 \omega C_S + \frac{1}{R_2 \omega C_S} \end{aligned}\right\}$$

from which it is easily seen that

$$\left.\begin{aligned} \tan\delta &= \frac{1}{r_X \omega C_X} = R_2 \omega C_S \\ C_X = \left(\frac{C_S}{R_K}\right)&(R_S + R_1)\cos^2\delta \simeq \frac{C_S}{R_K}(R_S + R_1) \end{aligned}\right\}$$

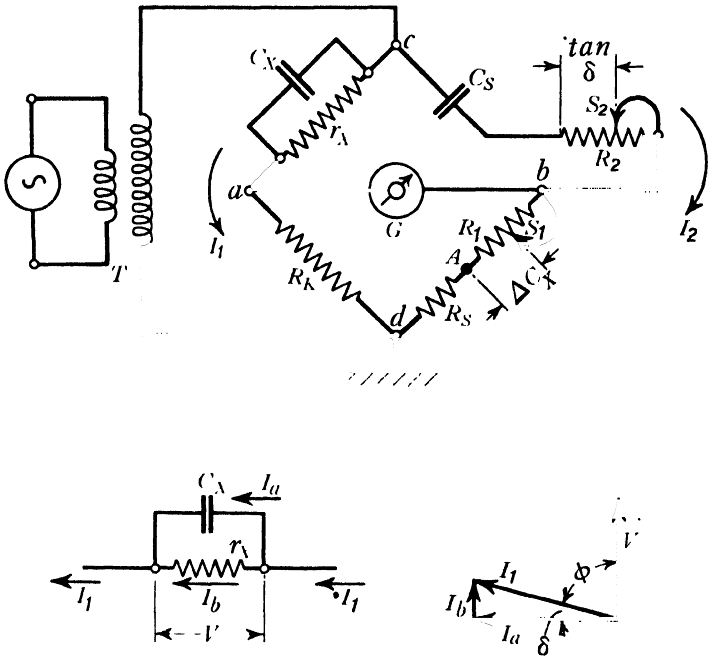


Fig. 88.

Reference to the circuit and vector diagram of Fig. 88 will make this clear, observing that

$$\mathfrak{V} = \frac{\mathfrak{I}_a}{\omega C_X} = \mathfrak{I}_b r_X$$

$$\mathfrak{I}_1 = \mathfrak{I}_a + \mathfrak{I}_b$$

so that
$$\tan\delta = \frac{\mathfrak{I}_b}{\mathfrak{I}_a} = \frac{1}{r_X \omega C_X}$$

The change ΔC_X of the capacitance C_X and the change of the loss factor $\tan\delta$ with the frequency or with the temperature can be satisfactorily measured in this way by reading the respective positions of the slide contacts S_1 and S_2 on the corresponding resistance scales. It will be seen that the capacitance C_X and the loss factor $\tan\delta$ each bear a linear relationship to the positions of the respective slide contacts as read on the corresponding resistance scales.

The Thomson Double Bridge

If the normal type of Wheatstone Bridge (e.g. Fig. 6, page 120) is used to measure low resistances, considerable errors may be introduced owing to the resistance of the connections from the bridge arms to the conductor under test. In order to eliminate this difficulty, Lord Kelvin (Sir W. Thomson) devised the Double-Bridge system, the principle of which will be seen by reference to Fig. 89. The characteristic feature of this bridge is that it comprises a duplication of the normal Wheatstone Bridge system, and in this way the contact resistances of the bridge connections are eliminated from the test, which is primarily intended for the measurement of very low resistances.

In Fig. 89 R_x represents the resistance which is under test, and in Fig. 90 is shown the practical arrangement of the connections. A source of current, which may be either d.c. or a.c., is connected so that a sufficiently large current can be sent through the unknown resistance R_x to give an adequate pressure drop along the resistance for measurement purposes. It is the pressure drop in the connections for coupling this unknown resistance in series with the heavy current circuit, which is eliminated from the measurement by means of this double-bridge system.

Referring to the diagram of Fig. 89, the resistance R_0 corresponds to the heavy current connection lead shown in Fig. 90, and the relationships which must exist between the resistances of the bridge arms can be expressed by stating that the balance of the bridge must be independent of the resistance R_0. Thus, when balance is obtained and the galvanometer shows no deflection, any change of the resistance R_0 will not

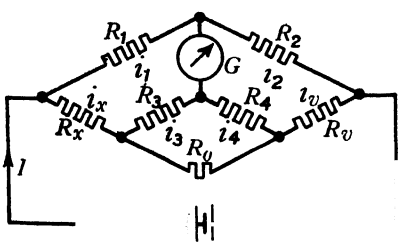


Fig. 89.

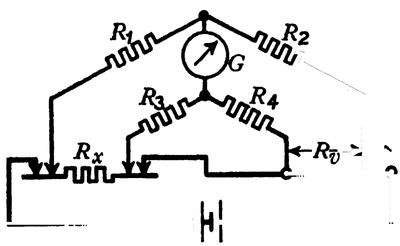


Fig. 90.

affect the balance. Hence, considering the two extreme cases, (i) $R_0 = \infty$, (ii) $R_0 = 0$, then

$$\text{for } R_0 = \infty : \frac{R_1}{R_2} = \frac{R_x + R_3}{R_v + R_4} \quad . \quad . \quad . \quad (83)$$

and

$$\text{for } R_0 = 0 : \frac{R_1}{R_2} = \frac{R_x}{R_v} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Consequently, these two relationships must be simultaneously true, whatever the value of R_0 may be, that is to say, the following relationship must also be true, viz.,

$$\frac{R_x}{R_v} = \frac{R_x + R_3}{R_v + R_4}$$

that is,

$$\frac{R_x}{R_v} = \frac{R_3}{R_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Hence the required conditions for a balanced system, so that no current will pass through the galvanometer, are

$$\boxed{\frac{R_1}{R_2} = \frac{R_3}{R_4} = \frac{R_x}{R_v}} \quad . \quad . \quad . \quad . \quad . \quad (86)$$

In the older model of the double bridge, such as is shown in Fig. 90, the ratio resistances $\frac{R_1}{R_2}$ $\frac{R_3}{R_4}$ were all kept constant, whilst the comparison resistance R_v was adjustable and in the form of a calibrated slide wire of 0·01-ohm resistance. In order to obtain sufficiently high sensitivity for the measurement of small resistances, the current in the main circuit must be sufficiently large and, for a short period of time, the slide wire

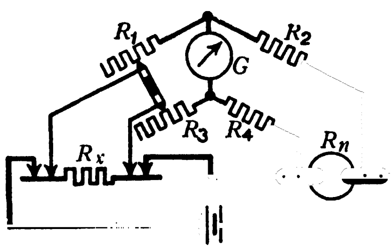


Fig. 91.

is capable of standing a current strength up to 20 amperes. The sensitivity of the bridge is the greater the smaller the resistance R_0 (Fig. 89), that is to say, the smaller the resistance of the connecting lead between R_v and R_x. The test result is obtained from the reading of R_v on the slide wire multiplied by the set ratio $\frac{R_1}{R_2}$ which, in accordance with the

expression (86), must always be equal to the ratio $\frac{R_3}{R_4}$, that is,

$$R_x = R_v \frac{R_1}{R_2} \qquad \qquad (87)$$

In the later model of the double bridge (Fig. 91), a standard resistance R_n is used as the comparison resistance instead of the adjustable slide wire of Fig. 90, and this resistance R_n is kept constant, whilst the ratios $\frac{R_1}{R_2}$ and $\frac{R_3}{R_4}$ are varied in magnitude but are maintained equal to one another. These ratio resistances are so arranged that R_2 and R_4 can only be varied in relatively large steps, whilst R_1 and R_3 can be altered simultaneously with one another by fine adjustment, the required conditions (86) being maintained throughout. When making a test, the resistances are set so that $R_2 = R_4$ and the resistances R_1 and R_3 are adjusted so that the galvanometer shows no deflection, then

$$R_x = R_n \frac{R_1}{R_2} \qquad \qquad (88)$$

In the diagrams of Figs. 89–91 the source of supply is indicated as a d.c. accumulator. The Thomson Double Bridge is, however, suitable for both a.c. and d.c. measurements and one example of a.c. measurements carried out by this means is given in the Test Paper for Chapter IX, Example 14.

Chapter X

OSCILLATING SYSTEMS

BEFORE considering some problems of electrical oscillating circuits it will be helpful, as a basis of comparison and reference, to derive the equations of motion for a damped and an undamped spring system respectively.

The Oscillation of an Undamped Spring System

In Fig. 1 is shown diagrammatically a frictionless spring system which can execute oscillations in a horizontal straight line. A mass of

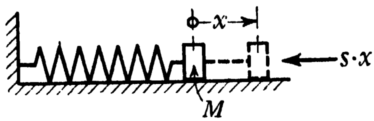


Fig. 1.

m gm. is attached to the spring of strength S dynes per centimetre extension. If x cm. is the displacement of the mass from its stable position of rest the equation of motion will be

$$m\frac{d^2x}{dt^2} = -S.x. \qquad (1)$$

If the mass is M kg. and the strength of the spring is S in kilograms per metre extension, the equation of motion will then be

$$10^3M\frac{d^2x}{dt^2} = -S.x\frac{981 \times 10^3}{100}$$

that is

$$\frac{M}{9{\cdot}81}\frac{d^2x}{dt^2} = -S.x \qquad (2)$$

The solution of this equation is

$$x = A \cos \omega_0 t \qquad (3)$$

where A is the displacement x when $t = 0$ and

$$\omega_0 = \frac{2\pi}{\tau_0} = 2\pi f_0.$$

The periodic time of the natural free oscillations of this system will then be

$$\tau_0 = 2\pi\sqrt{\frac{\left(\frac{M}{9{\cdot}81}\right)}{S}} \text{ sec.} \qquad (4)$$

The Oscillations of a Damped Spring System

If the damping force is proportional to the velocity and of numerical value of a kg. for a velocity of 1 metre per second, the equation of motion will then be

$$\frac{M}{9{\cdot}81}\frac{d^2x}{dt^2}+a\frac{dx}{dt}+S.x=0$$

or
$$\frac{d^2x}{dt^2}+a\left(\frac{9{\cdot}81}{M}\right)\frac{dx}{dt}+S\frac{9{\cdot}81}{M}x=0 \quad . \quad . \quad . \quad (5)$$

For relatively small damping, that is, when $\left(\frac{a9{\cdot}81}{M}\right)^2<4\left(S\frac{9{\cdot}81}{M}\right)$ the solution of this equation is

$$x=Ae^{-\frac{1}{2}a\frac{9\cdot 81}{M}t}\cos(\omega_a t+\alpha),$$

where A and α are arbitrary constants, the respective magnitude of which will depend upon the initial conditions. The circular frequency of the oscillation will be given by

$$\omega_a=\sqrt{S\frac{9{\cdot}81}{M}-\frac{1}{4}\left(a\frac{9{\cdot}81}{M}\right)^2}=\sqrt{\omega_0{}^2-\alpha^2} \quad . \quad . \quad (6)$$

where
$$\alpha=\tfrac{1}{2}\left(a\frac{9{\cdot}81}{M}\right). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the periodic time of the oscillations will be

$$\tau_a=\frac{2\pi}{\omega_a}=\frac{2\pi}{\sqrt{S\frac{9{\cdot}81}{M}}\sqrt{1-\frac{1}{4}\left(\frac{a^2}{S}\frac{9{\cdot}81}{M}\right)}} \quad . \quad . \quad (8)$$

that is,
$$\tau_a=\frac{\tau_0}{\sqrt{1-\Delta^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where
$$\Delta=\alpha.\tau_a\simeq\alpha.\tau_0=\frac{a\pi}{\sqrt{S\frac{M}{9{\cdot}81}}} \quad . \quad . \quad . \quad . \quad (10)$$

and is the "logarithmic decrement" as will be seen by reference to expression (29) and Fig. 7, page 321.

The Free Oscillations of a Circuit comprising Resistance, Capacitance, and Inductance in Series

In Fig. 2 is shown a series circuit, and by means of the switch S, the condenser can first be charged to the pressure V_0 and then discharged by short-circuiting the terminals A and B. The current which will flow when the condenser discharges through the resistance and inductance will now be considered.

When the system is short-circuited by the switch S, as shown in Fig. 2, the equation for the total pressure in the system may be written,

$$e_R + e_L + e_C = 0$$

that is,

$$Ri + L\frac{di}{dt} + \frac{q}{C} = 0 \quad . \quad . \quad . \quad . \quad (11)$$

or, writing, $i = \frac{dq}{dt}$ the equation (11) becomes

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0 \quad . \quad . \quad . \quad . \quad (12)$$

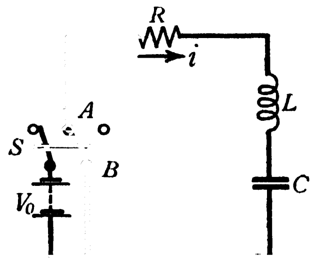


Fig. 2.

The general solution of this differential equation may be written

$$q = A_1e^{\gamma_1 t} + A_2e^{\gamma_2 t} \quad . \quad . \quad . \quad . \quad (13)$$

where γ_1 and γ_2 are the roots of the quadratic equation,

$$\gamma^2 + \frac{R}{L}\gamma + \frac{1}{LC} = 0 \quad . \quad . \quad . \quad . \quad (14)$$

viz.

$$\left.\begin{matrix}\gamma_1\\ \gamma_2\end{matrix}\right\} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad . \quad . \quad . \quad (15)$$

Two distinct cases are to be distinguished in interpreting the expressions (15): (*i*) when $\left(\frac{R}{2L}\right)^2$ is $> \frac{1}{LC}$, (*ii*) when $\left(\frac{R}{2L}\right)^2$ is $< \frac{1}{LC}$.

Case I.—$\left(\frac{R}{2L}\right)^2$ is $> \frac{1}{LC}$. In this case the roots γ_1 and γ_2 given by the expressions (14) are both real and negative, that is to say,

$$q = A_1e^{\gamma_1 t} + A_2e^{\gamma_2 t},$$

the arbitrary constants A_1 and A_2 being determined by the initial conditions. The discharge of the condenser will then be non-oscillatory (non-periodic) and the current will also be non-oscillatory.

EXAMPLE.—$L = 0{\cdot}005$ henry : $C = 200 \times 10^{-6}$ farad : $R = 20$ ohms : so that

$$\frac{R}{L} = 4{,}000 : \quad \frac{1}{4}\left(\frac{R}{L}\right)^2 = 4 \times 10^6 : \quad \frac{1}{LC} = 10^6.$$

The corresponding values for the roots of the quadratic equation (14) will then be $\gamma_1 = -3{,}730$: $\gamma_2 = -270$. Assuming as initial conditions that when $t = 0$: $i = 0$ and that the condenser is charged to a p.d. of 200 volts, so that $q_0 = 0{\cdot}04$ coulomb, then the equation for the quantity at any time t will be

$$q = 0{\cdot}04312e^{-270t} - 0{\cdot}00312e^{-3730t} \text{ coulomb.}$$

The graph of this equation is shown in Fig. 3. The corresponding equation of the current will be

$$i = 11{\cdot}6e^{\ 3730t} - 11{\cdot}6e^{\ 270t} \text{ amperes,}$$

and the graph of this equation is shown in Fig. 4.

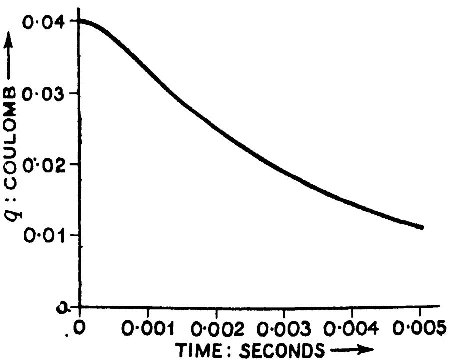


Fig. 3.

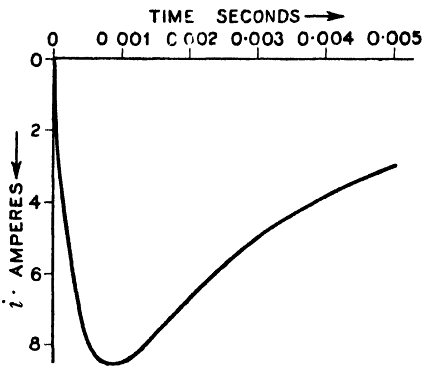


Fig. 4.

CASE II.—$\left(\frac{R}{2L}\right)^2$ is $< \frac{1}{LC}$. In this case it will be seen from the expressions (15) that the roots of the quadratic equation (14) are complex quantities

$$\left.\begin{matrix}\gamma_1 \\ \gamma_2\end{matrix}\right\} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad -\alpha \pm j.\nu \quad . \quad . \quad (16)$$

where α is written for $\frac{R}{2L}$ and ν for $\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$. The quantity $\frac{R}{2L}$ is also equal to $\frac{1}{2T}$ where T is the time constant (see page 270), so that

$$\alpha = \frac{R}{2L} = \frac{1}{2T}$$

The quantity ν is the "natural circular frequency" of the circuit. It has already been seen on pages 270, 274 and 285, Chapter IX, that if there were no resistance in the circuit the natural frequency of oscillation would be given by

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

so that (see Fig. 5) $\nu^2 = \omega_0{}^2 - \alpha^2 : \ \omega_0{}^2 = \nu^2 + \alpha^2$ . . . (17)

The general solution of equation (16) will have two arbitrary constants and may be written in the form

$$q = A_1 e^{(-\alpha + j\nu)t} + A_2 e^{(-\alpha - j\nu)t}$$

that is,

$$q = e^{-\alpha t}[A_1 e^{j\nu t} + A_2 e^{-j\nu t}].$$

Now, since $e^{\pm j\nu t} = \cos \nu t \pm j \sin \nu t$ (see Appendix No. I, page 523), and further, since only the real components of this expression will have any physical significance (see Chapter IX, page 295), the general solution of equation (16) may be written in the form,

$$q = e^{-\alpha t}[G \cos \nu t + H \sin \nu t] \quad . \quad . \quad . \quad (18)$$

where the values of the arbitrary constants G and H will be determined by the initial conditions. If the initial conditions are defined by

$$t = 0, \ q = V.C : \ i = 0,$$

then substitution in equation (18) for $t = 0 : \ q = V.C$ gives

$$G = V.C$$

where V is the pressure to which the condenser is initially charged and C is the capacitance of the condenser.

Also, by differentiating equation (18) with respect to t and substituting $i = 0 : \ t = 0$, that is

$$\Big] i \Big]_{t=0} = \frac{dq}{dt}\Big]_{t=0} = 0$$

it is found that

$$H = G.\frac{\alpha}{\nu} = V.C.\frac{\alpha}{\nu}$$

so that equation (18) now becomes

$$q = V.Ce^{-\alpha t}\left[\cos \nu t + \frac{\alpha}{\nu} \sin \nu t\right] \quad . \quad . \quad . \quad (19)$$

which reduces to

$$q = V.C\frac{\omega_0}{\nu}e^{-\alpha t} \cos (\nu t - \delta) \quad . \quad . \quad . \quad (20)$$

where
$$\tan \delta = \frac{\alpha}{\nu}$$
$$\cos \delta = \frac{\nu}{\omega_0}$$

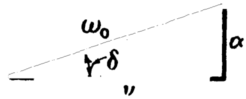


Fig. 5.

as shown in Fig. 5.

The equation for the current is then,

$$i = \frac{dq}{dt} = V.C\frac{\omega_0}{\nu}e^{-\alpha t}[-\alpha \cos (\nu t - \delta) - \nu \sin (\nu t - \delta)]$$

which reduces to

$$i = V.C\frac{\omega_0^2}{\nu}e^{-\alpha t} \cos \left(\frac{\pi}{2} + \nu t\right) \quad . \quad . \quad . \quad (21)$$

The equation for the rate of change of the current is

$$\frac{di}{dt} = - V.C\frac{\omega_0^2}{\nu}e^{-\alpha t}\left[\alpha \cos \left(\frac{\pi}{2} + \nu t\right) + \nu \sin \left(\frac{\pi}{2} + \nu t\right)\right]$$

which reduces to

$$\frac{di}{dt} = - V.C\frac{\omega_0^3}{\nu}e^{-\alpha t} \cos (\pi + \delta + \nu t). \quad . \quad . \quad (22)$$

It follows, therefore, that the pressures across the respective components of the series circuit at any moment t, are as follows:

For the condenser C,

$$e_C = \frac{q}{C} = V\frac{\omega_0}{\nu}e^{-\alpha t} \cos (\nu t - \delta)$$

For the resistance R,

$$e_R = i.R = V.C.R\frac{\omega_0^2}{\nu}e^{-\alpha t} \cos\left(\nu t + \frac{\pi}{2}\right) \quad . \quad . \quad (23)$$

For the inductance L,

$$e_L = - L\frac{di}{dt} = V.C.L\frac{\omega_0^3}{\nu}e^{-\alpha t} \cos (\nu t + \delta + \pi)$$

In Fig. 6 the three vectors for these component pressures are shown in their correct relative positions for the moment $t = 0$, the magnitudes of the respective vectors then being a maximum, viz.,

$$\left.\begin{aligned} E_C &= V.\frac{\omega_0}{\nu} \\ E_R &= \frac{V.C.R\omega_0^2}{\nu} = \frac{V}{L.\nu}R \\ E_L &= \frac{V.C.L.\omega_0^3}{\nu} = V.\frac{\omega_0}{\nu} \end{aligned}\right\} \quad . \quad . \quad . \quad (24)$$

Reference should also be made to the treatment of this same problem in Chapter IX, pages 275–277, Fig. 40.

The maximum pressure across the condenser terminals will be $e_C]_{t=0} = V = \frac{Q}{C}$ volts, and the maximum peak value of the current will be obtained when $t = \frac{\pi}{2\nu}$ sec., that is,

$$i]_{t=\frac{\pi}{2\nu}} = I_0 = Q\frac{\omega_0^2}{\nu}e^{-\frac{\alpha\pi}{\nu\,2}} \simeq \frac{Q.\omega_0}{\cos\delta} = \frac{Q}{\sqrt{LC}} = V\sqrt{\frac{C}{L}} \quad . \quad . \quad (25)$$

The foregoing results have been obtained under the assumption that

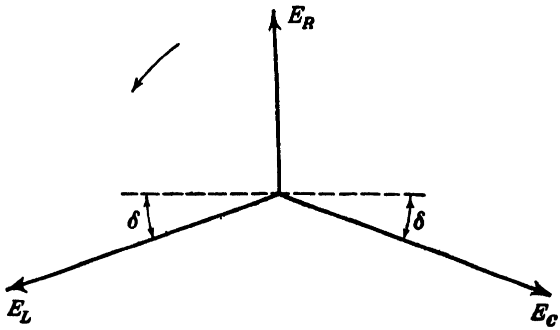


Fig. 6.

the circuit resistance is relatively small; the energy relationships are then approximately as follows:

(i) The electromagnetic energy which is stored in the magnetic field of the inductance when the current has its maximum peak value I_0 amperes is

$$\tfrac{1}{2}LI_0^2 \text{ joules} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

(ii) The electrostatic energy stored in the dielectric of the condenser when the pressure across the terminals has its maximum peak value V is

$$\tfrac{1}{2}CV^2 \text{ joules} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and since the loss of power in the circuit is relatively very small, it follows that

$$\tfrac{1}{2}LI_0^2 \simeq \tfrac{1}{2}CV^2$$

that is

$$\frac{V}{I_0} \simeq \sqrt{\frac{L}{C}},$$

as in expression (25).

The expression (21) for the current in the circuit shows that the oscillations are damped in accordance with the factor $e^{-\alpha t}$ and the damping may be expressed in terms of the ratio of the magnitudes of

two successive positive (or two successive negative) peaks, for example, I_1 and I_2, as shown in Fig. 7. The damping is then defined by the ratio

$$\frac{I_2}{I_1} = \frac{e^{-\alpha(t+\tau)}}{e^{-\alpha t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

that is, $$\frac{I_2}{I_1} = e^{-\alpha\tau} = e^{-\Delta}$$

or $$\Delta = \alpha . \tau = \log_e \frac{I_2}{I_1}$$

where Δ is the *logarithmic decrement*, viz. $\quad . \quad (29)$

$$\Delta = \alpha . \tau \simeq \frac{R}{2L} 2\pi \sqrt{LC} \simeq \pi R \sqrt{\frac{C}{L}}$$

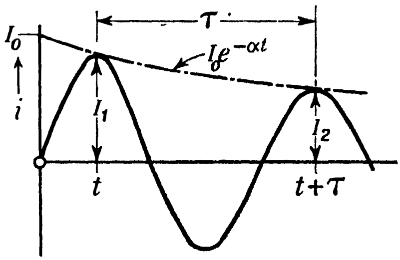


Fig. 7.

that is, the logarithmic decrement for two successive positive amplitudes as shown in Fig. 7, it being assumed that α is small in comparison with ω_0, that is, in comparison with $\frac{1}{\sqrt{LC}}$.

From the expressions (29) it is easily seen that the following relationships exist between successive positive peak values of the current wave

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = \quad . \ . \ . \quad = \frac{I_{n-1}}{I_n} = e^{\Delta} . \quad . \quad . \quad . \quad (30)$$

so that $$\frac{I_1}{I_n} = e^{(n-1)\Delta}$$

and consequently

$$\Delta = \frac{1}{(n-1)} \log_e \frac{I_1}{I_n} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It will also be apparent from the expression (30) that the peak value of the oscillatory current, which would be obtained if there were no resistance in the circuit, would be (see Fig. 5)

$$I_0 = I_1 e^{\Delta/4} = I_n e^{(n-3/4)\Delta} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In the case of a *ballistic galvanometer* (see also pages 339–342), the same relationships will exist between the magnitudes of the successive deflections as are defined by the expressions (30), for the successive positive peak values of the oscillatory current. Thus, if in Fig 6, $\theta_1 : \theta_2 : \ldots \theta_n \ldots$ represent the successive maximum deflections on the same side of the galvanometer scale zero, the logarithmic decrement Δ will be given by the expression

$$\Delta = \frac{1}{n-1} \log_e \frac{\theta_1}{\theta_n} \quad . \quad . \quad . \quad . \quad (33)$$

By observing the 1st and the nth deflection on the same side of the scale zero, the logarithmic decrement is at once found from expression (33).

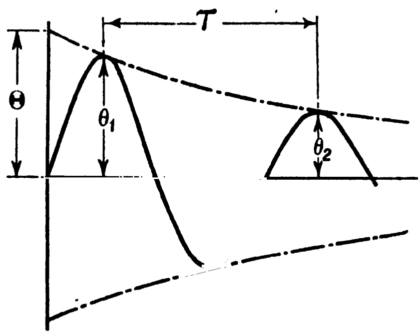


Fig. 8.

The true deflection of the galvanometer as corrected for damping will then be (see also Fig. 8)

$$\Theta = \theta_1 e^{\Delta/4} \simeq \theta_1\left(1 + \frac{\Delta}{4}\right) \quad . \quad . \quad . \quad . \quad (34),$$

It is of interest to note here that the decay factor of the exponential curve shown by the broken line in Fig. 8 (see also Fig. 7) is

$$\alpha = \frac{R}{2L} = \frac{1}{2T},$$

where $T = \frac{L}{R}$, and in the *electro-magnetic time constant* of the circuit, whereas the decay factor in the case of an inductance discharging through a resistance as has been considered on page 268, Chapter IX, is $\frac{1}{T}$.

It is also interesting to note that

$$\alpha = \frac{R}{2L} = \frac{1}{\sqrt{LC}}$$

that is,

$$\alpha = \omega_0$$

and is the critical condition which distinguishes between an oscillatory discharge and an aperiodic discharge of a series circuit of condenser, inductance and resistance.

If the damped oscillatory current of expression (23), viz.

$$i = I_0 e^{-\alpha t} \cos\left(\nu t + \frac{\pi}{2}\right),$$

is measured by means of a hot-wire ammeter, the effective heating value of the current will be given by

$$\frac{H}{R} = \int_0^\infty i^2\,dt = \int_0^\infty I_0^2 e^{-2\alpha t} \cos^2\left(\nu t + \frac{\pi}{2}\right)dt = \frac{I_0^2}{4\alpha} + \frac{I_0^2}{4\omega_0^2} \simeq \frac{I_0^2}{4\alpha} \quad . \quad (35)$$

If there is a train of a complete oscillations per second of the wave of the discharging process, the heating effect of the current will be given by

$$H = R\frac{a}{4\alpha}I_0^2 \text{ joules} \quad . \quad . \quad . \quad . \quad (36)$$

Example.—Find the number of complete cycles in the oscillatory discharge of a condenser after which the current peak value will be less than x per cent. of the first peak value of the discharge current.

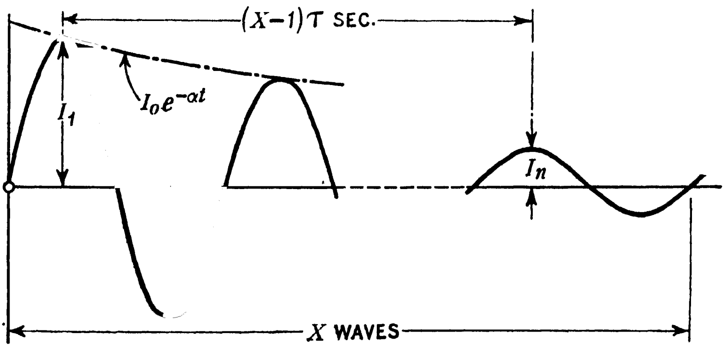


Fig. 9

Referring to Fig. 9, suppose that after X complete cycles have passed, the peak value is less than x per cent. of its initial peak value, then :

$$\frac{I_n}{I_1} = e^{-\alpha(X-1)\tau} = \frac{x}{100}$$

so that
$$\log_e \frac{100}{x} = \alpha(X-1)\tau$$

or
$$X.\Delta - \Delta = \log_e \frac{100}{x} : \text{ (since } \Delta = \alpha.\tau)$$

so that
$$X = \frac{\log_e \frac{100}{x} + \Delta}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

If the constants of the oscillatory circuit have the following respective values,

$$C = 2{,}250 \text{ cm.} = 2{\cdot}5 \times 10^{-9} \text{ farad} : \quad L = 10^5 \text{ cm.} = 10^{-4} \text{ henry} :$$

$$R = 20 \text{ ohms} : \quad V = 1{,}000 \text{ volts, then} :$$

$$\alpha = \frac{R}{2L} = \frac{20 \times 10^4}{2} = 10^5 : \quad \omega_0{}^2 = \frac{1}{L.C} = \frac{10^{13}}{2{\cdot}5} = 4 \times 10^{12}.$$

$$f = \frac{\omega_0}{2\pi} = 318{,}000 \text{ hz.} : \quad \nu = \sqrt{\omega_0{}^2 - \alpha^2} \simeq 2 \times 10^6 :$$

$$\tau = \frac{1}{f} = 3{\cdot}14 \times 10^{-6} \text{ sec.} : \quad \Lambda = \pi R \sqrt{\frac{C}{L}} = \alpha.\tau = 0{\cdot}314.$$

The number of cycles which will be passed through before the current amplitude falls to 1 per cent. of its initial value is

$$X = \frac{\log_e 100 + 0{\cdot}314}{0{\cdot}314} \simeq 16 \text{ cycles},$$

so that the time required for the current amplitude to fall to 1 per cent. of its initial value is

$$X.\tau = 16 \times 3{\cdot}14 \times 10^{-6} \simeq 5 \times 10^{-5} \text{ second.}$$

Since the maximum pressure at the condenser terminals is $V_0 = 1{,}000$ volts, the maximum peak value of the oscillating current will be

$$I_0 \simeq V_0 \sqrt{\frac{C}{L}} \simeq 1{,}000 \sqrt{\frac{2{\cdot}5 \times 10^{-9}}{10^{-4}}} = 5 \text{ amperes},$$

and the heating effect H of this current will then be given by

$$\frac{H}{R} = \frac{I_0{}^2}{4\delta} = \frac{25}{4 \times 10^5} = 6{\cdot}25 \times 10^{-5},$$

so that the heat energy dissipated in the circuit resistance of 20 ohms will be

$$H = 6{\cdot}25 \times 10^{-5} \times 20 = 1{,}250 \times 10^{-6} \text{ joules.}$$

It is convenient to tabulate for reference purposes the more important formulae relative to oscillatory circuits, assuming that the damping factor α of the circuit is relatively very small (Table I, page 325).

Forced Oscillations

In Fig. 10 is shown a circuit comprising a resistance of R ohms, an inductance of L henry, and a capacitance of C farad in series and connected across a supply pressure which is defined by the vector

$$\mathfrak{V} = Ve^{j\omega t} = V \cos \omega t \quad . \quad . \quad . \quad . \quad (38)$$

TABLE I

Natural circular frequency .	ω_0	$\dfrac{1}{\sqrt{L.C}}$
Natural frequency	f_0	$\dfrac{1}{2\pi\sqrt{L.C}}$ hz.
Periodic time . . .	τ_0	$2\pi\sqrt{L.C}$ sec.
Logarithmic decrement . .	Λ	$\pi R\sqrt{\dfrac{C}{L}} = \dfrac{\pi R}{L\cdot\omega_0}$ neper *
Wave length	λ	$2\pi c\sqrt{L.C}$ cm. (for L in henry and C in farad)
Velocity of propagation . .	c	3×10^{10} cm. per sec
Wave length	λ	$2\pi\sqrt{L_{cm} \times C_{cm}}$ cm.
Electrostatic unit of capacitance		1 cm. $= \dfrac{1}{9 \times 10^{11}}$ farad
Electrostatic unit of inductance.		1 cm. $= \dfrac{1}{10^9}$ henry

* Note: 1·15 nepers = 10 decibels.

From the results already obtained in the foregoing, the vector of the circuit impedance is

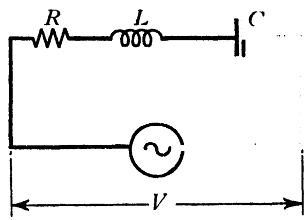


Fig. 10.

$$\mathfrak{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = Ze^{j\phi}$$

where $\tan\phi = \dfrac{\omega L - \dfrac{1}{\omega C}}{R}$

and $$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \text{ ohms,}$$

so that the vector of the current will be

$$\mathfrak{J} = \frac{\mathfrak{V}}{\mathfrak{Z}} = \frac{Ve^{j\omega t}}{Ze^{j\phi}} = \frac{V}{Z}e^{j(\omega t - \phi)}$$

that is $$\mathfrak{J} = I\cos(\omega t - \phi)$$

where $$I = \frac{V}{Z}.$$

When $\omega L = \dfrac{1}{\omega C}$, that is, when $\omega^2 = \dfrac{1}{L.C} = \omega_0{}^2$, the current will have its maximum amplitude $I_0 = \dfrac{V}{R}$ and the condition is then one of

pressure resonance. Hence, at any frequency ω the amplitude I of the current will be given by

$$\left(\frac{I}{I_0}\right)^2 = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad . \quad . \quad . \quad (39)$$

Denoting, as before, $\frac{R}{2L}$ by α and writing $\omega_0^2 = \frac{1}{L.C}$ the natural frequency ν of the circuit is then given by (see also pages 270 and 318)

$$\nu^2 = \omega_0^2 - \alpha^2 \quad . \quad . \quad . \quad . \quad (40)$$

and the logarithmic decrement by

$$\Delta = \alpha.\tau = \frac{\pi R}{L\nu}$$

so that

$$R = \frac{\Delta L\nu}{\pi} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Substituting this value for R in the expression (39) gives

$$\left(\frac{I}{I_0}\right)^2 = \frac{\Delta^2\left(\frac{L\omega\nu}{\pi}\right)^2}{\Delta^2\left(\frac{L\nu}{\pi}\right)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \frac{\Delta^2}{\Delta^2 + \pi^2\frac{\omega_0^4}{\omega^2\nu^2}\left(\frac{\omega^2}{\omega_0^2} - 1\right)^2} \quad . \quad (42)$$

If α^2 is small in comparison with ω_0^2 then from expression (40) $\nu \simeq \omega_0$ and the expression (42) becomes

$$\left(\frac{I}{I_0}\right)^2 = \frac{\Delta^2}{\Delta^2 + \pi^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \quad . \quad . \quad . \quad (43)$$

EXAMPLE.—Let $R = 0{\cdot}25$ ohm : $L = 0{\cdot}005$ henry : $C = 200 \times 10^{-6}$ farad : $V = 200$ r.m.s. volts : then

$$I_0 = \frac{V}{R} = 800 \text{ r.m.s. amperes} : \quad \alpha = \frac{R}{2L} = 25 :$$

$$\omega_0 = \frac{1}{\sqrt{L.C}} = 1{,}000 : \quad \nu^2 = \omega_0^2 - \alpha^2 = 10^6 - 625 \simeq 10^6 :$$

so that $\nu \simeq \omega_0 = 1{,}000$: $\Delta = \frac{\pi R}{L.\omega_0} = 0{\cdot}157$: $\Delta^2 = 0{\cdot}026$: and

$$\left(\frac{I}{I_0}\right)^2 = \frac{0{\cdot}026}{0{\cdot}026 + \pi^2\left(\frac{\omega}{10^3} - \frac{10^3}{\omega}\right)^2} \quad . \quad . \quad . \quad (44)$$

If this relationship is plotted as in Fig. 11 with the frequency f as the abscissae, the curve shown gives the r.m.s. value of the current I. In

Fig. 11 is also shown the relationship of I and f for the case in which the circuit resistance is $R = 1$ ohm, so that the maximum current which is obtained at resonance in this case is $I = \frac{V}{R} = 200$ r.m.s. amperes.

Referring now to the graph in Fig. 12 which shows the relationship between the ratio $\left(\frac{I}{I_0}\right)$ and the circular frequency ω as a general case.

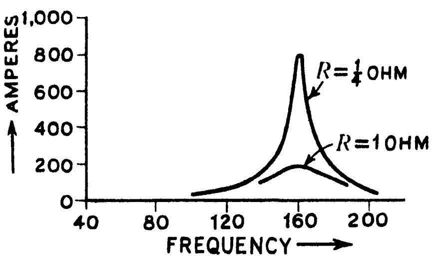


Fig. 11.

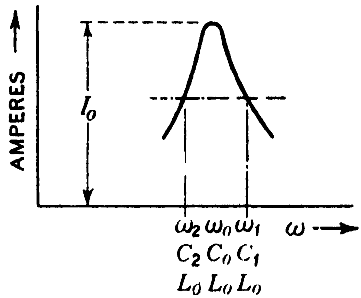


Fig. 12.

If a horizontal line be drawn such as AB, it is seen that the same current ratio $\left(\frac{I}{I_0}\right)$ is obtained for each of the frequencies ω_1 and ω_2 and the practical significance of this result will now be considered. From the expression (43) it is easily seen that for the supply frequency ω_1

$$\left(\frac{I_0}{I}\right)^2 - 1 = \frac{\pi^2\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right)^2}{\Delta^2} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

that is
$$\Delta\sqrt{\left(\frac{I_0}{I}\right)^2 - 1} = \pi\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right),$$

and, after substituting $\Delta = \frac{\pi R}{L\omega_0}$ in this expression, it will be found that, for a supply frequency ω_1

$$\omega_1\frac{R}{L}\sqrt{\left(\frac{I_0}{I}\right)^2 - 1} = \omega_1{}^2 - \omega_0{}^2 \quad . \quad . \quad . \quad (46)$$

In a precisely similar way it will be found that for a supply frequency ω_2 that

$$\omega_2\frac{R}{L}\sqrt{\left(\frac{I_0}{I}\right)^2 - 1} = \omega^2{}_0 - \omega_2{}^2.$$

After adding the two expressions (45) and (46) and rearranging the terms, the result is obtained that

$$\frac{R}{L}\sqrt{\left(\frac{I_0}{I}\right)^2 - 1} = \omega_1 - \omega_2 = 2\pi(f_1 - f_2) \quad . \quad . \quad (47)$$

If, therefore, the supply frequency is adjusted so that for each of the values f_1 and f_2 the same value of the ratio $\frac{I_0}{I}$ is obtained, then the expression (47) will give the value of $\frac{R}{L}$ and hence the logarithmic decrement

$$\Delta = \frac{\pi R}{L\omega_0}.$$

EXAMPLE.—Suppose a carrier wave of frequency f_0 is modulated with a maximum modulation frequency of 5 kHz., so that $f - f_0$ has the constant value of 5,000 hz., then:

(*i*) For a logarithmic decrement $\Delta = 1$: $f_0 = 10^4$ hz., and $f = 1{\cdot}5 \times 10^4$ hz., substitution in the expression (43) shows that

$$\frac{I}{I_0} = \frac{1}{\sqrt{1 + \pi^2(1{\cdot}5 - 0{\cdot}66)^2}} = 36 \text{ per cent.}$$

(*ii*) For a logarithmic decrement $\Delta = 0{\cdot}4$: $f_0 = 10^4$ hz., and $f = 1{\cdot}5 \times 10^4$ hz.

$$\frac{I}{I_0} = \frac{0{\cdot}4}{\sqrt{0{\cdot}16 + \pi^2(1{\cdot}5 - 0{\cdot}66)^2}} = 15{\cdot}2 \text{ per cent.}$$

EXAMPLE.—From the data given in the table on page 325 it is seen that the following expression holds for the wave-length in metres,

$$\lambda \text{ (metres)} = 1{,}885\sqrt{(C \text{ in } \mu\text{F}) \times (L \text{ in } \mu\text{H})} \quad . \quad . \quad (48)$$

For a given coil, the capacitance was measured and found to be $C = 475 \times 10^{-6}\,\mu$F and the resonance wave-length was 450 metres, so that

$$450^2 = (1{,}885)^2(475 \times 10^{-6})(L \text{ in } \mu\text{H}),$$

from which it is found that the inductance of the coil was

$$L = 119{\cdot}3\ \mu\text{H (i.e. microhenry).}$$

Hence, from expression (47),

$$R = 2\pi L(f_1 - f_2)\sqrt{\frac{I^2}{I_0^2 - I^2}} \text{ for } f_1 \text{ and } f_2 \text{ in hz.,}$$

that is, $$R = 750 \times 10^{-6}(f_1 - f_2)\sqrt{\frac{I^2}{I_0^2 - I^2}} \text{ for } f_1 \text{ and } f_2 \text{ in hz.,}$$

or $$R = 0{\cdot}75(f_1 - f_2)\sqrt{\frac{I^2}{I_0^2 - I^2}} \text{ for } f_1 \text{ and } f_2 \text{ in kHz.,}$$

so that if the current at resonance is known and also the respective values of the frequencies f_1 and f_2 which give the same current values (see Fig. 12), then the resistance of the coil can at once be calculated. This procedure forms a very convenient method for measuring the effective resistance of a coil at very high frequencies for which the "skin effect" becomes prominent (see also Chapter XIV). The following numerical data were obtained in an actual test:

$$(i)\ \begin{aligned} I_0 &= 103 \times 10^{-3} \text{ ampere} \\ I &= \ \ 51 \times 10^{-3} \text{ ampere} \end{aligned} \left\{ \begin{aligned} & f_0 = 433 \text{ kHz.}: \ f_1 = 443 \text{ kHz.}: \\ & f_2 = 424 \text{ kHz.} \\ & \quad (I_0^2 - I^2) = 8{,}000 \times 10^{-6}: \\ & \sqrt{\frac{I^2}{I_0^2 - I^2}} = 0{\cdot}57, \end{aligned} \right.$$

so that $\qquad R = 0{\cdot}75 \times 19 \times 0{\cdot}57 = 8{\cdot}13$ ohms.

$$(ii)\ \begin{aligned} I_0 &= 70 \times 10^{-3} \text{ ampere} \\ I &= 40 \times 10^{-3} \text{ ampere} \end{aligned} \left\{ \begin{aligned} & f_0 = 782 \text{ kHz.}: \ f_1 = 792 \text{ kHz.}: \\ & f_2 = 770 \text{ kHz.} \\ & \quad I_0^2 - I^2 = 6{,}170 \times 10^{-6}: \\ & \sqrt{\frac{I^2}{I_0^2 - I^2}} = 0{\cdot}696, \end{aligned} \right.$$

so that $\qquad R = 11{\cdot}5$ ohms.

$$(iii)\ \begin{aligned} I_0 &= 88 \times 10^{-3} \text{ ampere} \\ I &= 40 \times 10^{-3} \text{ ampere} \end{aligned} \left\{ \begin{aligned} & f_0 = 596 \text{ kHz.}: \ f_1 = 608 \text{ kHz.}: \\ & f_2 = 580 \text{ kHz.}: \\ & \sqrt{\frac{I^2}{I_0^2 - I^2}} = 0{\cdot}5, \end{aligned} \right.$$

so that $\qquad R = 9{\cdot}15$ ohms.

and from these results it will be seen how the resistance of the coil depends upon the frequency of the current.

The logarithmic decrement Δ may also be defined in terms of the wavelengths λ_0 and λ since from the table on page 325 it is seen that $\frac{\omega_0}{\omega} = \frac{\lambda}{\lambda_0}$, and hence

$$\Delta = \pi\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{\lambda_0}{\lambda} - \frac{\lambda}{\lambda_0}\right) \quad . \quad . \quad . \quad . \quad (49)$$

Similarly, if for a given value of L the corresponding values of the capacitances are respectively C_0 and C, so that $C_0L = \frac{1}{\omega_0^2}$: $CL = \frac{1}{\omega^2}$ and consequently $\frac{\lambda_0}{\lambda} = \sqrt{\frac{C_0}{C}}$, then

$$\Delta = \pi\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{C_0 - C}{C_0}\right)\sqrt{\frac{C_0}{C}} \quad . \quad . \quad . \quad (50)$$

and in the same way it follows that

$$\Delta = \pi\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{L_0 - L}{L_0}\right)\sqrt{\frac{L_0}{L}} \quad . \quad . \quad . \quad (51)$$

Again, suppose the capacitance to be so adjusted that (see Fig. 12),

(*i*) the frequency $\dfrac{1}{\sqrt{C_1 L_0}} = \omega_1$ and is $> \omega_0$,

(*ii*) ,, ,, $\dfrac{1}{\sqrt{C_2 L_0}} = \omega_2$ and is $< \omega_0$,

the magnitude I of the current being the same in each case. Then

$$\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = \left(\frac{\omega_0}{\omega_2} - \frac{\omega_2}{\omega_0}\right) : \frac{C_0 - C_1}{\sqrt{C_1 . C_0}} = \frac{C_2 - C_0}{\sqrt{C_2 . C_0}}$$

and $\qquad C_0 - C_1 = C_2 - C_0 = \tfrac{1}{2}(C_2 - C_1)$

$$\left.\begin{aligned} &\text{then} && \Delta = \frac{\pi}{2}\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{C_2 - C_1}{C_0}\right) \\ &\text{and similarly} && \Delta = \frac{\pi}{2}\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{L_2 - L_1}{L_0}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (52)$$

EXAMPLE.—Suppose a series circuit comprises a resistance of $R = 20$ ohms, a capacitance $C = 2{,}250$ cm. $= 2 \cdot 5 \times 10^{-9}\,F$, and an inductance $L = 10^5$ cm. $= 10^{-4}$ henry. If the peak value of the applied p.d. is $V_0 = 1{,}000$ volts, the resonance frequency will be

$$\omega_0 \simeq \frac{1}{\sqrt{L . C}} = 2 \times 10^6 \text{ hz.},$$

so that at resonance, $I_0 = \dfrac{V_0}{20} = 50$ amperes. The pressure across the capacitance is then $V_C = \dfrac{I_0}{\omega C}$ and across the inductance $V_L = I_0 \omega L$, and since at resonance $\omega L = \dfrac{1}{\omega C}$, then

$$V_C = V_L = 50 \times 2 \times 10^6 \times 10^{-4} = 10{,}000 \text{ volts},$$

so that these individual pressures are each 10 times the supply pressure

If, now, the supply frequency is changed by 2 per cent. from the resonance value, that is, if the supply frequency is $\omega = 2 \cdot 04 \times 10^6$ hz., the impedance vector will then be

$$\mathfrak{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = 20 + j8 = (21 \cdot 5)e^{j21 \cdot 6^\circ}$$

and the current vector will be

$$\mathfrak{I} = \frac{V}{\mathfrak{Z}} = \frac{1{,}000}{21{\cdot}5e^{j21{\cdot}6^\circ}} = (46{\cdot}5)e^{-j21{\cdot}6^\circ} \text{ amperes (crest value),}$$

as shown in Fig. 13.

The logarithmic decrement is then

$$\Delta = \pi\sqrt{\frac{I^2}{I_0^2 - I^2}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = 0{\cdot}32.$$

At resonance, the power supplied to the circuit will be

$$W_0 = \frac{1{,}000}{\sqrt{2}} \times \frac{50}{\sqrt{2}} = 25 \text{ kW}.$$

When the supply frequency is raised by 2 per cent. above the resonance value, the impedance vector is found to be

$$\mathfrak{Z} = 21{\cdot}5e^{j21{\cdot}6^\circ}$$

and the active power supplied to the circuit will be (see Chapter X)

$$W_a = \frac{1{,}000}{\sqrt{2}} \times \frac{46{\cdot}5}{\sqrt{2}} \times \cos 21{\cdot}6^\circ = 21{\cdot}6 \text{ kW}.$$

and the inductive reactive power will be

$$W_r = \frac{1{,}000}{\sqrt{2}} \times \frac{46{\cdot}5}{\sqrt{2}} \times \sin 21{\cdot}6^\circ = 8{\cdot}6 \text{ kVA}.$$

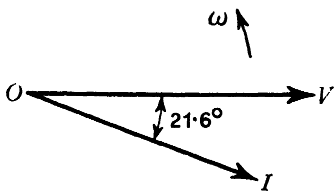


Fig. 13.

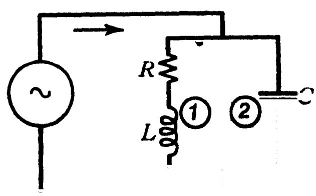


Fig. 14.

Resonance in Parallel Circuits

In Fig. 14 is shown a system which comprises two circuits in parallel, of which the branch 1 contains a resistance of R ohms in series with an inductance of L henry, and the branch 2 contains a condenser of capacitance C farad.

The vector of the impedance of branch 1 is $\mathfrak{Z}_1 = R + j\omega L$, and the vector of impedance of branch 2 is $\mathfrak{Z}_2 = -j\frac{1}{\omega C}$, so that the vector of the impedance of the two branches in parallel will be (see Chapter IX, page 305)

$$\mathfrak{Z}_T = \frac{1}{\dfrac{1}{\mathfrak{Z}_1} + \dfrac{1}{\mathfrak{Z}_2}} = \frac{\mathfrak{Z}_1 \times \mathfrak{Z}_2}{\mathfrak{Z}_1 + \mathfrak{Z}_2} = \frac{(R + j\omega L)\left(-j\dfrac{1}{\omega C}\right)}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)},$$

that is

$$\mathfrak{Z}_T = \frac{R\dfrac{1}{\omega^2 C^2} - j\dfrac{1}{\omega C}\left\{R^2 + \omega^2 L^2 - \dfrac{L}{C}\right\}}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2} \quad . \quad . \quad . \quad (53)$$

At resonance, the current I_0 in the mains will be in phase with the supply pressure, that is to say, the "imaginary" term in the expression (53) will then be zero, so that

$$R^2 + \omega^2 L^2 - \frac{L}{C} = 0$$

and hence

$$\omega^2 = \nu^2 = \frac{1}{L.C}\left\{1 - R^2\frac{C}{L}\right\} = \omega^2_0 - \left(\frac{R}{L}\right)^2$$

When the quantity $R^2\dfrac{C}{L}$ is small in comparison with unity, then $\omega^2 \simeq \omega_0{}^2 = \dfrac{1}{L.C}$ and the impedance of the parallel system will be

$$\mathfrak{Z}_T = \frac{1}{R}\frac{L}{C}. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and this is generally known as the "resonance resistance" of the circuit At resonance, $\mathfrak{Z}_T = \infty$ when $R = 0$, so that the current in the mains will be zero, but there will be a circulating current in the closed circuit formed by the two branch circuits. When R is small the circulating current will be

$$I_C = \frac{V}{\omega L} = V\omega C,$$

so that the magnitude of this current will be the greater as L becomes smaller and C correspondingly greater, the product $L.C$ remaining constant to maintain resonance.

As has been pointed out on page 265, Chapter IX, this type of circuit is known as a "rejector circuit". The effect of the resistance of the circuit of Fig. 14 is to produce damping, so that there will then be a current in the mains when resonance is established, the magnitude of which is $\mathfrak{I} = \dfrac{\mathfrak{V}}{\mathfrak{Z}_T}$, the impedance vector $\mathfrak{Z}_T$ being given in expression (53).

EXAMPLE.—For the parallel circuit system of Fig. 14 the following numerical data apply:

$$R = 20 \text{ ohms}: \ L = 10^{-4} \text{ henry}: \ C = 2{\cdot}5 \times 10^{-9} \text{ farad}.$$

If the applied a.c. pressure wave has a peak value $V_m = 1{,}000$ volts, find the input impedance at resonance and the current in each branch of the circuit

$$R^2\frac{C}{L} = \frac{20^2 \times 2\cdot5 \times 10^{-9}}{10^{-4}} = 0\cdot01$$

and since this is a very small quantity in comparison with unity the circular frequency at resonance will be very closely given by

$$\omega_0 = \frac{1}{\sqrt{L.C}} = \frac{1}{\sqrt{10^{-4} \times 2\cdot5 \times 10^{-9}}} = 2 \times 10^6$$

and the input impedance is then

$$Z_i = \frac{1}{R}\frac{L}{C} = \frac{10^{-4}}{20 \times 2\cdot5 \times 10^{-9}} = 2{,}000 \text{ ohms.}$$

The current taken from the supply mains is then

$$I_0 = \frac{V_m}{\sqrt{2}Z_i} = \frac{1{,}000}{\sqrt{2} \times 2{,}000} = 0\cdot35 \text{ r.m.s. amperes}$$

and the current in the two parallel branches will be

$$I_L = I_C = \frac{V_m}{\sqrt{2}\,\omega_0 L} = \frac{1{,}000}{\sqrt{2} \times 2 \times 10^6 \times 10^{-4}} = 3\cdot4 \text{ r.m.s. amperes.}$$

In Fig. 15 is shown a parallel system in which no resistance is included. The vector of the total impedance of this combination is found as before, as follows:

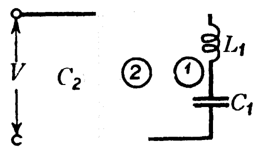


Fig. 15.

$$\mathfrak{Z}_T = \frac{\mathfrak{Z}_1 \cdot \mathfrak{Z}_2}{\mathfrak{Z}_1 + \mathfrak{Z}_2} = \frac{j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\left(-j\frac{1}{\omega C_2}\right)}{j\left(\omega L_1 - \frac{1}{\omega C_1}\right) - j\frac{1}{\omega C_2}}$$

$$= \frac{\frac{L_1}{C_2} - \frac{1}{\omega^2 C_1 C_2}}{j\omega L_1 - j\left(\frac{1}{\omega C_1} + \frac{1}{\omega C_2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Taking the following numerical data, viz. $L = 0\cdot3 \times 10^{-5}$ henry, $C_1 = 2\cdot2 \times 10^{-9}$ farad, $C_2 = 4 \times 10^{-9}$ farad, $V = 1{,}000$ volts (peak value), then:

(*i*) *For current resonance.*—

$\mathfrak{Z}_T = \infty$, so that $\omega L_1 = \left(\frac{1}{\omega C_1} + \frac{1}{\omega C_2}\right) = \frac{1}{\omega C}$ where

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

or
$$\omega^2 = \frac{C_1 + C_2}{L_1 C_1 C_2} = 2{\cdot}35 \times 10^{14}$$

and
$$\omega = 15{\cdot}3 \times 10^6.$$

The circulating current in the closed loop formed by the two branch circuits will be

$$I_C = V\omega C_2 = \frac{V}{\omega L_1 - \dfrac{1}{\omega C_1}} = 61{\cdot}2 \text{ amperes (peak value).}$$

It is to be observed from expression (55) that this condition of resonance can be expressed otherwise as follows : If C is the capacitance of C_1 and C_2 in series, that is, if

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

and the *current resonance* condition is then defined by $\omega = \dfrac{1}{L_1 . C}$.

(*ii*) *For pressure resonance.*—In this case the total impedance of the parallel combination is zero, that is, $\mathfrak{Z}_T = 0$, so that, from expression (55) $L_1 = \dfrac{1}{\omega^2 C_1}$, that is, $\omega^2 = \dfrac{1}{L_1 . C_1}$, and the resonance frequency is determined solely by the constants of the branch circuit 1 in Fig. 15. For the numerical data already given in the foregoing, therefore, the condition of pressure resonance is that

$$\omega_{res} = 12{\cdot}4 \times 10^6.$$

When either branch circuit of Fig. 15 contains resistance, the resonance conditions can, in general, be most conveniently found by means of the method of " Inversion ", as explained in Chapter XI.

The Natural Frequencies of Coupled Oscillatory Circuits

In Fig. 16 is shown diagrammatically two coupled circuits, each of which may be an oscillatory circuit, and this type of coupling is of

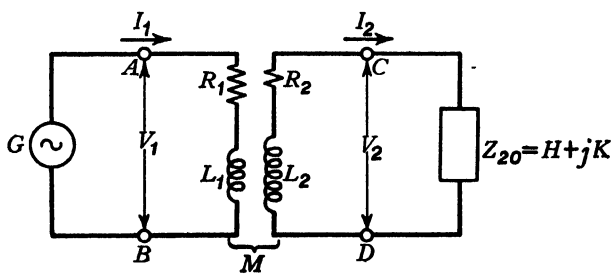


Fig. 16.

great importance as a transformer in high frequency technique. The two circuits are coupled by the mutual inductance M henry, where

$$M = (1 - \tau)\sqrt{L_1L_2} = k\sqrt{L_1L_2} \quad . \quad . \quad . \quad (57)$$

the factor τ being the "total" or Blondel leakage coefficient and the factor k the coupling coefficient, as has been considered already in Chapter VIII, page 239.

The primary or "input" terminals are shown at A and B, and the secondary or "output" terminals are shown at C and D. The simultaneous vector equations for this coupled system are

$$\left.\begin{aligned} \mathfrak{V}_1 &= (R_1 + j\omega L_1)\mathfrak{J}_1 - j\omega M\mathfrak{J}_2 \\ 0 &= \mathfrak{V}_2 + (R_2 + j\omega L_2)\mathfrak{J}_2 - j\omega M\mathfrak{J}_1 \end{aligned}\right\} \begin{aligned} &. \; . \; . \; (a) \\ &. \; . \; . \; (b) \end{aligned} \quad . \quad . \quad (58)$$

from which it is easily seen that

$$\left.\begin{aligned} \mathfrak{V}_1 &= \mathfrak{V}_2\left[\frac{R_1 + j\omega L_1}{j\omega M}\right] + \mathfrak{J}_2\left[\frac{(R_2 + j\omega L_2)(R_1 + j\omega L_1) + (\omega M)^2}{j\omega M}\right] \\ \mathfrak{J}_1 &= \mathfrak{V}_2\frac{1}{j\omega M} + \mathfrak{J}_2\left[\frac{R_2 + j\omega L_2}{j\omega M}\right] \end{aligned}\right\} \quad . \quad (59)$$

If the load on the secondary terminals C and D is defined by $\mathfrak{Z}_{20} = H + jK = \dfrac{\mathfrak{V}_2}{\mathfrak{J}_2}$, then the "Input Impedance" is

$$\mathfrak{Z}_1 = \frac{\mathfrak{V}_1}{\mathfrak{J}_1} = \frac{\mathfrak{V}_2(R_1 + j\omega L_1) + \mathfrak{J}_2[(R_1 + j\omega L_1)(R_2 + j\omega L_2) + (\omega M)^2]}{\mathfrak{V}_2 + \mathfrak{J}_2(R_2 + j\omega L_2)}$$

that is,

$$\mathfrak{Z}_1 = \frac{\mathfrak{Z}_{20}(R_1 + j\omega L_1) + [(R_1 + j\omega L_1)(R_2 + j\omega L_2) + (\omega M)^2]}{\mathfrak{Z}_{20} + (R_2 + j\omega L_2)} \quad . \quad (60)$$

After substituting for $\mathfrak{Z}_{20}$ the equivalent vector $H + jK$, it is found that

$$\mathfrak{Z}_1 = \left[R_1 + \frac{(\omega M)^2(H + R_2)}{\{(H + R_2)^2 + (K + \omega L_2)^2\}}\right]$$
$$+ j\left[\omega L_1 - \frac{(\omega M)^2(K + \omega L_2)}{\{(H + R_2)^2 + (K + \omega L_2)^2\}}\right] \quad . \quad (61)$$

and, writing this in the form

$$\mathfrak{Z}_1 = (R_1 + S) + j(\omega L_1 - T) \quad . \quad . \quad . \quad (62)$$

it is seen that the coupled system of Fig. 16 is equivalent to the simple series circuit of Fig. 17. The effect of the load on the secondary terminals CD of Fig. 16, therefore, is to *increase the effective resistance* and to *reduce the effective reactance* across the primary terminals A, B.

An important type of two coupled oscillatory circuits is shown in Fig. 18, and, inserting the corresponding reactances in equation (61) and

noting that $H = 0$, it is seen that the input impedance for this coupled system is

$$\mathfrak{Z}_1 = \left[R_1 + \frac{(\omega M)^2 R_2}{R_2^2 + \left(\omega L_2 - \dfrac{1}{\omega C_2}\right)^2}\right]$$

$$+ j\left[\left(\omega L_1 - \frac{1}{\omega C_1}\right) - \frac{(\omega M)^2\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{R_2^2 + \left(\omega L_2 - \dfrac{1}{\omega C_2}\right)^2}\right] \quad . \quad . \quad (63)$$

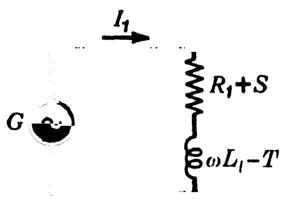


Fig. 17.

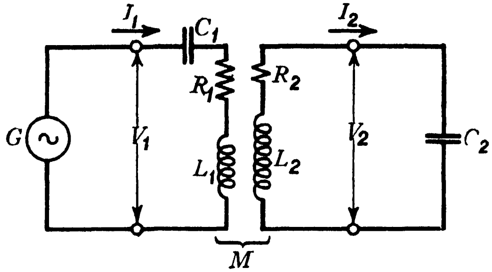


Fig. 18

At resonance, the j component of equation (63) must be zero, that is

$$\left(\omega L_1 - \frac{1}{\omega C_1}\right) - \frac{(\omega M)^2\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)}{R_2^2 + \left(\omega L_2 - \dfrac{1}{\omega C_2}\right)^2} = 0 \quad . \quad . \quad (64)$$

If, as is frequently the case in practice, R_2 is relatively very small, the equation (64) becomes

$$\left(\omega L_1 - \frac{1}{\omega C_1}\right) - \frac{\omega^2 k^2 L_1 L_2}{\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)} = 0 \quad . \quad . \quad . \quad (65)$$

since from expression (57), page 335, $M^2 = k^2 L_1 L_2$.

Equation (65) is a bi-quadratic in ω and writing

$$\omega_1^2 = \frac{1}{L_1 C_1} : \omega_2^2 = \frac{1}{L_2 C_2},$$

then

$$\omega = \sqrt{\frac{(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(1 - k^2)\omega_1^2\omega_2^2}}{2(1 - k^2)}} \quad . \quad . \quad (66)$$

If the two circuits of Fig. 18 are separately tuned to the same frequency

so that $$\omega_1 = \omega_2 = \omega_0,$$

then $$\omega = \omega_0\sqrt{\frac{1 \pm k}{1 - k^2}},$$

that is

$$\left.\begin{aligned} \omega_0' &= \omega_0\sqrt{\frac{1}{1 - k}} \quad . \quad . \quad . \quad (a) \\ \omega_0'' &= \omega_0\sqrt{\frac{1}{1 + k}} \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (67)$$

EXAMPLE.—For the circuit system of Fig. 19:

$$L_1 = 50{,}000 \text{ cm.} = 5 \times 10^{-5} \text{ henry},$$
$$L_2 = 200{,}000 \text{ cm.} = 2 \times 10^{-4} \text{ henry},$$
$$C_2 = 114 \text{ cm.} = 126{\cdot}5 \times 10^{-12} \text{ farad}.$$

$$k = 0{\cdot}1: \quad M = k\sqrt{L_1L_2} = 0{\cdot}1 \times 10^{-4} \text{ henry},$$

$$\omega_0^2 = \omega_2^2 = \frac{1}{L_2C_2} = \frac{1}{2 \times 10^{-4} \times 126{\cdot}5 \times 10^{-12}} = \frac{10^{16}}{253},$$

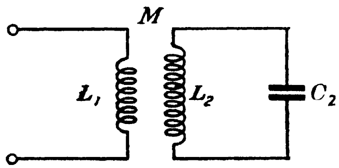


Fig. 19.

or $$\omega_0 = 6{\cdot}3 \times 10^6 \text{ circular frequency},$$

$$f = \frac{\omega_0}{2\pi} \simeq 10^6 \text{ hz.},$$

and since $$\lambda . f = (3 \times 10^{10}) \text{ cm. per second},$$

the corresponding wavelength is

$$\lambda = \frac{3 \times 10^{10}}{10^6} \text{ cm.} = 300 \text{ metres}.$$

In the foregoing treatment the coupling of the two circuits has been assumed to be electro-magnetic, that is, corresponding to the mutual inductance M. There are, however, other types of coupling and a generalised definition of the coupling coefficient is given by the expression

$$k = \frac{X_K}{\sqrt{X_1X_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

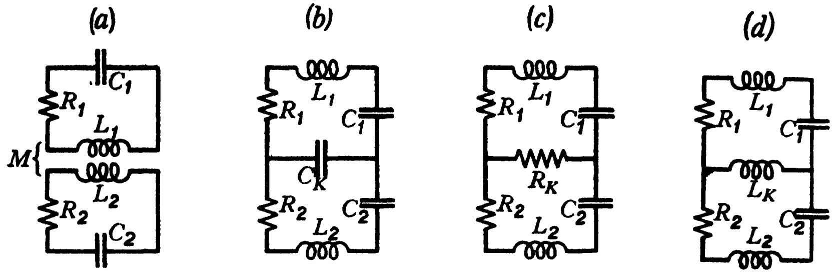
in which, X_K ohms is the reactance to which the coupling is due,

X_1 and X_2 ohms are the respective reactances for the individual circuits, these reactances being of the same type as X_K.

In the diagrams below are shown the circuit connections for four different types of coupling:

(*a*) Inductive coupled circuits.
(*b*) Capacitance coupled circuits.
(*c*) Resistance coupled circuits.
(*d*) Circuits coupled by means of a common inductance L_K.

The respective values of the four components k, X_K, X_1, X_2, are given on page 534, Appendix IV.



For each of these different types of coupled circuits two simultaneous differential equations may be written down, as, for example, for the system (*a*),

$$\left.\begin{aligned} R_1 i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + \frac{q_1}{C_1} &= 0 \quad . \quad . \quad . \quad (a) \\ R_2 i_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + \frac{q_2}{C_2} &= 0 \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (69)$$

where q_1 and q_2 coulombs are the respective charges of the capacitances C_1 and C_2.

If the resistances of the two circuits are relatively very small, so that $R_1 \simeq R_2 \simeq 0$, then, after substituting $i_1 = \dfrac{dq_1}{dt}$: $i_2 = \dfrac{dq_2}{dt}$ the equations (68) become

$$\left.\begin{aligned} L_1\frac{d^2q_1}{dt^2} + M\frac{d^2q_2}{dt^2} + \frac{q_1}{C_1} &= 0 \\ L_2\frac{d^2q_2}{dt^2} + M\frac{d^2q_1}{dt^2} + \frac{q_2}{C_2} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (70)$$

Differentiating each of these equations twice with respect to t, gives

$$\left.\begin{aligned} L_1\frac{d^4q_1}{dt^4} + M\frac{d^4q_2}{dt^4} + \frac{1}{C_1}\,\frac{d^2q_1}{dt^2} &= 0 \\ L_2\frac{d^4q_2}{dt^2} + M\frac{d^4q_1}{dt^4} + \frac{1}{C_2}\,\frac{d^2q_2}{dt^2} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (71)$$

from the solution of which the natural frequencies of oscillation of the coupled circuits are found to be given by

$$\omega = \sqrt{\frac{(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2) - 4(1 - k^2)\omega_1^2\omega_2^2}}{2(1 - k^2)}} \quad . \quad (72)$$

which is the same result as was obtained previously in equation (66). As before, if $\omega_1 = \omega_2 = \omega_0$
then

$$\left.\begin{aligned} \omega_0' &= \frac{\omega_0}{\sqrt{1 - k}} \\ \omega_0'' &= \frac{\omega_0}{\sqrt{1 + k}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (73)$$

When the coupling factor is $k = 0$, then $w_0' = \omega_0'' = \omega_0$ and both oscillation frequencies become equal. Further, if the coupling factor is $k = 1$, then

$$\left.\begin{aligned} \omega_0' &= \infty \\ \omega_0'' &= \frac{1}{\sqrt{2}}\omega_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (74)$$

that is to say, there will be in practice only one frequency of oscillation, viz. $\omega_0'' = 0{\cdot}71\omega_0$.

The Ballastic Galvanometer

An instructive example of a mechanically oscillating system which is used for electrical measurement purposes is the *ballistic galvanometer* (see also page 206). This type of galvanometer is used to measure the quantity of electricity which passes round a circuit when a transient current of extremely short time duration flows in the circuit. Thus, when a condenser is discharged through a galvanometer of which the moving system is only very slightly damped, a deflection is produced and the moving system will then come to rest after a series of oscillations of gradually diminishing amplitude. The angular displacement that is the " angle of throw " of the first deflection is then a measure of the *quantity* of electricity which has passed through the circuit by the condenser discharge.

The conditions which a galvanometer, suitable for measuring electric quantity, should fulfil are :

(i) The moving system should have a large moment of inertia so that the discharge through the galvanometer shall be completed before the moving system has become appreciably deflected.

(ii) The damping of the moving system should be small so that, other conditions being the same, the " throw " or first full deflection shall be as large as possible.

If, for example, the moving coil of the galvanometer is wound on a metal frame, this frame should be split along a line at right-angles to the plane of the coil, so that the electromagnetic damping effect shall be small. The significance of this condition of small damping will be apparent from the theory of the moving-coil ballistic galvanometer in what follows (see also Fig. 8, page 322).

Let K be the moment of inertia of the moving system,
,, θ ,, ,, deflection at any moment t,
,, $a.\frac{d\theta}{dt}$,, ,, damping couple when the angular velocity is $\frac{d\theta}{dt}$,
,, $b.\theta$,, ,, controlling couple which operates to cause the moving system to return to its zero position.

In the case of a moving-coil instrument the controlling couple will be provided by the torsion of the suspension wire or fibre.

If i amperes is the discharge current at any moment t, the deflecting couple will be $k.i$ where k is a constant of the instrument. Hence (see also expression (41), page 387),

$$k.i = K\frac{d^2\theta}{dt^2}$$

or

$$k\int i\,dt = k.Q = K\frac{d\theta}{dt}$$

so that the initial angular velocity Ω_{max} of the moving system will be

$$\Omega_{max} = \left.\frac{d\theta}{dt}\right]_{t=0} = \frac{k}{K}Q \quad . \quad . \quad . \quad . \quad (75)$$

where Q coulombs is the quantity of electricity which has passed through the instrument. The moving system, therefore, starts off with an angular velocity $\Omega_{max.}$ and will oscillate until its kinetic energy has become dissipated by friction and other damping effects. The equation of motion is then

$$K\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + b.\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{a}{K}\frac{d\theta}{dt} + \frac{b}{K}\theta = 0.$$

When the damping is relatively very small (and this is an essential feature of the galvanometer), the solution of this equation is for the initial conditions already stated in the foregoing (see Fig. 20),

$$\theta = \Theta\, e^{-\frac{1}{2}\frac{a}{K}t} \sin\sqrt{\frac{b}{K}}.t$$

In this expression Θ is the maximum undamped deflection and, writing the expression in the form,

$$\theta = \Theta e^{-\alpha t} \sin \omega_0 t \quad . \quad . \quad . \quad . \quad . \quad (76)$$

where

$$\alpha = \frac{1}{2}\frac{a}{K} : \ \omega_0 = \sqrt{\frac{b}{K}}.$$

The periodic time of oscillation is then,

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{K}{b}} \quad . \quad . \quad . \quad . \quad . \quad (77)$$

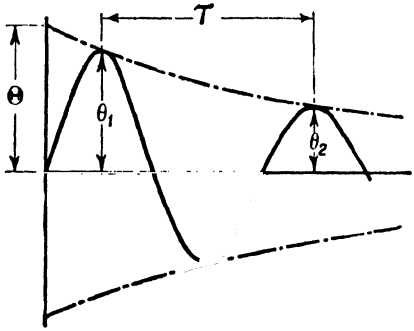


Fig. 20.

Differentiating the equation (75) with respect to t and putting $t = 0$, it is found that the maximum velocity is

$$\Omega_{max} = \left[\frac{d\theta}{dt}\right]_{t=0} = \Theta\sqrt{\frac{b}{K}}$$

so that, from expression (75) it follows that

$$Q = \Theta\frac{\sqrt{b.K}}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The time occupied in the first swing is $t = \frac{1}{4}\tau_0$, so that the deflection of the first (positive) swing as shown in Fig. 20 is

$$\theta_1 = \Theta e^{-\frac{1}{4}\alpha.\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

and for the second (positive) swing is,

$$\theta_2 = \Theta e^{-\frac{5}{4}\alpha\tau}$$

and for the nth (positive) swing is

$$\theta_n = \Theta e^{-\frac{4n-3}{4}\alpha\tau}.$$

Hence the logarithmic decrement is

$$\Delta = \log_e \frac{\theta_1}{\theta_2} = \log_e e^{\alpha\tau}$$

that is
$$\Delta = \alpha . \tau \text{ nepers}$$

or,
$$\Delta = \frac{1}{n-1} \log_e \frac{\theta_1}{\theta_n} \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where θ_n is the deflection of the nth (positive) swing, as has been found already earlier in this Chapter (see expressions (29) and (33), pages 321 and 322).

The undamped deflection is given by expression (79), viz.,

$$\Theta = \theta_1 e^{\frac{1}{4}\Delta} \simeq \theta_1\left(1 + \frac{\Delta}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad (81)$$

and from expression (78), the quantity of electricity which has been discharged through the galvanometer is

$$Q = \Theta\frac{\sqrt{b.K}}{k} = \theta_1\left(1 + \frac{\Delta}{4}\right)\frac{\sqrt{b.K}}{k}$$

where $\frac{\sqrt{b.K}}{k}$ is a constant of the instrument, and there are several methods available for the determination of this constant, two of which are as follows :

(i) Suppose a condenser is charged with a known quantity Q coulombs, that is
$$Q = V.C,$$
where V volts is the d.c. pressure applied to the condenser of capacitance C farad. If the charged condenser is discharged through the ballistic galvanometer and if θ_1 is the corresponding first deflection, then

$$\frac{\sqrt{b.K}}{k} = \frac{Q}{\theta_1(1 + \frac{1}{4}\Delta)} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

(ii) If a known steady current of I amperes flows through the galvanometer and produces a steady deflection Θ^* then (see page 340)

$$\frac{k}{b} = \frac{\Theta^*}{I} \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and since the periodic time of oscillation is, from expression (77),

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{K}{b}},$$

it follows that the required constant of the galvanometer is

$$\frac{\sqrt{b.K}}{k} = \frac{I}{\Theta^*}\cdot\frac{\tau}{2\pi} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Chapter XI

SINGLE-PHASE AND THREE-PHASE ALTERNATING CURRENT SYSTEMS:

ALTERNATING CURRENT POWER

Power in a Single-Phase Circuit

THE standard wave-form for alternating currents and pressures is the sinusoidal function of the time which, in the general sense, also include, of course, the cosine function of the time. For example, the instantaneous value of such a wave-form may be expressed as

$$i = I_m \cos \omega t \text{ amperes} \quad . \quad . \quad . \quad . \quad (1)$$

as shown in Fig. 1. In this expression,

i is the current at the moment, t
I_m ,, ,, peak value of the wave,
ω circular frequency,
f cyclic frequency in hertz,
τ the "periodic time", that is, the time for one cycle,
ωt angle measured in electrical radians so that ω is an angular velocity measured in electrical radians per second.

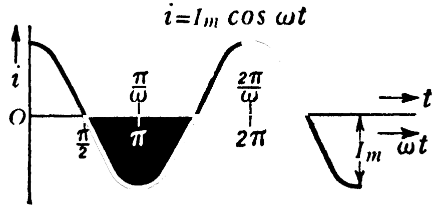


Fig. 1.

It has also been seen (page 297) that the expression (1) can be defined as the projection on the "real" axis of a vector of magnitude I_m rotating

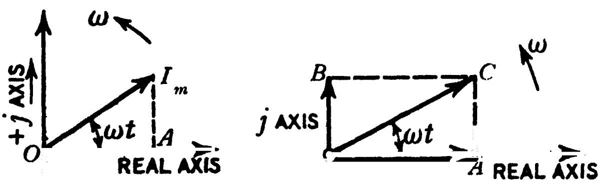


Fig. 2.

with the angular velocity ω in the counter-clockwise direction as shown in Fig. 2, so that an alternative form for this expression is the vector

$$\mathfrak{I} = I_m e^{j\omega t} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that OA is the instantaneous value of the current at the moment t. Similarly, the instantaneous value of an alternating pressure may also be represented in the general form,

$$v = V_m \cos (\omega t + \phi) \quad . \quad . \quad . \quad . \quad (3)$$

or, in the equivalent form,

$$\mathfrak{V} = V_m e^{j(\omega t + \phi)} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Suppose now that an alternating current defined as $i = I_m \cos \omega t$ is flowing in a resistance of R ohms as shown in Fig. 3. The p.d. across the resistance at any moment t will then be given by

$$v = iR = I_m R \cos \omega t = V_m \cos \omega t$$

where $V_m = I_m R$.

In Fig. 4 are shown the current and pressure vectors, respectively, for this circuit. The current and pressure waves are in phase with one

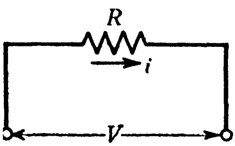


Fig. 3.

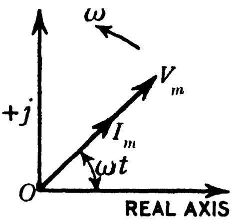


Fig. 4.

another, this being a characteristic feature of a purely resistance circuit. For the conditions shown in Fig. 4 the current vector is

$$\mathfrak{I} = I_m e^{j\omega t} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the pressure vector is

$$\mathfrak{V} = I_m R e^{j\omega t} = V_m e^{j\omega t} \quad . \quad . \quad . \quad . \quad (6)$$

where $V_m = I_m R$.

The electric power supplied to the circuit of Fig. 3 at any moment t will be, by Joule's Law (see Chapter II, page 59)

$$w = i^2 R \text{ watts.}$$

and this power will be completely dissipated in heating the resistance, viz.

$$w = i^2 R \text{ joules per sec.} = 0{\cdot}24\ i^2 R \text{ gm.-calories per sec.}$$

The mean power supplied, that is, the mean power for one cycle, will then be given by

$$W = \frac{1}{\tau}\int_0^\tau i^2 R\, dt = \frac{RI_m{}^2}{\tau}\int_0^\tau \cos^2 \omega t\, dt$$

$$= \frac{I_m{}^2 R}{\tau}\int_0^\tau \tfrac{1}{2}(1 + \cos 2\omega t)\, dt$$

that is

$$W = \tfrac{1}{2}I_m{}^2R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R \text{ watts} \quad . \quad . \quad . \quad . \quad (7)$$

This result shows that the heating effect of the alternating current of peak value I_m is the same as that of a steady direct current of magnitude $\frac{I_m}{\sqrt{2}}$. From the procedure by means of which this expression (7) has been obtained, it will be seen that the quantity $\frac{I_m}{\sqrt{2}}$ has been obtained by taking the square root of the mean square of the instantaneous value of the current and consequently this quantity is termed the "root mean square" (r.m.s.) value of the current, and it is this value which is measured in practice by an ammeter. (See also Chapter IX, page 265). In what follows, the r.m.s. value of the current will be denoted by the symbol I.

The expression for the power may be written in another form as follows:

$$w = i^2R = vi \text{ watts}$$

that is,

$$w = vi = (V_m \cos \omega t)(I_m \cos \omega t) = V_mI_m \cos^2 \omega t \quad . \quad . \quad (8)$$

and the mean power is then given by

$$W = \frac{1}{T}\int_0^T V_mI_m \cos^2 \omega t = \tfrac{1}{2}V_mI_m = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} = VI \text{ watts} \quad . \quad (9)$$

so that the mean a.c. power in a purely resistance circuit is given by the expression

$$W = (\text{r.m.s. pressure}) \times (\text{r.m.s. current}),$$

and this is the quantity which is measured in practice by a wattmeter.

In Fig. 5 is shown the instantaneous power w as a function of the time t in accordance with the expression (8). By expanding this expression (8) as follows:

$$w = vi = V_mI_m \cos^2 \omega t = \tfrac{1}{2}V_mI_m(1 + \cos 2\omega t) = VI(1 + \cos 2\omega t) \quad . \quad (10)$$

it is seen that the instantaneous power is given by the following components:

(i) The quantity VI watts, which is represented by the horizontal line AB in Fig. 5 and is the mean power in the circuit in accordance with the expression (9).

(ii) The quantity $VI \cos 2\omega t$, which is a cosine function of double the supply frequency and which is symmetrically placed with regard to the horizontal line AB of Fig. 5.

The instantaneous power, therefore, varies from a peak value $2VI$ to zero, and its cyclic frequency is double that of the supply frequency.

Next, suppose that an alternating current is flowing in a purely inductive circuit of which the coefficient of self-induction is L henry as

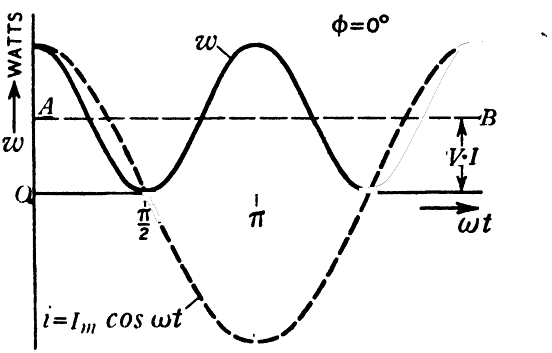


Fig. 5.

shown in Fig. 6, the instantaneous value of the current being again given by the expression,

$$i = I_m \cos \omega t.$$

It has been shown in Chapter VIII, page 241, that the back e.m.f. of self-induction is

$$e = -L\frac{di}{dt} = -L\frac{d}{dt}(I_m \cos \omega t) = \omega L I_m \sin \omega t \quad . \quad . \quad (11)$$

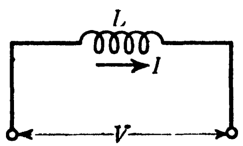


Fig. 6.

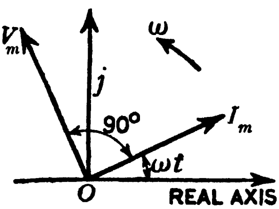


Fig. 7.

so that the applied p.d. which is necessary to overcome this back e.m.f. is

$$v = -e = -\omega L I_m \sin \omega t = \omega L I_m \cos\left(\omega t + \frac{\pi}{2}\right),$$

or, expressed in vector form,

$$\mathfrak{V} = V_m e^{j(\omega t + \pi/2)},$$

the corresponding vector form of the current being

$$\mathfrak{J} = I_m e^{j\omega t}.$$

These two vectors are shown in their correct relative positions in Fig. 7, from which it will be seen that the current vector lags by 90° on the pressure vector and this phase relationship is the characteristic feature

of a purely inductive circuit. The instantaneous power in the circuit is then

$$w = vi = V_m I_m \cos \omega t \cos \left(\omega t + \frac{\pi}{2}\right),$$

that is,

$$w = \frac{1}{2} V_m I_m \left\{\cos \frac{\pi}{2} + \cos \left(2\omega t + \frac{\pi}{2}\right)\right\} = VI \cos 2\left(\omega t + \frac{\pi}{4}\right),$$

and the graphical representation of this expression is shown in Fig. 8. The value of the mean power in the circuit is now seen to be zero, and the instantaneous power is given by the cosine curve which is symmetrically placed with regard to the abscissa axis. This cosine curve is of double the supply frequency, and the interesting feature of this relationship is that, during one half-cycle of this double-frequency wave, the power is negative, that is, power is being supplied from the circuit

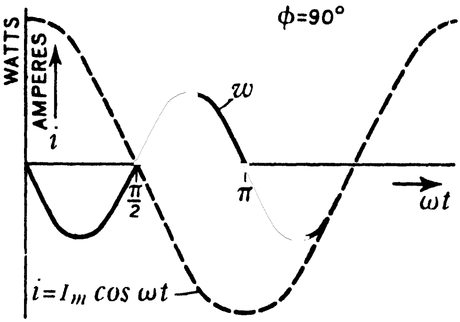


Fig. 8.

back into the mains and during the next half-cycle the mains are supplying power to the circuit. That is to say, energy is being stored in the electromagnetic field of the inductance during one half-cycle, and during the next half-cycle this field is decaying and the corresponding energy is being returned to the supply mains, so that on the whole the circuit is neither taking power from the mains nor returning power to the mains. For reference purposes, the broken line wave in Fig. 8 represents the current in the circuit.

If the diagram of Fig. 8 is compared with that of Fig. 5 it will be seen that the amplitude of the double frequency wave is the same in both cases, but in the case of Fig. 8 this wave has been lowered so that its horizontal axis of symmetry becomes coincident with the abscissa axis, and the wave has also been moved towards the left-hand side by one-quarter cycle of the double frequency wave, that is, by the angle $\frac{\pi}{4}$ on the abscissa scale of the current wave supply frequency.

The more general case in which a circuit contains an inductance and resistance in series is shown in Fig. 9. The current is again represented by

$$i = I_m \cos \omega t,$$

or, in the vector form,

$$\mathfrak{I} = I_m e^{j\omega t}.$$

The impedance vector for this circuit has already been shown to be (see Chapter IX, p. 254)

$$\mathfrak{Z} = R + j\omega L = \sqrt{R^2 + (\omega L)^2}\, e^{j\phi} = |\mathfrak{Z}|\, e^{j\phi},$$

where $|\mathfrak{Z}| = \sqrt{R^2 + (\omega L)^2}$ and is the magnitude of the impedance

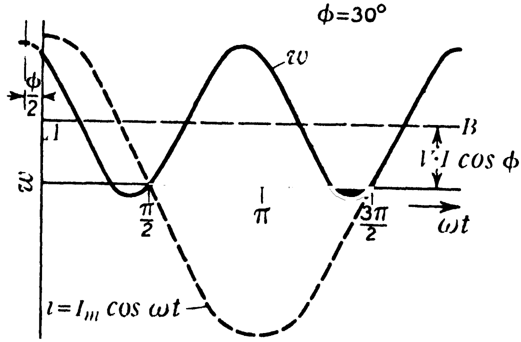


Fig. 9.

vector: this can also be represented by Z. The applied p.d. will be given by the vector,

$$\mathfrak{V} = \mathfrak{I}.\mathfrak{Z} = I_m \sqrt{R^2 + (\omega L)^2}.e^{j(\omega t + \phi)} = V_m e^{j(\omega t + \phi)}$$

or, the instantaneous value of the applied p.d. will be

$$v = V_m \cos(\omega t + \phi).$$

where $\tan \phi = \dfrac{\omega L}{R}$.

The power supplied to the circuit in this case will be

$$w = vi = V_m I_m \cos \omega t \cos(\omega t + \phi) = \tfrac{1}{2} V_m I_m [\cos\phi + \cos(2\omega t + \phi)],$$

that is

$$w = VI[\cos\phi + \cos(2\omega t + \phi)] \text{ watts} \quad . \quad . \quad . \quad (12)$$

so that the instantaneous power comprises the components:

(*i*) The constant quantity $VI \cos\phi$, which is the mean value of the power supplied to the circuit, and,

(*ii*) The double frequency quantity $VI \cos(2\omega t + \phi)$.

The expression (12) is shown graphically in Fig. 9, from which it will be seen that the horizontal axis of symmetry AB for the double frequency wave is at a height $VI \cos \phi$ above the abscissa axis. Comparison of the diagram of Fig. 9 with the corresponding one of Fig. 5 will show that the amplitude of the double frequency wave is the same in both cases and that Fig. 9 can be derived from Fig. 5 by lowering the horizontal line AB so that its height above the abscissa axis is $VI \cos \phi$ and, in addition, moving the double frequency wave towards the left-hand side by the angle ϕ measured on the double frequency scale, that is, by the angle $\frac{\phi}{2}$ on the supply frequency scale to which the current and pressure waves have been drawn. The special case represented by Figs. 7 and 8 is obtained by substituting $\phi = \frac{\pi}{2}$ in the expression (12).

Symmetrical Three-Phase Systems

A symmetrical three-phase e.m.f. system is one in which three separate e.m.f.s of the same magnitude and of the same frequency act, these

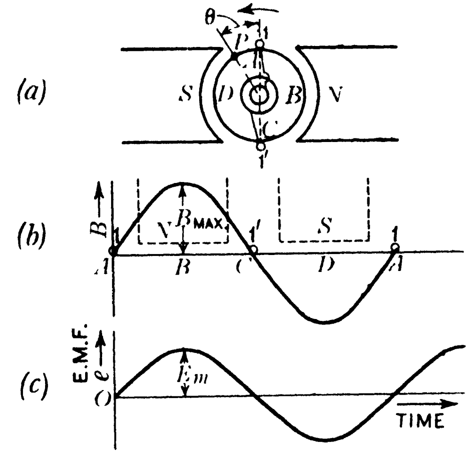


Fig. 10.

e.m.f.s being mutually displaced in phase by one-third of a period. The load on a three-phase system is said to be balanced if the load on each phase is the same as regards current and power factor. The phase windings of a three-phase generator or motor may be arranged in either of two ways: (*i*) Star, or (*ii*) Mesh (also termed Delta). Before considering the power relationship for three-phase systems the principles of the connections of such system will be briefly outlined.

Consider in the first place the arrangement shown in Fig. 10*a*, which

gives a diagrammatical representation of a single-phase generator. A coil of insulated wire consisting of w closely wound turns is fixed to the armature core so that the coil sides $1 : 1'$ lie diametrically opposite, the ends of the coil being brought out, each to a slip-ring as shown. Assuming that the magnetic flux density in the air-gap between the periphery of the armature core and the faces of the poles is sinusoidally distributed, then Fig. 10*b* will be a developed diagram of the arrangement of Fig. 10*a*, the sine wave giving the flux density at every point on the surface of the armature core. It will be seen from what follows that this assumption of a sine wave of flux density distribution implies that a sine wave of e.m.f. will be induced in the armature coil. In practice, however, the requisite sine wave of induced e.m.f. is obtained, in general, by suitably distributing the turns of each coil over the periphery of the armature core instead of using a concentrated winding as is shown in Fig. 10*a*.

Suppose now that the armature is driven at a speed of n revs. per second in a counter-clockwise direction, and let a cm. be the radius of the armature core and l cm. its axial length. The flux density in the air-gap is defined by the expression,

$$B_\theta = B_m \sin \theta \qquad . \quad . \quad . \quad . \quad . \quad (13)$$

where B_m is the peak value of the sine wave of flux density distribution, so that if θ is the angle which any point P on the armature core makes with the vertical axis of the pole system, and if the zero of time is taken to be that moment at which the coil side 1 is in the position shown in Fig. 10*a*, then at any time t the conductors of coil side 1 will lie in a magnetic field of density $B_m \sin \omega t$, where $\omega = 2\pi n$, and is the angular velocity of the armature in radians per second, so that $\omega t = \theta$. The e.m.f. induced in each of the conductors of coil side 1 will then be (see Chapter VIII, page 229)

$$e_1 = \frac{vB_\theta l}{10^8} = \frac{vlB_m}{10^8} \sin \omega t \text{ volts} \qquad . \quad . \quad . \quad (14)$$

The e.m.f. induced in each of the conductors of the coil side $1'$ will be, at every moment, opposite to that induced in the coil side 1, so that the e.m.f.s induced in all the conductors will act in the same direction round the coil, and consequently the total e.m.f. at any moment t for a coil of w turns, that is, $2w$ conductors, will be

$$e = \frac{2wvlB_m}{10^8} \sin \omega t = E_m \sin \omega t \text{ volts} \qquad . \quad . \quad . \quad (15)$$

where $E_m = \frac{2w \cdot v \cdot lB_m}{10^8}$ volts and is the magnitude of the peak value of the sine wave of induced e.m.f. as is shown in Fig. 10*c*. One complete cycle of this e.m.f. wave will be passed through for each revolution of the armature, so that for a speed of n revs. per second there will be n cycles

per second, i.e. n hz. For a two-pole machine, as shown in Fig. 10, the frequency of the induced e.m.f. will be

$$f = n = \frac{\omega}{2\pi} \text{ hz.}$$

EXAMPLE.—Suppose, $B_m = 10{,}000$ lines per sq. cm. (i.e. gauss): $a = 40$ cm.: $l = 30$ cm.: $n = 8$ revs. per second: $w = 10$: $v = 2\pi an = 2{,}000$ cm. per second: $\omega = 2\pi n = 50$ radians per second (circular frequency), then $E_m = 120$ volts and $e = 120 \sin 50t$ volts.

The r.m.s. value of the induced e.m.f. is $\frac{E_m}{\sqrt{2}} = 84$ volts and the frequency is $f = \frac{\omega}{2\pi} = 8$ cycles per second (hz.).

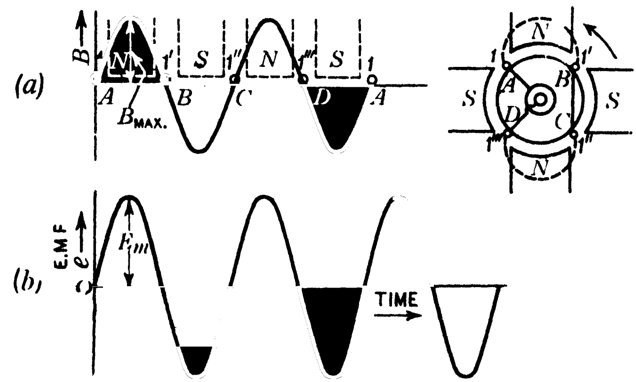


Fig. 11.

If the generator has $2p$ poles (that is, p = pairs of poles), then, in order that the e.m.f.s in the two sides of a coil shall be additive, they must be displaced on the periphery of the armature core by the angle $\frac{180}{p}$ *geometrical degrees* as is shown in Fig. 11*a*, in which $p = 2$. It is usual to call the angle between the axes of two consecutive poles 180 *electrical degrees*, so that, for a machine with p pairs of poles:

$$1 \text{ geometrical degree} = p \text{ electrical degree.}$$

For a machine with p pairs of poles and an armature speed of n revs. per second, it will be clear from Fig. 11*a* that the value of θ in the expression (13) is to be measured in electrical degrees, and the angular velocity ω is also to be measured in electrical radians per second, so that

$$\omega = 2\pi . n . p \text{ electrical radians per second.}$$

The e.m.f. induced in each conductor at any moment t will then be

$$e_1 = \frac{vlB_m}{10^8} \sin \omega t \text{ volts}$$

as before (see expression (14)). If the winding is arranged as in Fig. 11b so that there are $2p$ coil sides, the induced e.m.f. will be

$$e = \frac{2p.w.v.lB_m}{10^8} \sin \omega t \text{ volts,}$$

when each coil side comprises w conductors. The frequency of the induced e.m.f. in this case will be seen from Fig. 11b to be given by the relationship

$$f = n.p = \frac{\omega}{2\pi} \text{ hz.,}$$

where $\omega = 2\pi f$ electrical radians per second.

(i) A THREE-PHASE STAR-CONNECTED GENERATOR WINDING.—In Fig. 12a are shown three coils, 1 1′, 2 2′, 3 3′, each wound diametrically

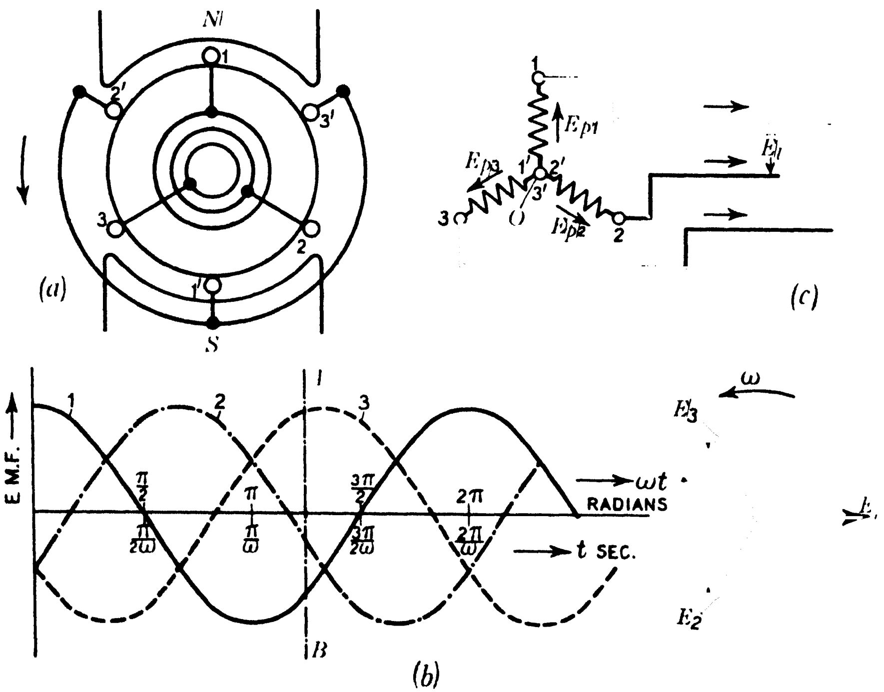


Fig. 12

on the armature core of a two-pole machine. The coils are mutually spaced on the core at 120°, that is to say, the distance between the conductors 1 and 2 is one-third of the periphery of the armature core, as is also the distance between the conductors 2 and 3. In what follows the *undashed* numerals 1, 2, and 3, are used to denote the beginnings of the respective coils and the *dashed* numerals 1′, 2′, 3′, are used to denote the ends of the respective coils.

The beginnings of the coils 1, 2, and 3 are brought out, each to a separate slip-ring on the armature shaft. The ends 1′, 2′, 3′ are connected together to form a common junction known as the *neutral point*, or the *star point* of the winding. This common junction may also be brought to a separate slip-ring on the armature shaft if so desired, although this is not necessary for the operation of the machine. If the armature of Fig. 12*a* rotates in a counter-clockwise direction, the relative phase displacements of the waves of e.m.f. induced in the coils 1 1′, 2 2′, 3 3′, will be as shown in Fig. 12*b*, on the assumption that the magnetic flux density is sinusoidally distributed in the air gap. A diagrammatical representation of a star-connected generator is shown in Fig. 12*c*, the neutral point being marked *O*. The e.m.f. in each phase is termed positive when it acts in the direction from *O* outwards, that is, from the finishing end of each coil winding towards the beginning end. If the e.m.f. induced in phase 1 1′ is denoted by the expression

$$e_1 = E_m \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

this will correspond to the zero of time being taken as the moment at which the conductors of coil 1 1′ are in the position shown in Fig. 12*a*. The e.m.f.s of the three coils will then be respectively expressed as follows :

$$\left.\begin{aligned} e_1 &= E_m \cos \omega t \\ e_2 &= E_m \cos \left(\omega t - \frac{2\pi}{3}\right) \\ e_3 &= E_m \cos \left(\omega t - \frac{4\pi}{3}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and these e.m.f.s are shown in the vector diagram of Fig. 12*b*.

Reference to Fig. 12*b* will show that at any moment the algebraic sum of the e.m.f.s in the three phases will be zero, that is, for any ordinate such as *AB* the sum of the individual ordinates for the three phases is zero. This result is also seen from the expression (17),

$$e_1 + e_2 + e_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In Fig. 13*a* is shown diagrammatically a star-connected generator supplying a star-connected balanced load, and it is assumed that the load is non-inductive so that $\cos \phi = 1$ for each phase. The ve

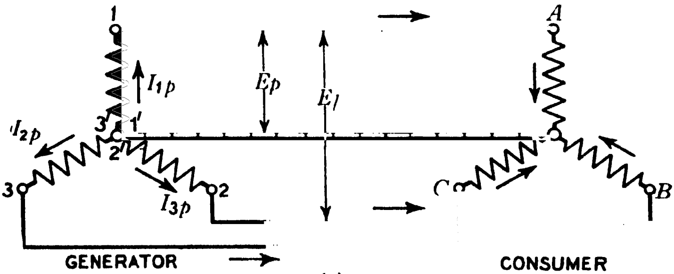
diagram is shown in Fig. 12*b*, for which the respective phase e.m.f.s are given by the expressions 13 *b*.

$$\left.\begin{aligned} e_1 &= E_m \sin \omega t \\ e_2 &= E_m \sin\left(\omega t - \frac{2\pi}{3}\right) \\ e_3 &= E_m \sin\left(\omega t - \frac{4\pi}{3}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19)$$

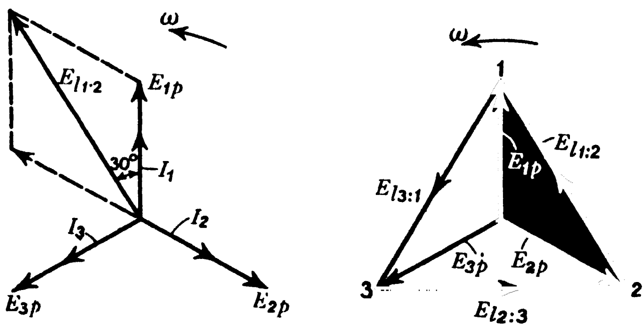
and the respective phase currents are given by the expressions

$$\left.\begin{aligned} i_1 &= I_m \sin \omega t \\ i_2 &= I_m \sin\left(\omega t - \frac{2\pi}{3}\right) \\ i_3 &= I_m \sin\left(\omega t - \frac{4\pi}{3}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

also $E_m = I_m \times R$, where R ohms per phase is the load resistance.



(*a*)



(*b*) (*c*)

Fig. 13.

It is to be observed that, since the convention has been adopted in the foregoing, to term the *generator* induced e.m.f. positive when it acts from the star point *outwards*, this convention also applying of course to the phase currents, it follows that the positive direction for the phase p.d.s and the phase currents in the *load* must be from the terminals *towards* the star point as is shown in Fig. 13*a*. The suffix p in Fig. 13 denote "phase" values and the suffixes l denote line values.

Since from expression (20) it follows that the sum of the phase currents is zero, that is,

$$i_1 + i_2 + i_3 = 0 \quad . \quad . \quad . \quad . \quad (21)$$

this implies that the total current which flows away from the generator star point at any moment is zero and similarly, that the total current flowing towards the load star point is zero, and consequently no current will flow along a conductor joining the two star points. Another way of stating this result is to say that since from expression (19) the generator star point e.m.f. is always zero and, similarly, the p.d. of the load star point is always zero, there will be no p.d. between the two star points so

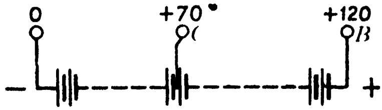


Fig. 14.

that no current will flow along a conductor which connects them. As will be seen later, however (Chapter XIII), this result does not apply to harmonics of the current which are a multiple of three.

Referring again to the vector diagram of Fig. 13*b*, which shows the vectors of the phase e.m.f.s with respect to the generator star point, it is now necessary to consider what will be the e.m.f. between any two terminals, that is, the e.m.f. between any two lines. Before considering the method of deriving this e.m.f. it will be helpful to consider a simple example of the potential difference between two points of a direct-current system. Fig. 14 shows a battery of accumulators of which O is the negative end, B is the positive end, and C is an intermediate tapping point, the pressure of the respective points being as shown in Fig. 14.

Then the pressure of	B	with respect	to	O	is	+ 120	volts
,, ,,	C	,,	,,	O	,,	+ 70	,,
,, ,,	O	,,	,,	B	,,	− 120	,,
,, ,,	O	,,	,,	C	,,	− 70	,,
,, ,,	C	,,	,,	B	,,	− 50	,,

Inspection of these results shows that, if the pressure $V_{B:O}$ of any point B with respect to the datum point O is known and the pressure $V_{X.O}$ of any other point X with respect to the same datum, then the pressure of

B with respect to X is obtained by the following rule, *reverse the sign of the datum pressure and add to the pressure of the point B*, thus,

$$V_{B X} = + V_{B O} - V_{X.O} \quad . \quad . \quad . \quad . \quad (22)$$

and this rule will be found to apply to all the foregoing numerical examples. For the very simple conditions of Fig. 14 only pressure *magnitudes* are involved and no rule in fact, is then necessary in practice, since the foregoing results are obtained almost instinctively. In the case of alternating currents and pressures, however, both magnitude and phase are involved and the corresponding problem of determining the pressure difference between two points is by no means so obvious. The foregoing rule, however, as exemplified in the expression (22), will be found helpful in such cases and will avoid much confusion which otherwise might arise. For example, in Fig. 13*b*, the pressure vector diagram defines the respective phase pressures with respect to the star point O. If it is desired to find the pressure of the terminal 1 with respect to the pressure of the terminal 2, that is to say, the magnitude and phase of the line e.m.f. $E_{1\ 2}$, the procedure is as follows. Fig. 13*b* shows the e.m.f. vectors $E_{1\ 0} : E_{2\ 0} : E_{3\ 0}$, and in order to obtain the line e.m.f. of 1 with respect to the line 2 it is necessary, in accordance with the foregoing rule, to reverse the datum pressure vector, that is, the vector $E_{2\ 0}$, and to add to the pressure vector $E_{1\ 0}$, and this procedure is shown in Fig. 13*b*, viz.

$$(E_l)_{1\ 2} = E_{1\ 0} - E_{2\ 0}.$$

It is then easily seen that the magnitude of the line e.m.f. will be $\sqrt{3}$ times the magnitude of the phase pressure, that is,

$$E_l = \sqrt{3}E_p \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and also that the line e.m.f. $(E_l)_{1\ 2}$ is 30° in advance of the phase e.m.f. E_1. In Fig. 13*c* is shown a combined diagram giving the phase e.m.f.s and the line e.m.f.s for the three terminals of the machine.

(*ii*) A Three-phase Mesh connected System.—In Fig. 15*a* is shown diagrammatically a two-pole generator having three coils exactly similar and similarly situated as those of the star-connected generator of Fig. 12*a*. In Fig. 15*a*, however, the coils are connected to form a closed circuit as follows. The beginning of coil 1 is connected to the finishing end 3′ of coil 3, the beginning of coil 3 is connected to the finishing end 2′ of coil 2, and the beginning end of coil 2 is connected to the finishing end 1′ of coil 1, the three junctions so formed being connected respectively to a separate slip-ring which is mounted on the armature shaft. A diagrammatic representation of this generator winding is shown in Fig. 15*b*.

Since the coils 1 1′ : 2 2′ : 3 3′ in Fig. 15*a* are exactly similarly situated on the armature core as the corresponding coils in Fig. 12*a*, the e.m.f.s induced in the individual coils will be the same for both the star-connected and the mesh-connected generators, that is to say, the positive direction

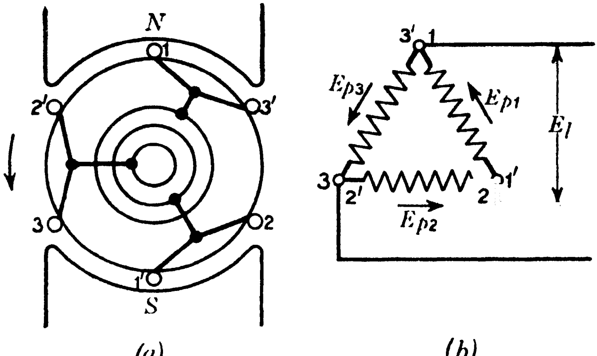


Fig. 15.

of the induced e.m.f.s act from the finishing end towards the beginning end of the coil, and it will be seen in Fig. 15*b* that the positive direction

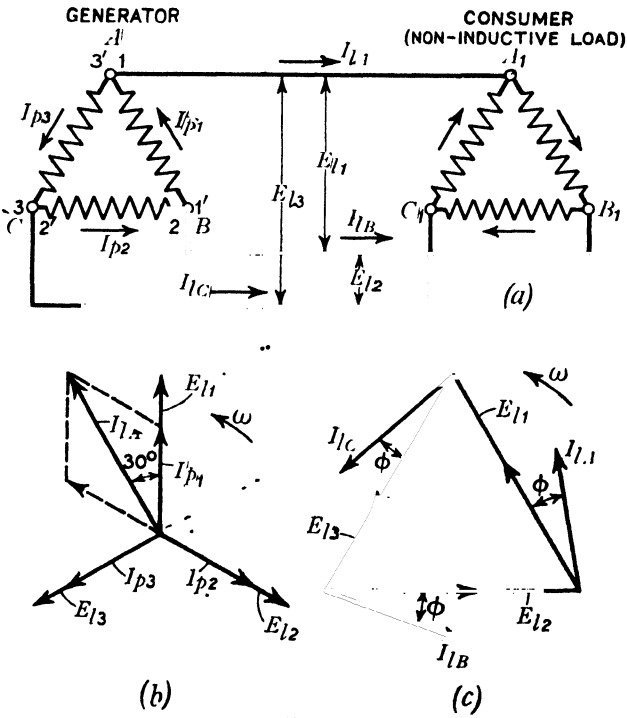


Fig 16

of the e.m.f.s defined in this way will be *counter-clockwise* round the mesh. It will also be seen that the e.m.f. between any two lines is the same as the e.m.f. in that phase across which the lines are connected.

In Fig. 16*a* is shown a mesh-connected generator supplying a balanced mesh-connected load, and it is assumed that the load is non-inductive and of a resistance R ohms per phase so that the power factor $\cos\phi = 1$ for each phase. From what has been said in the foregoing section (*ii*) with regard to the positive directions of current and pressure in a star connected load, it will be seen that, since the positive direction of the e.m.f. and the current in the generator is *counter-clockwise* round the mesh, the positive direction for the pressure and the current in the load must be *clockwise* round the mesh, as is shown in the diagram of Fig. 16*a*. In order to derive the phase and magnitude of the line current vector in Fig. 16*a* it is to be observed that, in accordance with Kirchhoff's rule (1) (Chapter V, page 124), the current flowing towards a junction in a network must be equal to the current flowing away from that junction. For example, at the junction A the following equation must hold:

$$\mathfrak{J}_{p1} = \mathfrak{J}_{lA} + \mathfrak{J}_{p2} \quad . \quad . \quad . \quad . \quad (24)$$

that is to say, the line current vector $\mathfrak{J}_{lA}$ is equal to the vector difference $\mathfrak{J}_{p1} - \mathfrak{J}_{p2}$ of the two adjoining phase currents, and this vector difference is shown in Fig. 16*b*, from which it will be seen that the magnitude of the line current is $\sqrt{3}$ times the magnitude of the phase current and that the vector of the line current I_{lA} is 30° ahead of the vector of the phase current I_{p1}. In Fig. 16*c* is shown the vector diagram for the line pressures and the line currents of a balanced three-phase system for which the power factor of the load is $\cos\phi$.

Referring again to Fig. 15*b*, it is seen that the positive direction of the e.m.f. induced in each phase is counter-clockwise round the mesh. Since, however, the sum of the instantaneous values of the induced e.m.f.s is zero at every moment, that is, since $e_1 + e_2 + e_3 = 0$, the resultant e.m.f. round the mesh is zero at every moment. If, however, there is a third harmonic component in the wave of induced e.m.f., this component will be in phase for all three coils of the winding so that the closed mesh will then form a short-circuit for this component harmonic as well as for every harmonic of the e.m.f. wave which is a multiple of 3 (see Chapter XIII).

Power of a Balanced Three-Phase System

If the star-connected generator system G of Fig. 17 is supplying a balanced load L of power factor $\cos\phi$, then the mean power for each phase will be given by the expression (12), page 348, that is,

$$W_p = V_p I_p \cos\phi \text{ watts} \quad . \quad . \quad . \quad . \quad (25)$$

where V_p volts is the r.m.s. value of the pressure per phase and I_p amperes is the r.m.s. value of the current per phase. Substituting the line pressure

for the phase pressure [see expression (23)], and noting that the phase current is also the line current in a star-connected system, the total mean value for the three-phase power may then be expressed as follows,

$$W = 3V_pI_p \cos\phi = \sqrt{3}V_lI_l \cos\phi \text{ watts} \quad . \quad . \quad (26)$$

It is of interest to examine this expression for the total mean power

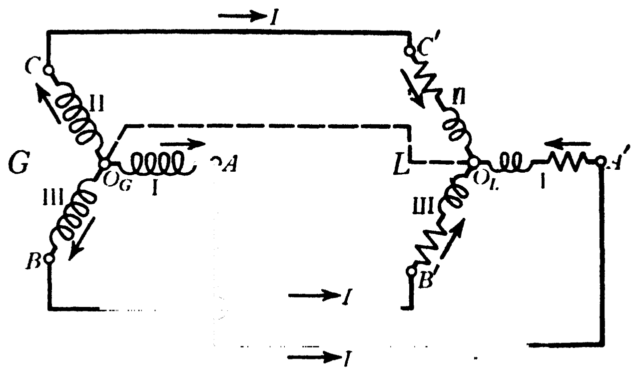


Fig. 17.

in comparison with the power at any moment t. In this case the sum of the values of the instantaneous power in the respective phases is

$$w = v_1i_1 + v_2i_2 + v_3i_3 \text{ watts.} \quad . \quad . \quad . \quad (27)$$

where v and i denote the instantaneous pressure and the instantaneous current in the individual phases at any moment t. Hence,

$$w = V_mI_m\left[\cos\omega t\cos(\omega t - \phi) + \cos\left(\omega t - \frac{2\pi}{3}\right)\cos\left(\omega t - \frac{2\pi}{3} - \phi\right)\right.$$
$$\left. + \cos\left(\omega t - \frac{4\pi}{3}\right)\cos\left(\omega t - \frac{4\pi}{3} - \phi\right)\right]. \quad (28)$$

The quantity in square brackets on the right-hand side of this expression reduces to $\frac{3}{2}\cos\phi$, so that the total instantaneous power at any moment t is given by the expression

$$w = \frac{3}{2}V_mI_m\cos\phi = 3V_pI_p\cos\phi \text{ watts} \quad . \quad . \quad (29)$$

That is to say, the total instantaneous power is at every moment equal to the total mean power for the three phases. For a balanced three-phase load, therefore, there is no pulsating component in the expression for the instantaneous power. This result should be contrasted with the result obtained on page 348, Fig. 9, for the single-phase system, in which case a double frequency component appears.

Measurement of the Power and Power Factor of a Three-Phase Load by Means of One Wattmeter

The circuit connections for this measurement are shown in Fig. 18, in which the current coil of the wattmeter is connected in series with the line I, whilst the pressure coil is connected at one end to the line I and at the other end it is connected alternately to the lines II and III by means of the two-way switch. Let

$W_{\mathrm{I\ II}}$ be the wattmeter reading when the switch is on 2
$W_{\mathrm{I\ III}}$,, ,, ,, ,, ,, ,, ,, ,, ,, 3

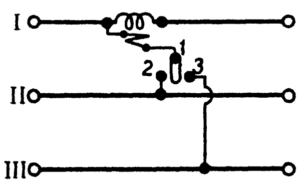


Fig. 18

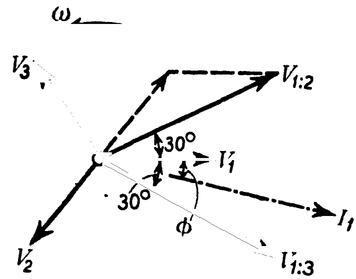


Fig. 19.

then the total power which is being supplied to the load will be given by the sum of the two wattmeter readings,

$$W_{\mathrm{I\ II}} + W_{\mathrm{I\ III}} \text{ watts.}$$

If one of the wattmeter readings, say, $W_{\mathrm{I\ II}}$, is negative, then the pressure-coil terminals should be interchanged by means of a change-over switch in order to get a reading on the scale of the instrument and the sign of the reading $W_{\mathrm{I\ II}}$ is then taken to be negative, so that the total three-phase power would then be given by the difference,

$$W_{\mathrm{I\ III}} - W_{\mathrm{I\ II}}.$$

That the total three-phase power is, in fact, given by the sum (or the difference) of the two wattmeter readings can be seen from the following considerations. In the vector diagram of Fig. 19 the vectors $OV_1 : OV_2 : OV_3$ respectively represent the phase pressures of the lines I, II, III. The vector OI_1 represents the current in the line I and this current is assumed to be lagging by the angle ϕ on the vector of the phase pressure OV_1.

The wattmeter readings will then be

$$W_{\mathrm{I\ II}} = OV_{1\cdot 2} \times OI_1 \cos(30^\circ + \phi) : \ W_{\mathrm{I\ III}} = OV_{1\cdot 3} \times OI_1 \cos(30^\circ - \phi)$$

but $\qquad OV_{1\cdot 2} = OV_{1\cdot 3} = V_l$ and $I_1 = I_l$,

so that

$$W = W_{\mathrm{I\ II}} + W_{\mathrm{I\ III}} = V_l I_l[\cos(30^\circ + \phi) + \cos(30^\circ - \phi)]$$
$$= \sqrt{3}\, V_l I_l \cos\phi \quad . \quad . \quad . \quad . \quad (30)$$

that is to say, this expression gives the total three-phase power which is supplied to the balanced load.

In order to determine the power factor from the two wattmeter readings, the following relationships may be considered,

$$W_{\mathrm{I\ III}} = V_l I_l \cos(30^\circ - \phi) : \quad W_{\mathrm{I\ II}} = V_l I_l \cos(30^\circ + \phi),$$

so that

$$\begin{aligned} W_{\mathrm{I\ III}} - W_{\mathrm{I\ II}} &= V_l I_l[\cos(30^\circ - \phi) - \cos(30^\circ + \phi)] \\ &= 2V_l I_l \sin 30^\circ \sin\phi \\ &= V_l I_l \sin\phi \end{aligned}$$

also, since

$$W_{\mathrm{I\ III}} + W_{\mathrm{I\ II}} = \sqrt{3}V_l I_l \cos\phi$$

it follows that

$$\tan\phi = \sqrt{3}\,\frac{1 - \dfrac{W_{\mathrm{I\ II}}}{W_{\mathrm{I\ III}}}}{1 + \dfrac{W_{\mathrm{I\ II}}}{W_{\mathrm{I\ III}}}} \quad . \quad . \quad . \quad (31)$$

so that if the ratio $\dfrac{W_{\mathrm{I\ II}}}{W_{\mathrm{I\ III}}}$ is known, the value of $\tan\phi$ and hence the power factor $\cos\phi$ can be derived at once from the expression (31).

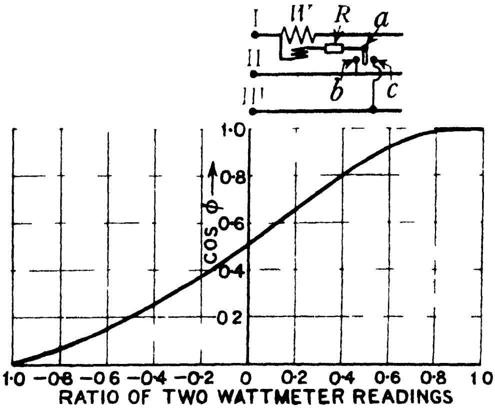


Fig 20.

It is convenient to plot the ratio $\dfrac{W_{\mathrm{I\ II}}}{W_{\mathrm{I\ III}}}$ as a function of the corresponding value of $\cos\phi$ as is shown in Fig. 20, so that, in any given case, the power factor may be read off the curve for the given ratio of the two wattmeter readings without the necessity of having to refer to trigonometrical tables.

Measurement of the Total Three-Phase Power by Means of Two Wattmeters whether the Load is Balanced or Unbalanced

In Fig. 21 are shown the circuit connections for this method of measurement and, as an example, a star-connected load is assumed, although the results which are derived in what follows are equally

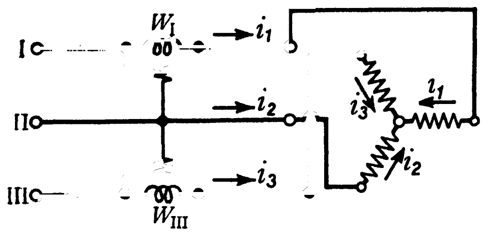


Fig. 21.

applicable when the load is mesh-connected. The power supplied to the circuit at any instant

$$w = v_I i_I + v_{II} i_{II} + v_{III} i_{III} \text{ watts,}$$

where $v_I : v_{II} : v_{III}$, respectively, represent the phase pressures of the lines I, II, III. It is also known that

$$i_I + i_{II} + i_{III} = 0 : \text{ so that, } i_{II} = -(i_I + i_{III}),$$

which means, of course, that one line forms the return path for the currents in the other two lines. Hence the power at any instant is

$$w = v_I i_I - v_{II} i_I - v_{II} i_{III} + v_{III} i_{III} = i_I(v_I - v_{II}) + i_{III}(v_{III} - v_{II}),$$

that is,

$$w = v_{I:II} i_I + v_{III:II} i_{III} \text{ watts} \quad . \quad . \quad . \quad (32)$$

But the mean value of $v_{I:II} i_I$ is given by the wattmeter in line I and the mean value of $v_{III:II} i_{III}$ is given by the reading of the wattmeter in line III, so that the algebraical sum of the two wattmeter readings is a measure of the total power supplied to the load and the foregoing relationships are true, whether the load is balanced or not.

Power Loss in the Dielectric of a Three-Phase Cable

Reference to the dielectric power loss in cables * has been made in Chapter II (page 84), and it is stated there that this power loss is defined by the "angle of loss" δ. Fig. 22 shows the vector diagram per phase of a three-phase cable, that is, the applied pressure per phase and the corresponding capacitance current vector, from which it is seen that the current vector leads on the vector of applied pressure by the angle ϕ. The "angle of loss" is then defined as

$$\delta^\circ = 90^\circ - \phi^\circ.$$

* See also *Engineering*, Vol. 158, December 1, 1944, p. 422.

In Fig. 23 is shown the value of tan δ as a function of the applied pressure per phase, for a constant temperature of 17° C. and of 40° C., and it is seen that the angle of loss is fairly constant over a wide range of applied pressure values. The power loss in the dielectric will increase as the angle of loss increases and, for a cable with three lead-sheathed cores (see Chapter IV, page 108) the value of the angle δ will usually lie with the range of about 0·003–0·008 radian.

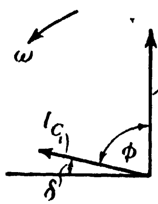


Fig. 22.

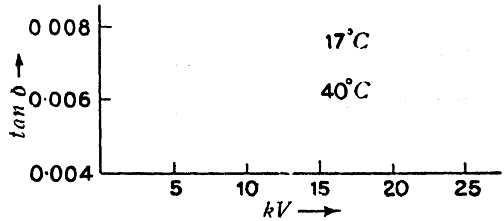


Fig. 23.

Reference to Fig. 22 shows that the dielectric loss can be expressed as follows :

$$W_{diel.} = 3V_1 I_{C1} \cos\phi = 3V_1 I_{C1} \sin\delta$$

or

$$W_{diel.} \simeq 3V_1 . I_{C1} . \delta \text{ watts per phase per km.} \quad . \quad . \quad (33)$$

where $I_{C1} = V_1\omega C$ amperes is the "charging current" of the cable per phase per kilometre, C farad is the effective operating capacity per phase per kilometre of the cable, and V_1 volts per phase is the applied pressure. Hence the dielectric power loss may be expressed by the relationship

$$W_{diel.} = (3V_1{}^2\omega C) \times \delta \text{ watts per phase per kilometre.} \quad . \quad (34)$$

For the method of measuring the angle of loss by means of the Wien Bridge, see Chapter IX, page 309.

The Vector Representation of Alternating Current Power

It has been seen that the mean power in a circuit in which the pressure is defined by the vector $\mathfrak{V} = Ve^{j(\omega t+\phi)}$ and the current by the vector $\mathfrak{J} = Ie^{j\omega t}$, is given by the expression

$$W = VI \cos\phi \text{ watts,}$$

where V volts is the r.m.s. magnitude of the pressure and I amperes is the r.m.s. value of the current. This expression may be written in the form

$$W = VI \cos\phi = V \times OC,$$

where OC (Fig. 24) is the projection of the current vector on the pressure vector. The quantity $VI \cos\phi$ is termed the "active" power and $OC = I \cos\phi$ is the "watt component" of the current. For many

investigations, however, it is of great importance to take into consideration the wattless component of the current, that is, the quantity $VI \sin \phi = V \times AC$ in Fig. 24. This product is termed the "reactive" power and the vector of power may then be expressed as follows:

$$\mathfrak{W} = W_a + jW_r = VI \cos \phi + jVI \sin \phi \quad . \quad . \quad (35)$$

the magnitude VI of this quantity being termed the "apparent power".

The problem will now be considered as to how the vector of power may be algebraically expressed in terms of the vectors of pressure and current. In Chapter IX, page 296, it has been seen that a sinusoidal current wave may be expressed in the form,

$$i = I_m \cos \omega t = I_m e^{j\omega t} \text{ amperes,}$$

on the assumption that only the real component of the vector $I_m e^{j\omega t}$ was to be taken into account and that the imaginary or j component had no practical

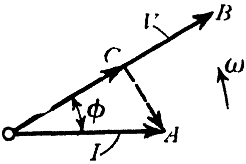


Fig. 24.

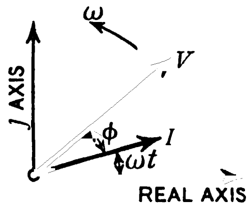


Fig. 25.

significance and was neglected. For the investigation of the power relationships, however, the vector $I_m e^{j\omega t}$ will be expressed in its complete mathematical form,

$$I_m e^{j\omega t} = I_m(\cos \omega t + j \sin \omega t),$$

as is shown in Fig. 25.

Now write the vectors of pressure and current respectively as follows:

$$\left.\begin{aligned} \mathfrak{I} &= Ie^{j\omega t} = I(\cos \omega t + j \sin \omega t) \\ \mathfrak{V} &= Ve^{j(\omega t + \phi)} = V\{\cos (\omega t + \phi) + j \sin (\omega t + \phi)\} \end{aligned}\right\} . \quad . \quad (36)$$

where I and V are respectively the r.m.s. current in amperes and the r.m.s. pressure in volts. The product of the two vectors of expression (36) is then,

$$\begin{aligned} \mathfrak{V}.\mathfrak{I} = VI\{&\cos (\omega t + \phi) \cos \omega t - \sin (\omega t + \phi) \sin \omega t \\ &+ j[\cos (\omega t + \phi) \sin \omega t + \sin (\omega t + \phi) \cos \omega t]\} \end{aligned}$$

and this reduces to

$$\mathfrak{V}.\mathfrak{I} = VI\{\cos (2\omega t + \phi) + j \sin (2\omega t + \phi)\} \quad . \quad . \quad (37)$$

That is to say, the product of the vectors of current and pressure, as defined by the expressions (36), only gives information relating to the double frequency terms of the power components and gives no information as

regards the mean power. This product, therefore, will not lead to the required power vector as defined by the expression (35).

Now consider, for example, the vector which is conjugate to the current vector $\mathfrak{J} = Ie^{j\omega t}$, viz.

$$\mathfrak{J}^* = Ie^{-j\omega t} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

as shown in Fig. 26 (see also Fig. 69, page 296). The product of the vectors $\mathfrak{J}^*$ and $\mathfrak{B}$ is then

$$\mathfrak{B}.\mathfrak{J}^* = VI[\cos(\omega t + \phi) + j\sin(\omega t + \phi)][\cos(-\omega t) + j\sin(-\omega t)]$$

and this reduces to

$$\mathfrak{B}.\mathfrak{J}^* = VI[\cos\phi + j\sin\phi] \quad . \quad . \quad . \quad (39)$$

This expression is now identically the same as that for the power vector $\mathfrak{W}$ of expression (35). It will be noted that in the foregoing investigation, the condition has been assumed that the current lags on the applied pressure, and that as a consequence of this assumption the result is obtained that the reactive component of the power vector in expression (39) is *positive*. It is easily shown, however, that if, instead of taking the product of the pressure vector and the *conjugate current vector*, as has been done in the expression (39), the product is taken of the *conjugate pressure vector* and the current vector, then a *lagging current* will lead to the result,

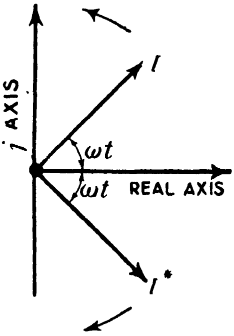


Fig. 26.

$$\mathfrak{B}^*\mathfrak{J} = VI[\cos\phi - j\sin\phi] \quad . \quad . \quad . \quad (40)$$

whilst a *leading current* gives the result,

$$\mathfrak{B}^*\mathfrak{J} = VI(\cos\phi + j\sin\phi) \quad . \quad . \quad . \quad (41)$$

and it is merely a matter of convenience which of these two alternative representations is selected in any particular problem.

EXAMPLE.—As one typical example of the application of these results may be taken a simple case illustrative of the problems which arise in heavy current transmission systems and also in radio transmission and reception.

Suppose in the series circuit of Fig. 27 the consumer's terminals are AB and the supply is provided by the generator of which the e.m.f. is defined by the vector $\mathfrak{E} = Ee^{j\omega t}$. The impedance of the generator, of the leads, and of any other supplementary apparatus in front of the consumer's terminal AB being

$$\mathfrak{Z}_i = R_i + jX_i = Z_ie^{j\phi_i} \quad . \quad . \quad . \quad . \quad (42)$$

The impedance of the consumer's circuit is defined by the vector

$$Z_0 = R_0 + jX_0 = Z_0e^{j\phi_0} \quad . \quad . \quad . \quad . \quad (43)$$

The pressure across the consumer's terminals is $\mathfrak{V}_0$ and the current in the circuit is $\mathfrak{I}$, so that

$$\mathfrak{I} = Ee^{j\omega t}\frac{1}{\mathfrak{Z}_R} = E\frac{1}{Z_R}e^{j(\omega t - \alpha)} \quad . \quad . \quad . \quad (44)$$

where $$\mathfrak{Z}_R = \mathfrak{Z}_i + \mathfrak{Z}_0 = Z_R e^{j\alpha}$$

also $$\mathfrak{V}_0 = \mathfrak{I}\,\mathfrak{Z}_0 = E\frac{Z_0}{Z_R}e^{j(\omega t + \phi_0 - \alpha)} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and, reference to Fig. 28 shows that

$$Z_R^2 = Z_0^2 + Z_i^2 + 2Z_0Z_i \cos(\phi_i - \phi_0) \quad . \quad . \quad (46)$$

The vector of the power supplied to the consumer is then

$$\mathfrak{W} = \mathfrak{V}_0\mathfrak{I}^* = \frac{E^2Z_0}{Z_R^2}e^{j\phi_0} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

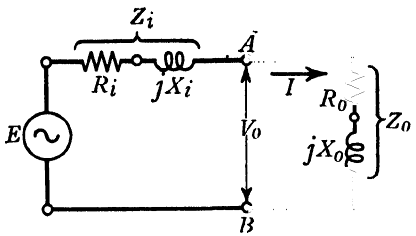


Fig. 27.

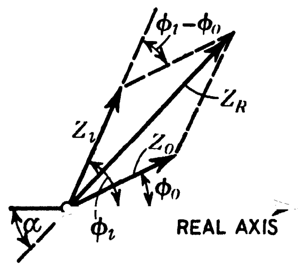


Fig. 28.

The modulus of this vector is the apparent power, viz.

$$|\,\mathfrak{W}\,| = \frac{E^2Z_0}{Z_0^2 + Z_i^2 + 2Z_0Z_i\cos(\phi_i - \phi_0)} \quad . \quad . \quad (48)$$

The active power is then

$$W_a = |\,\mathfrak{W}\,| \cos\phi_0 \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and the reactive power is

$$W_r = |\,\mathfrak{W}\,| \sin\phi_0 \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From these results, for example, the condition for which the active power is a maximum when the angle ϕ_0 is given, can easily be found, viz. :

$$\frac{d}{dZ_0}W_a = 0$$

which leads to the result that

$$Z_0 = Z_i.$$

The d.c. counterpart of this important result is given in Example 20 of the Test Paper for Chapter II.

In Chapter XV further use is made of this method for representing the vector of power.

Chapter XII

SOME GRAPHICAL METHODS

INVERSION; FLYWHEEL LOAD EQUALISER; ACCELERATION OF A MOTOR; ROTATING MAGNETIC FIELD: SYMMETRICAL COMPONENTS

The Principle of Inversion

THE method of " Inversion " provides a simple and easily surveyed graphical representation of the solution of many problems which would otherwise involve much more complicated treatment, with the corresponding loss of clarity of interpretation of the physical significance of the solutions, as well as of the intermediate stages by which the solutions have been reached.

Consider in Fig. 1 any curve such as Q and let O be a point some distance from the curve as shown. If P is any point on the curve Q and if another point p is found on the line OP such that,

$$OP \times Op = k^2,$$

where k^2 is a constant, then the point p will lie on some such curve as q. The curve q is termed the *inverse* of the curve Q with respect to the point O,

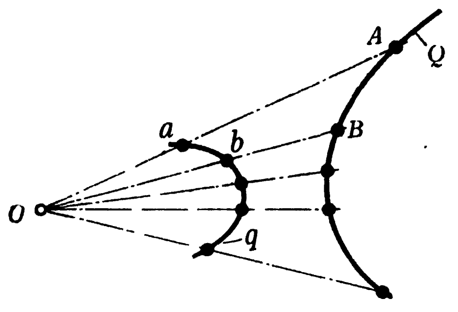


Fig. 1.

the constant k^2 is the *constant of inversion* and the point O is the *centre of inversion*. The diagram of Fig. 1 has been drawn for the condition that $k^2 = 42$ units.

PROPOSITION I. THE INVERSE OF A STRAIGHT LINE IS A CIRCLE THROUGH THE CENTRE OF INVERSION.—From the centre of inversion O in Fig. 2, draw a line OA perpendicular to the straight line CD which it is required to invert. Mark off the point a so that

$$OA \times Oa = k^2,$$

where k^2 is the given constant of inversion, and on Oa as diameter, draw the circle shown in Fig. 2. Draw any other straight line such as OR which cuts the circle at r and the straight line CD at R. Join ar, then it will be seen that

the triangles Oar and ORA are similar,

so that
$$\frac{Or}{OA} = \frac{Oa}{OR} \text{ and } OR \times Or = OA \times Oa = k^2.$$

Consequently, Or is the inverse of OR with respect to O and this relationship will hold for any other point on the circle. Hence the proposition is proved.

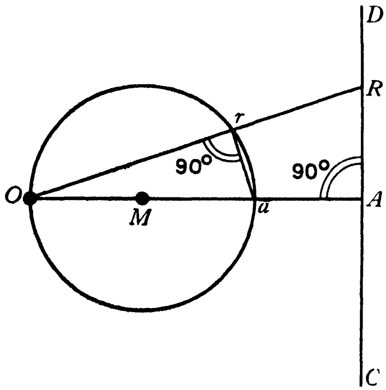


Fig. 2.

PROPOSITION 2. THE INVERSE OF A CIRCLE WITH RESPECT TO A POINT OUTSIDE THE CIRCLE IS ANOTHER CIRCLE.—In Fig.3 O is the centre of inversion and Q is the circle which it is required to invert with respect to the point O. Draw a line OA through the centre of the circle Q and cutting this circle at F and G. Let Of be the inverse of OF and let Og be the inverse of OG. Draw the circle q on fg as diameter.

From O draw the line OT tangential to the circle Q, then if this line is also tangential to the circle q, the following condition must hold,

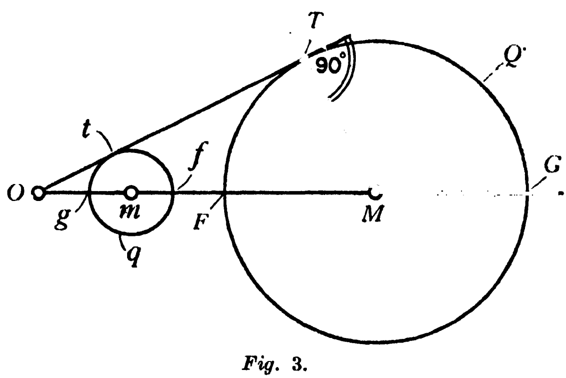


Fig. 3.

$$Ot^2 = Og \times Of: \ OT^2 = OG \times OF$$

and hence
$$Ot^2 \times OT^2 = (OG \times Og) \times (OF \times Of) = k^4,$$

so that
$$Ot \times OT = k^2,$$

that is to say, Ot is the inverse of OT, and consequently the circle q is the inverse of the circle Q.

An important special case of this proposition is that in which the

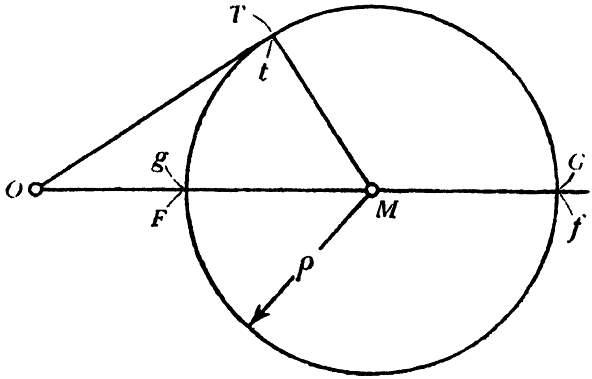


Fig. 4.

circle Q inverts into itself as shown in Fig. 4. In this case,

$$OF \times Of = OG \times Og = k^2 = OT^2 = OM^2 - MT^2,$$

that is $$OM^2 = MT^2 + OF \times Of$$

or $$OM^2 = \rho^2 + k^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

That is to say, in order that a circle shall invert into itself, the position of the centre of inversion O must be related to the radius ρ of the circle and the constant of inversion k^2 by the expression (1).

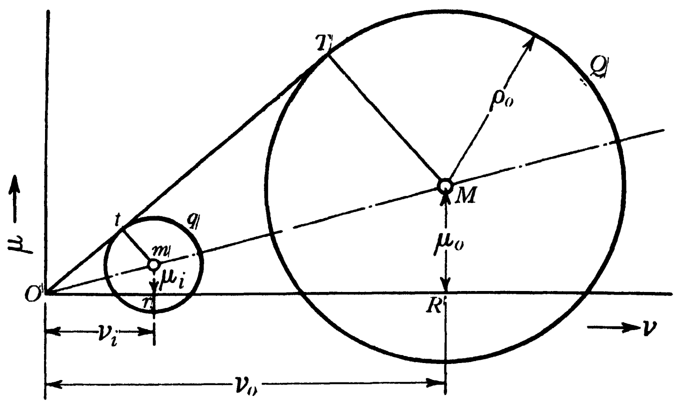


Fig. 5.

Example.—As an important practical example of Proposition 2, suppose in Fig. 5 it is required to find the co-ordinates of the centre and the radius of the inverted circle q with respect to the co-ordinates of the

centre and the radius of the original circle Q, when the constant of inversion is $k^2 = 1$. Let the centre of inversion O be taken as the origin of co-ordinates as shown in Fig. 5.

Then :

$$\text{For the original circle } Q\begin{cases}\text{Centre of circle is at} & M\\ \text{Co-ordinates of centre are} & \nu_0 : \mu_0\\ \text{Radius of the circle is} & \rho_0\end{cases}$$

$$\text{For the inverted circle } q\begin{cases}\text{Centre of circle is at} & m\\ \text{Co-ordinates of centre are} & \nu_i : \mu_i\\ \text{Radius of the circle is} & \rho_i.\end{cases}$$

It will be seen from Fig. 5 that
the triangles $OMT : Omt$ are similar, and the triangles $OMR : Omr$ are similar, so that

$$\frac{OT}{Ot} = \frac{\mu_0}{\mu_i},$$

also

$$OT \times Ot = k^2 = 1,$$

hence

$$\left.\begin{aligned}\mu_i &= \mu_0\frac{Ot}{OT} = \mu_0\frac{1}{OT^2}\\ \nu_i &= \qquad\quad \nu_0\frac{1}{OT^2}\\ \rho_i &= \qquad\quad \rho_0\frac{1}{OT^2}\\ OT^2 &= \mu_0^2 + \nu_0^2 - \rho^2\end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

also (the last line above)

Proposition 3.—*If there is a pair of curves such as the circles $Q : Q_1$ in Fig. 6, and if these curves be inverted with respect to the point O, the value of the ratio $\frac{DC}{DA}$ for an elementary area $DABC$ of the first pair of curves will be the same as the value of the ratio $\frac{dc}{da}$ for the area $dabc$ of inversion of the second pair of curves $q : q_1$.*

This proposition may be proved as follows: $OD \times Od = k^2 = OC \times Oc$ so that the triangles OCD and Ocd are similar and hence

$$\frac{DC}{dc} = \frac{OC}{Od},$$

similarly

$$\frac{DA}{da} = \frac{OA}{Od},$$

from which it follows that

$$\frac{DA}{DC} = \frac{OA}{OC} \times \frac{da}{dc} = \frac{da}{dc},$$

since $OA = OC$ when the dimensions of the elementary areas are indefinitely small.

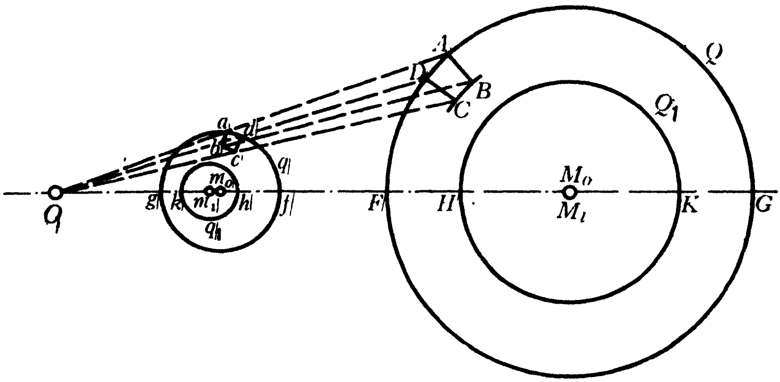


Fig. 6.

EXAMPLE.—An overhead transmission is represented by the circle C in Fig. 7*a*, and the surface of the earth by the line EF, the height of the line above the earth being h cm. and the diameter of the wire d cm.

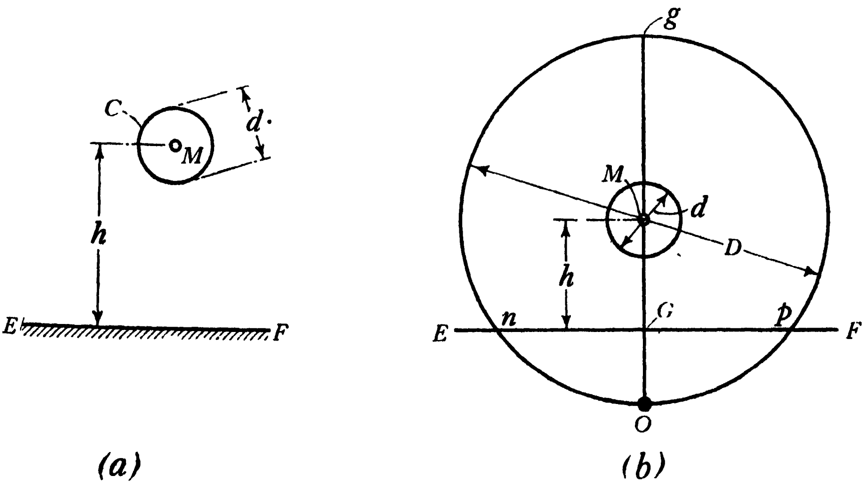


Fig. 7.

Now invert the circle C and the line EF so that the circle inverts into itself and the line inverts into a circle which is concentric with the circle C. From what has been said in the foregoing, it will be seen that the capacitance of the electrostatic field per centimetre length between these two concentric cylinders will be the same as the capacitance per

centimetre length between the cylindrical surface represented by the circle and the plane EF in Fig. 7a. Since the capacitance between two concentric circles is given by the expression (12), page 103, Chapter IV, the problem is at once solved.

In Fig. 7b let D cm. be the diameter of the circle which is the inverse of the line EF. From Proposition 1 (see Fig. 2) it is known that the centre of inversion O must lie on the circle npg. Then

$$\left(\frac{D}{2}\right)^2 = OM^2 = \left(\frac{d}{2}\right)^2 + k^2$$

and $\quad OG \times Og = k^2$: that is, $\left(\frac{D}{2} - h\right)D = \left(\frac{D}{2}\right)^2 - \left(\frac{d}{2}\right)^2,$

Solving this quadratic equation in $\frac{D}{2}$ gives,

$$\frac{D}{2} = h + \sqrt{h^2 - \left(\frac{d}{2}\right)^2} \simeq 2h,$$

when d is small compared with h and, from expression (12), on page 103, the required capacitance of the two concentric cylinders is,

$$C = \frac{\varepsilon}{2 \log_e \frac{D}{d}} \text{ electrostatic units per centimetre length.}$$

This provides one proof of the formula which is stated on page 104, expression (13).

Example.—As an example of Proposition 3 may be taken the capacitance between a cable sheath and one core of a three-core cable as illustrated by the two eccentric circles in Fig. 8. For this case, the centre of inversion is so chosen that the large circle H inverts into a straight line and the small circle inverts into itself. That is to say, the centre of inversion O must lie on the large circle as is shown in Fig. 8, and the straight line obtained by inverting this circle will cut the diameter through O at the point S where

$$OS = \frac{OF^2 - \left(\frac{d}{2}\right)^2}{OS_1}.$$

Since from Proposition 2, page 368,

$$OF^2 = \left(\frac{d}{2}\right)^2 + k^2.$$

and also $\quad OS \times OS_1 = k^2$

and, making use of the solution of Example 14, in the Test Paper for

Chapter IV, it is then easily seen that the capacitance of the field between the core and the eccentric sheath will be

$$C = \frac{\varepsilon}{41 \log_{10}\left[\frac{2FS + \sqrt{4FS^2 - d^2}}{d}\right]} \mu\text{F per km. length} \quad . \quad (3)$$

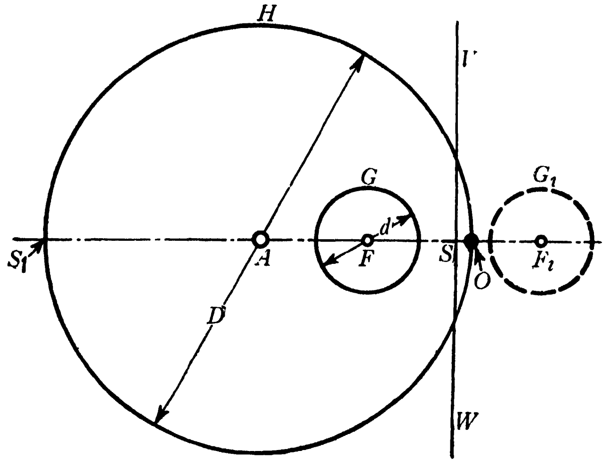


Fig. 8.

To Find the Input Admittance of a Circuit comprising a Fixed Inductance of *L* Henry and a Variable Resistance of *R* Ohms

This circuit is shown in Fig. 9 and the vector of impedance of the circuit will then be given by

$$\mathfrak{Z} = R + j\omega L = R + jX = Ze^{j\phi} \text{ ohms} \quad . \quad . \quad (4)$$

where $Z = \sqrt{R^2 + X^2}$ ohms: and $\tan \phi = \frac{X}{R}$.

The admittance vector $\mathfrak{Y}$, is the inverse of the impedance vector $\mathfrak{Z}$ so that

$$\mathfrak{Z}\cdot\mathfrak{Y} = 1 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

that is to say, the constant of inversion is $k^2 = 1$. Then,

$$\mathfrak{Y} = \frac{1}{\mathfrak{Z}} = \frac{1}{Z}e^{-j\phi} = Ye^{-j\phi} \text{ siemens} \quad . \quad . \quad . \quad (6)$$

In Fig. 10 the impedance vector is shown as OB making the angle $+\phi$ with the real axis so that, in accordance with the expression (4) OA is drawn to scale equal to X ohms in the direction of the j axis and the resistance R ohms is drawn to scale in the direction of the real axis.

Since the resistance R is variable, that is, the length AB is variable,

it follows that the extremity B of the impedance vector will move along the vertical straight line AB. The admittance vector will therefore be given by the inverse of this vertical line with respect to the origin O and, in accordance with Proposition I, the inverse of the line AB will be a circle which passes through O. Now reference to the expression (6) shows that the angle defining the position of the admittance vector will be $-\phi$ when the angle which defines the position of the impedance vector is $+\phi$. That is to say, the admittance circle will lie in the opposite quadrant to the impedance vector as is shown in Fig. 10.

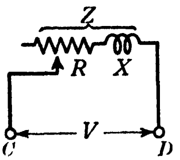


Fig. 9.

The relationship between the current vector $\mathfrak{I}$ and the pressure vector $\mathfrak{V}$ (Fig. 9) may be written in either of two ways,

$$\mathfrak{V} = \mathfrak{I}\cdot\mathfrak{Z} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$\mathfrak{I} = \mathfrak{V}\cdot\mathfrak{Y} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The expression (7) states that, if the *current is constant*, the necessary pressure at the terminals CD (Fig. 9) will be given in magnitude and phase by the vector OB when $\mathfrak{I}$ is the datum vector and is drawn along the real axis, that is the vertical axis in Fig. 10.

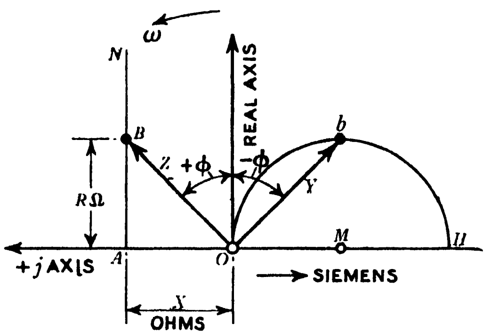


Fig. 10.

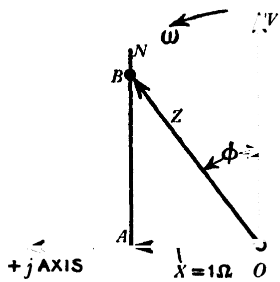


Fig. 11.

Alternatively, the expression (8) states that, if the *supply pressure* $\mathfrak{V}$ *is constant* and is taken to be the datum vector and drawn along the real axis, the current vector will be given in phase and magnitude by the vector Ob in the admittance diagram shown in Fig. 10.

As a numerical example, suppose the reactance $X = \omega L = 1$ ohm and choose a scale of 3 cm. = 1 ohm for the resistance, then in Fig. 11 mark off $OA = X = 1$ ohm = 3 cm. and draw the vertical line AN. Then, for example, when the resistance is $R = \frac{4}{3}$ ohm, that is, $AB = 4$ cm.

$$OB = \sqrt{3^2 + 4^2} = 5 \text{ cm. and } \tan\phi = \tfrac{3}{4}.$$

The impedance vector will then be,

$$\mathfrak{Z} = OB = \frac{5}{3}e^{j\phi}$$

where
$$\tan\phi = \frac{3}{4}.$$

If the current is maintained constant at 10 amperes, then the magnitude of the pressure which is necessary at the supply terminals will be $10 \times \frac{5}{3} = \frac{50}{3}$ volts, so that the vector OB which is 5 cm. long will represent $\frac{50}{3}$ volts. That is to say, the scale for the pressure vector is 3 cm. = 10 volts.

Next, suppose the pressure at the terminals CD (Fig. 9) is maintained constant at 100 volts so that the current vector is given by the expression (8), that is,

$$\mathfrak{J} = 100 \times \mathfrak{Y},$$

where the admittance $\mathfrak{Y}$ is measured in siemens. Since the admittance

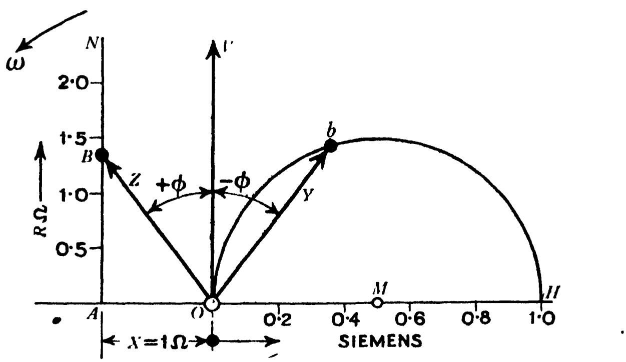


Fig. 12.

is the inverse of the impedance, that is, the inverse of the straight line AN of Fig. 11, the admittance diagram will be a circle.

The diameter OH (Fig. 12) of the admittance circle is given by the inverse of the reactance OA (Figs. 11 and 12), that is,

$$OH = \frac{1}{X} = 1 \text{ siemens},$$

the scale chosen for the admittance being 1 siemens = 9 cm. Since the supply pressure is 100 volts, the scale for the current will be

$$9 \text{ cms.} = 100 \times (1 \text{ siemens}) = 100 \text{ amperes},$$

that is, 1 cm. $= \frac{100}{9}$ amperes. Thus, the impedance vector OB of Figs. 11 and 12 is

$$\mathfrak{Z} = OB = \frac{5}{3}e^{j\phi} \text{ ohms}$$

The corresponding admittance vector is

$$\mathfrak{Y} = Ob = \frac{1}{\mathfrak{Z}} = \frac{3}{5}e^{-j\phi} \text{ siemens}$$

and the current represented by Ob will be

$$\mathfrak{I} = Ob = 100 \times 0{\cdot}6e^{-j\phi} = 60e^{-j\phi} \text{ amperes}$$

and this gives the scale for the current.

The Input Admittance Diagram for a Circuit comprising a Resistance, an Inductance, and a Capacitance in Series, and Supplied with Alternating Current at Variable Frequency

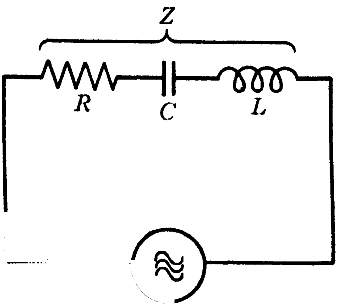


Fig. 13.

The circuit connections for this system are shown in Fig. 13, in which the inductance of L henry, the capacitance of C farad, and the resistance of R ohms are in series and supplied from a pressure of *variable frequency* as shown in Fig. 13. For any given value of the circular frequency ω, the impedance vector will be

$$\mathfrak{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = Ze^{j\phi} \text{ ohms} \quad . \quad . \quad (9)$$

where $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ and $\tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R}$.

The numerical values of the respective constants of the circuit are taken as follows for the purposes of an Example, viz.,

$R = 6$ ohms : $L = 4 \times 10^{-5}$ henry : $C = 5 \times 10^{-9}$ farad.

In Fig. 14 the resistance of R ohms is drawn along the real axis equal to OA to scale, and this quantity will, of course, remain of constant magnitude for all values of the circular frequency ω. For values of the frequency such that $\omega L > \frac{1}{\omega C}$ the reactance is to be marked off in the positive direction of the j axis, that is, along QP, whilst for values of the frequency ω such that $\omega L < \frac{1}{\omega C}$ the reactance is to be marked off

in the negative direction of the j axis, that is, in the direction AQ in Fig. 14. It is seen, therefore, that the impedance vector will be such

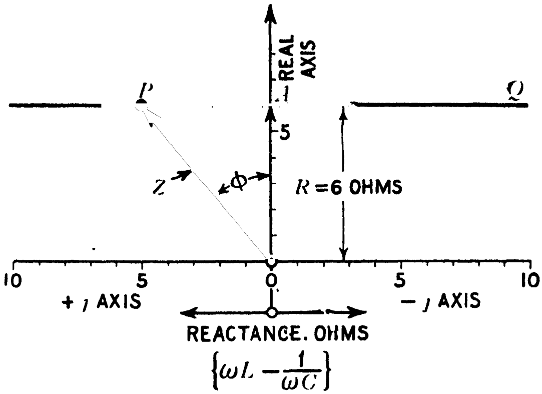


Fig. 14.

that its extremity P may lie anywhere along the horizontal line PQ. The admittance vector will therefore be given by the inverse of the

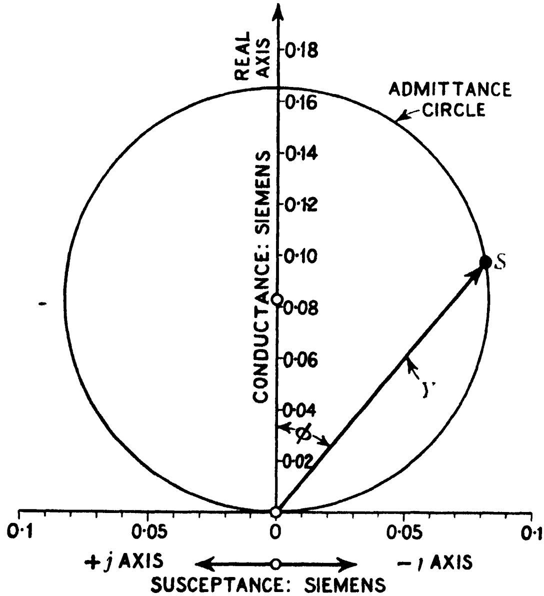


Fig. 15.

horizontal straight line PQ with respect to the origin O and is consequently defined by a circle which will pass through the origin O and of which

the centre will lie on the real axis, as is shown in Fig. 15. The point S in Fig. 15, for example, corresponds to the point P in Fig. 14.

The Input Impedance for the Compound Circuit of Fig. 16

This system comprises two component circuits 1 and 2 which are coupled by the mutual inductance M. The only variable quantity in the system is the resistance R_2. This system is typical of important practical problems such as the graphical representation of the performance of induction motors. The system is supplied with alternating current pressure at a fixed frequency connected across the terminals CD (Fig. 16). The following numerical data have been taken as an example of the detailed method for graphically finding the input admittance of the system, that is, with respect to the input terminals CD. The circuit constants are :

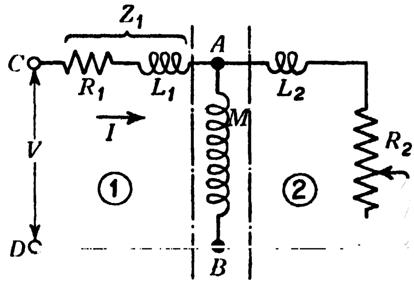


Fig. 16.

Reactance $X_1 = \omega L_1 = 2$ ohms : Resistance $R_1 = 1$ ohm
Reactance $X_2 = \omega L_2 = 5$ ohms : Resistance R_2 is a variable quantity
Mutual Reactance $X_M = \omega M = 50$ ohms.

STAGE I.—The impedance vector of the connection of the reactance of X_2 ohms in series with the resistance of R_2 ohms is

$$\mathfrak{Z}_2 = R_2 + jX_2 = R_2 + j5 = Z_2 e^{j\phi} \text{ ohms,}$$

and this impedance is represented graphically in Fig. 17 by the vertical line AB so that the extremity of any impedance vector such as OP will lie on this vertical line, whatever the value of the variable resistance R_2 may be. If this straight line AB is inverted with respect to the origin O, the admittance vector will be obtained, that is, $\mathfrak{Y}_2 = \frac{1}{\mathfrak{Z}_2}$, and this is defined by the circle C_0, of which the data are as follows, viz. :

Admittance Circle C_0 (*Fig.* 17).

Diameter : $OH = \frac{1}{5} = 0{\cdot}20$ siemens : Radius : $\rho_0 = 0{\cdot}10$ siemens.

Co-ordinates of the Centre $M_0 \begin{cases} \mu_0 = 0{\cdot}10 \text{ siemens.} \\ \nu_0 = 0. \end{cases}$

STAGE II.—The next step is to combine the admittance of the coupling branch AB (Fig. 16) with the admittance circle C_0. The admittance

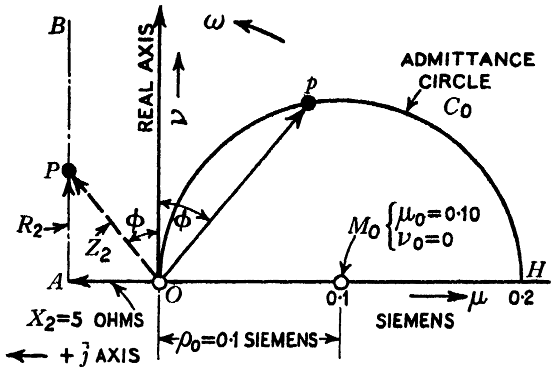


Fig. 17.

vector of the mutual inductance of M henry is $\mathfrak{Y}_M = -j\dfrac{1}{\omega M}$, so that the total admittance of the inductance M in parallel with the impedance $\mathfrak{Z}_2$ will be

$$\mathfrak{Y}_{\text{II}} = \mathfrak{Y}_2 + \mathfrak{Y}_M.$$

That is to say, the constant vector $\mathfrak{Y}_M$ is to be added to the admittance

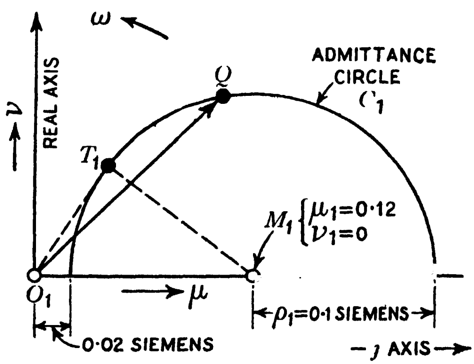


Fig. 18.

circle C_0, and this is most easily done by moving the origin O of Fig. 17 to the left-hand side by the amount $\dfrac{1}{\omega M} = 0{\cdot}02$ siemens, thus giving the new origin O_1 as shown in Fig. 18, in which the circle C_1 has the same radius as the circle C_0 in Fig. 17 but is referred to the new origin O_1. This circle C_1 is therefore defined by the following data :

Admittance Circle C_1 (*Fig.* 18).

Radius : $\rho_1 = 0{\cdot}10$ siemens.

Co-ordinates of the Centre $M_1 \begin{cases} \mu_1 = 0{\cdot}12 \text{ siemens.} \\ \nu_1 = 0. \end{cases}$

The length of the tangent to the circle C_1 drawn from the origin O_1 is given by

$$(O_1T_1)^2 = \mu_1{}^2 - \rho_1{}^2 = 0{\cdot}0044 \text{ (siemens)}^2.$$

STAGE III.—The admittance circle C_1 of Fig. 18 gives the graphical representation of the total admittance of the two parallel branches which are connected across the points A and B in the compound circuit

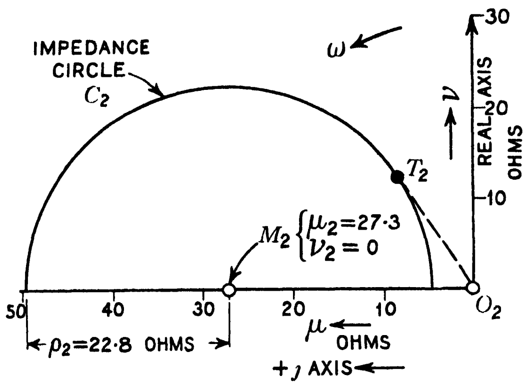


Fig. 19.

of Fig. 16. In order that this admittance may be added to the impedance

$$\mathfrak{Z}_1 = R_1 + jX_1$$

it is necessary to transform the admittance circle C_1 into the equivalent impedance circle C_2 of Fig. 19. This may be done at once by means of the expression (2), page 370, viz.,

$$\text{Radius : } \rho_2 = \frac{\rho_1}{(O_1T_1)^2} = 22{\cdot}8 \text{ ohms}$$

$$\text{Co-ordinates of the Centre } M_2 \begin{cases} \mu_2 = \dfrac{\mu_1}{(O_1T_1)^2} = 27{\cdot}3 \text{ ohms} \\[2ex] \nu_2 = \dfrac{\nu_1}{(O_1T_1)^2} = 0. \end{cases}$$

The circle C_2 of Fig. 19 is now the graphical representation of the total impedance of the two parallel branches which are connected across A and B in Fig. 16.

STAGE IV.—In order to find the total impedance of the compound circuit of Fig. 16 it is now necessary to add the constant impedance vector

$$\mathfrak{Z}_1 = R_1 + jX_1 = (1 + j2) \text{ ohms}$$

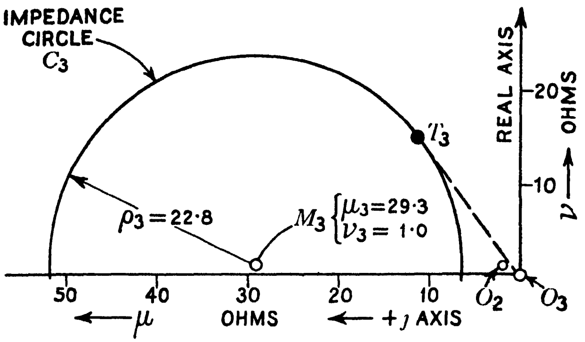


Fig. 20.

to the impedance given by the circle C_2 of Fig. 19. This addition is most easily obtained by moving the origin towards the right-hand side by an amount defined by the vector $\mathfrak{Z}_1$. This has been done in Fig. 20 in which

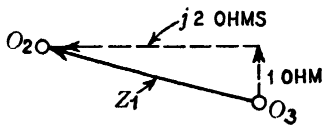


Fig. 20*a*.

C_3 is the same circle as C_2 but is now referred to the new origin O_3. An enlarged view of the vector $\mathfrak{Z}_1$ relating the two origins O_2 and O_3 is shown in Fig. 20*a*.

The circle C_3 referred to the origin O_3 is now defined by the following data :

$$\text{Radius : } \rho_3 = 22{\cdot}8 \text{ ohms}$$

$$\text{Co-ordinates of the Centre } M_3 \begin{cases} \mu_3 = 29{\cdot}3 \text{ ohms} \\ \nu_3 = \ \ 1{\cdot}0 \text{ ohm.} \end{cases}$$

Also, see Proposition 2, page 368,

$$(O_3T_3)^2 = (29{\cdot}3 + 22{\cdot}8) \times (29{\cdot}3 - 22{\cdot}8) = 339 \text{ (ohms)}^2.$$

STAGE V.—The total admittance of the compound circuit of Fig. 16, as measured between the input terminals C and D, is now obtained by inverting the impedance circle C_3 of Fig. 20 with respect to the origin O_3 by means of the expression (2). The result of this invertion is shown by the circle C_4 in Fig. 21 and this circle is then defined by the following data :

$$\text{Radius}: \rho_4 = \frac{\rho_3}{(O_3T_3)^2} = \frac{22\cdot8}{339} = 0\cdot066 \text{ siemens}$$

$$\text{Co-ordinates of the Centre } M_4 \begin{cases} \mu_4 = \dfrac{\mu_3}{(O_3T_3)^2} = \dfrac{29\cdot3}{339} = 0\cdot085 \quad ,, \\ \nu_4 = \dfrac{\nu_3}{(O_3T_3)^2} = \dfrac{1}{339} = 0\cdot0029 \quad ,, \end{cases}$$

If, then, the applied a.c. pressure at the input terminals CD of the compound circuit of Fig. 16 has the constant value V r.m.s. volts, the vector

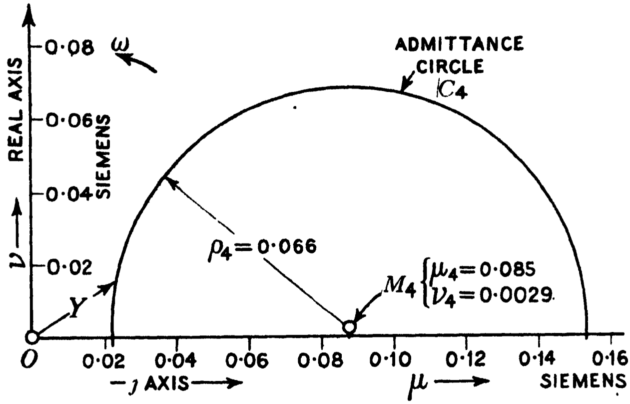


Fig. 21.

of the r.m.s. current which will then flow from the supply to this circuit will be given by

$$\mathfrak{J} = V.\mathfrak{Y},$$

where $\mathfrak{Y}$ is the admittance as defined by the circle C_4 of Fig. 21.

Distribution of Power in the Compound Circuit of Fig. 16

It is of interest and importance to consider the distribution of power between the resistances R_1 and R_2 respectively, when an alternating current pressure of constant r.m.s. value V volts is connected across the input terminals CD of the compound circuit shown in Fig. 16.

Referring to Fig. 22, the current circle MAF is shown for the applied p.d. of V volts, and from what has been said when deriving the admittance circle for the numerical example on pages 378–382, it will be understood that the current circle MAF of Fig. 22 is the same circle as C_4 of Fig. 21, but read to a scale of current in amperes instead of a scale of admittance in siemens.

Let x and y define the co-ordinates of any point A on the current circle of Fig. 22 and x_0, y_0 the co-ordinates of the centre M, then

$$(x - x_0)^2 + (y - y_0)^2 = \rho^2 \quad . \quad . \quad . \quad (10)$$

where ρ is the radius of the circle, that is,

$$x^2 + y^2 - 2x.x_0 - 2y.y_0 = \rho^2 - (x_0^2 + y_0^2 = -K^2 \quad . \quad (11)$$

where K^2 is a constant.

But the current I r.m.s. amperes which corresponds to the point A is given by

$$I^2 = x^2 + y^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The total power supplied to the circuit will be

$$W = VI \cos \phi = Vy \quad . \quad . \quad . \quad . \quad (13)$$

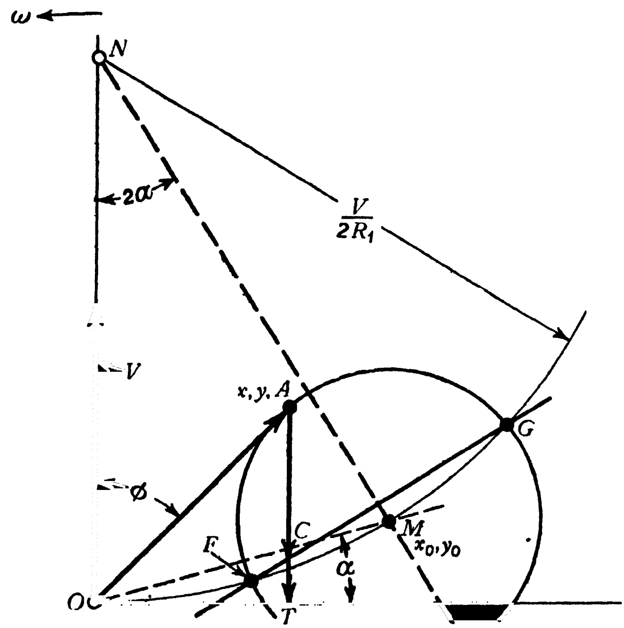


Fig. 22.

the power consumed by the resistance R_1 will be

$$W_1 = I^2R_1 = (2x.x_0 + 2y.y_0 - K^2)R_1 \quad . \quad . \quad (14)$$

and the power consumed by the resistance R_2 will be

$$W_2 = VI \cos \phi - I^2R_1$$

that is,

$$W_2 = VI \cos \phi - (2x.x_0 + 2y.y_0 - K^2)R_1 \quad . \quad . \quad (15)$$

Substituting y for $I \cos \phi$ in equation (15) gives

$$W_2 = V.y - (2x.x_0 + 2y.y_0 - K^2)R_1 . \quad . \quad . \quad . \quad (16)$$

that is

$$W_2 = (V - 2y_0R_1)\left[y - \left\{\frac{2x.x_0R_1}{V_0 - 2y_0R_1} - \frac{K^2R_1}{V - 2y_0R_1}\right\}\right] \quad . \quad (17)$$

This equation may now be written

$$W_2 = (V - 2y_0R_1)(y - y_1) \quad . \quad . \quad . \quad (18)$$

where

$$y_1 = \frac{x.x_0}{\frac{V}{2R_1} - y_0} - \frac{\frac{K^2}{2}}{\frac{V}{2R_1} - y_0} \quad . \quad . \quad . \quad (19)$$

The equation (19) represents a straight line and is shown in Fig. 22 by the line FG. Dropping the suffix in y_1, and re-writing the equation (19) in the form

$$y\left(\frac{V}{R_1} - 2y_0\right) - 2x.x_0 + K^2 = 0 \quad . \quad . \quad . \quad (20)$$

then, subtracting equation (20) from equation (11), gives

$$x^2 + y^2 - \frac{V}{R_1}y = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

that is

$$x^2 + \left(y - \frac{V}{2R_1}\right)^2 = \left(\frac{V}{2R_1}\right)^2 \quad . \quad . \quad . \quad (22)$$

The equation (22) represents a circle which is defined by the following data :

$$\left.\begin{aligned} \text{Radius} &= \frac{V}{2R_1} \\ \text{Co-ordinates of the Centre } N &\begin{cases} x_0 = 0 \\ y_0 = \dfrac{V}{2R_1} \end{cases} \end{aligned}\right\} \quad . \quad . \quad (23)$$

That is to say, the centre of the circle will lie at N on the ordinate axis (Fig. 22), at a height equal to the radius $\frac{V}{2R_1}$ above the origin O. The circle will, therefore, pass through the origin O and it is easily seen that it will also pass through the centre M of the current circle. The two circles will intersect at the points F and G.

Since the equation (22) has been derived from the straight line equation (20) and the equation (11) for the current circle, it follows that the straight line defined by the equation (20) must pass through the points of intersection of the two circles (equations (11) and (22)), that is, through the points F and G as shown in Fig. 22.

If, then, for any current vector OA, the vertical line AT be drawn to the abscissa axis, the total power supplied to the compound circuit of Fig. 18 will be given by

$$W = VI \cos \phi = V \times (AT) = V.y \text{ watts} \quad . \quad . \quad (24)$$

the power absorbed by the resistance R_2 will be

$$W_2 = (V - 2y_0R_1)(y - y_1) = (V - 2y_0R_1)(AC) \quad . \quad . \quad (25)$$

so that the power absorbed by the resistance R_1 will be

$$W_1 = W - W_2 = V \times (AT) - (V - 2y_0R_1)AC \quad . \quad . \quad (26)$$

that is

$$W_1 = V \times [(CT) + 2y_0R_1(AC)] \quad . \quad . \quad . \quad (27)$$

Two special cases should be noted as follows:

(*i*) When the resistance R_2 (Fig. 16) is open-circuited, the power supplied to R_2 will be zero and consequently the vector of the current taken from the supply will be defined by the point F, which may therefore be termed the "open-circuit point".

(*ii*) When the resistance R_2 is short-circuited, the power supplied to R_2 will again be zero, and consequently the vector of the current taken from the supply will then be defined by the point G which may therefore be termed the "short-circuit point".

The compound circuit shown in Fig. 16 *and the corresponding diagram of Fig.* 22 *represent respectively the equivalent circuit and the current circle diagram for a three-phase induction motor.*

Flywheel Equaliser for a Periodically Varying Load

As a preliminary step to the investigation of this problem it will be helpful to derive some dynamical relationships.

(*i*) LINEAR MOTION.—Suppose a constant force of F dynes is acting on a mass of m gm. which is initially at rest, and let x cm. be the distance

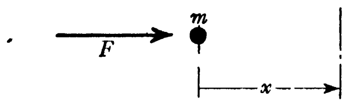


Fig. 23.

through which the mass moves in t seconds in the direction in which F acts, as shown in Fig. 23. The acceleration will then be given by

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} \text{ dynes} \quad . \quad . \quad . \quad (28)$$

where v cm. per second is the velocity at the time t. In practice, it is usually more convenient to measure the mass M in kilograms, in which case

$$F = M \times 10^3\frac{dv}{dt} \text{ dynes} \quad . \quad . \quad . \quad . \quad (29)$$

The velocity v which the mass acquires in the time t will then be given by

$$v = \int_0^t \frac{F}{M \times 10^3}dt = \frac{F}{M \times 10^3}t \text{ : cm. per second.} \quad . \quad (30)$$

that is to say, the velocity is proportional to the time, as is shown in

Fig. 24, and the distance through which the mass will have moved in the time t will then be

$$x = \int_0^t v\,dt = \frac{1}{2}\frac{F}{M \times 10^3}t^2 = \frac{1}{2}vt \text{ cm.} \quad . \quad . \quad . \quad (31)$$

that is, the distance x will be given by the shaded area of the graph in Fig. 24.

The work which must be expended in order that the mass shall attain the velocity v will be

$$u = F.x = \tfrac{1}{2}Fvt = \tfrac{1}{2}M \times 10^3 v^2 \text{ ergs} \quad . \quad . \quad . \quad (32)$$

and this is the *kinetic energy* which is acquired by the mass. If the velocity is V metres per second, the kinetic energy will be given by

$$U = \tfrac{1}{2}M \times 10^3 V^2 \times 10^4 = \tfrac{1}{2}MV^2\, 10^7 \text{ ergs} = \tfrac{1}{2}MV^2 \text{ joules} \quad . \quad (33)$$

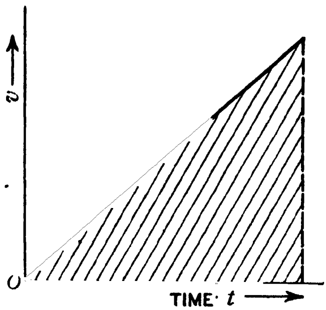


Fig. 24.

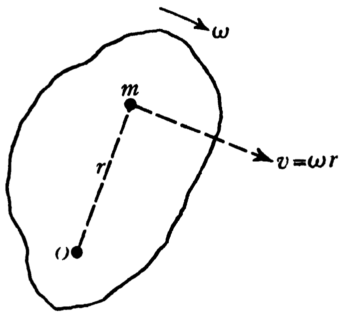


Fig. 25.

The power expended, that is, the rate of doing work will be

$$W = F.v \text{ ergs per sec.} = \frac{F.v}{981} \text{ gm.-cm. per sec.}$$

$$= \frac{F.v}{9{\cdot}81 \times 10^7} \text{ kg.-m. per sec.} \quad . \quad (34)$$

since the weight of 1 gm. is 981 dynes.

Also

$$W = F.v \text{ ergs per sec.} = \frac{F.v}{10^7} \text{ joules per sec.} \quad . \quad . \quad (35)$$

so that

$$\left.\begin{array}{ll} 9{\cdot}81 \text{ joules} & = 1 \text{ kg.-m.} \\ 9{\cdot}81 \text{ watts} & = 1 \text{ kg.-m. per sec.} \\ 1 \text{ joule} & = 0{\cdot}737 \text{ ft.-lb.} \\ 1 \text{ kg.-m.} & = 7{\cdot}22 \text{ ft.-lb.} \end{array}\right\} \quad . \quad . \quad . \quad (36)$$

also

(*ii*) Rotary Motion.—If a body is rotating at a uniform speed of n revs. per sec., that is, with a uniform angular velocity $\omega = 2\pi.n$ radians

per second about an axis O as shown in Fig. 25, the kinetic energy of a particle of mass m gm. at a distance r cm. from the axis will be given by the equation (32), viz.,

$$u = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\, m(\omega r)^2 \text{ ergs} \qquad (37)$$

so that the kinetic energy for the whole mass will be

$$U = \Sigma\tfrac{1}{2}mv^2 = \tfrac{1}{2}\omega^2\Sigma mr^2 \text{ ergs} \qquad (38)$$

that is

$$U = \frac{1}{2}J\omega^2 10^7 \text{ ergs} = \frac{1}{2}J\omega^2 \text{ joules} = \frac{J\omega^2}{2 \times 9{\cdot}81} \text{ kg.–m.} \qquad (39)$$

where J is the moment of inertia of the rotating mass about the axis O measured in kg.-(metre)^2 units. For example, a solid disc flywheel of mass M kg. and radius R metres has a moment of inertia

$$J = \tfrac{1}{2}M.R^2 \text{ kg.-m}^2\text{. units.}$$

If a torque of τ kg.-m. is applied to the mass of which the angular velocity is ω radians per second, the power expended will be

$$W = \tau.\omega = \frac{d}{dt}U = \frac{d}{dt}\left(\frac{J\omega^2}{2 \times 9{\cdot}81}\right) \text{ kg.-m. per sec.} \qquad (40)$$

so that the torque is given by

$$\tau = \frac{J}{9{\cdot}81}\frac{d\omega}{dt} \text{ kg.-m.} \qquad (41)$$

and the power extended may then be expressed as

$$W = \tau.\omega = \frac{J.\omega\, d\omega}{9{\cdot}81\ dt} \text{ kg.-m. per sec.} \qquad (42)$$

and the angular acceleration is therefore given by

$$\frac{d\omega}{dt} = 9{\cdot}81\frac{W.\omega}{J.\omega^2} = 9{\cdot}81\frac{W.\omega}{2(K.E.)} \qquad (43)$$

where $(K.E.)$ is the kinetic energy of the rotating mass in kilogram-metres and W the power in kilogram-metres per second. If the power is expressed in *watts*, then

$$\frac{d\omega}{dt} = \frac{W.\omega}{2(K.E.)}. \qquad (44)$$

(*iii*) Mechanical Time Constant.—A further useful relationship is obtained as follows: Suppose the normal full-load torque of a motor is τ_0 kg.-m. and the normal full-load speed is ω_0 radians per second. If the machine starts from rest without load and is allowed to accelerate under the constant torque τ_0 without friction or other losses, then the time required to reach the normal speed ω_0, is termed the " mechanical

time constant" of the machine. The magnitude of this time constant may then be expressed as follows. From the expression (41), the acceleration is given by

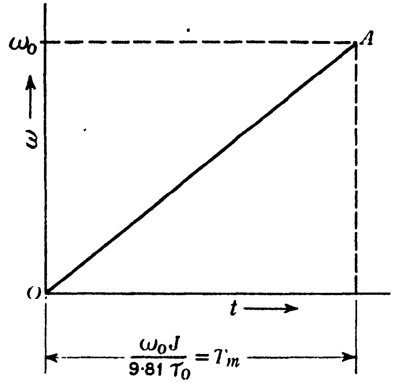


Fig. 26.

$$\tau_0 = \frac{J}{9{\cdot}81}\frac{d\omega}{dt} \text{ kg.-m.}$$

that is

$$\int_0^{\omega_0} d\omega = \frac{9{\cdot}81\tau_0}{J}\int_0^{T_m} dt \quad . \quad (45)$$

so that the mechanical time constant T_m is given by (see Fig. 26),

$$\omega_0 = 9{\cdot}81\frac{\tau_0}{J}T_m$$

that is, $$T_m = \frac{J\omega_0}{9{\cdot}81\tau_0} \text{ seconds}. \quad (46)$$

Example of Fly-Wheel Equaliser System for Cyclic Load Variations

In Fig. 27 is shown a typical load cycle for the winding installation of a colliery, the components of this cycle being as follows:

Time of the acceleration period OA	26·5 secs.
Time of the full-speed run BC	26·8 ,,
Time of the retardation period DE	10 ,,
Stationary period EF.	25·2 ,,

the total time for one complete load cycle thus being 88·5 seconds.

The load cycle of Fig. 27 will now be used as an example of the method for calculating the weight of the flywheel which will be necessary to screen the supply mains from the heavy peak loads. In the first place it is necessary to obtain the mean load, that is, to find the mean ordinate of the load cycle, and this is easily found by measuring the net positive area of the load diagram in (horse-power) × (seconds) units and then to divide this quantity by the time of one cycle, viz. 88·5 seconds. The value of the mean load so obtained is found to be 350 horse-power, and this is shown in the diagram by the horizontal line MN.

The shaded area in Fig. 27 then gives the peak load which is to be accounted for by the flywheel. The magnitude of this area is found to be 16,500 horse-power-seconds, that is, $1{\cdot}25 \times 10^6$ kg.-m. The stored kinetic energy of the flywheel will be

$$K.E. = \tfrac{1}{2}J\omega^2 \text{ kg.-m.} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where J kg.-m^2. units is the moment of inertia and $\omega = 2\pi n$ radians per second is the angular velocity. Now the energy which is represented by the shaded area in the load diagram of Fig. 27, that is, ΔU, is the demand which is in excess of the mean load and is the amount of the

stored kinetic energy of the flywheel which must be released in order to supply this peak load. If $\omega_1 = 2\pi n_1$ is the maximum speed at which the flywheel runs during the cycle that is, the speed corresponding to the point S on the load diagram of Fig. 27, and if $\omega_2 = 2\pi n_2$ is the minimum speed, that is, the speed corresponding to the point C on the

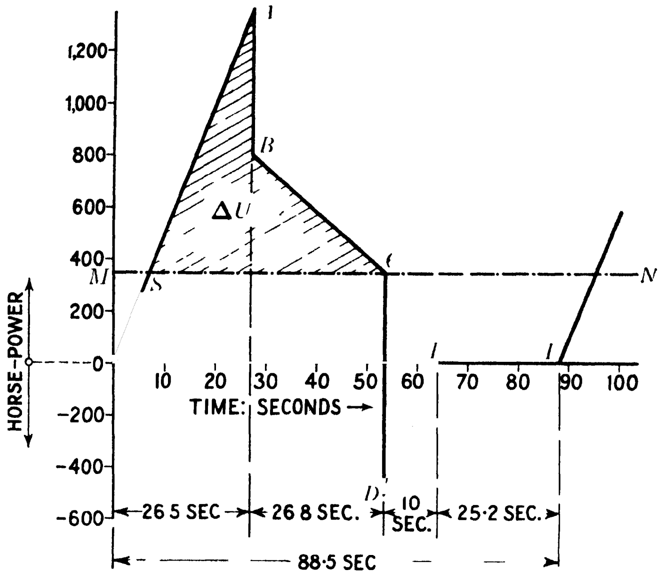


Fig. 27.

load diagram, then, equating the peak load energy ΔU and the released kinetic energy of the flywheel gives the equation

$$\Delta U = \frac{1}{2}\frac{J}{9{\cdot}81}\{\omega_1{}^2 - \omega_2{}^2\} \text{ kg.-m.} \quad . \quad . \quad . \quad (48)$$

so that $$J = \frac{2 \times 9{\cdot}81 \times \Delta U}{(2\pi)^2(n_1{}^2 - n_2{}^2)} = \frac{\Delta U}{2(n_1{}^2 - n_2{}^2)} \text{ kg.-m.}^2 \text{ units,}$$

noting that 9·81 is very approximately equal to π^2. Hence

$$J = \frac{\Delta U}{2(n_1 - n_2)(n_1 + n_2)} = \frac{\Delta U}{4n_m(n_1 - n_2)} = \frac{\Delta U}{4g.n_m{}^2} \quad . \quad (49)$$

where $g = \frac{n_1 - n_2}{n_m}$ and is the (fractional) percentage speed variation, and $n_m = \frac{1}{2}(n_1 + n_2)$ revs. per second, is the mean speed at which the flywheel runs, viz. $\frac{725}{60} = 12{\cdot}8$ revs. per second in this particular case.

If the drop in speed due to the peak load is $g = 6$ per cent. $= 0{\cdot}06$, then

$$J = \frac{\Delta U}{4 \times 12{\cdot}8^2 \times 0{\cdot}06} = \frac{1{\cdot}25 \times 10^6}{146 \times 0{\cdot}24} \text{ kg.-m.}^2 \text{ units,}$$

so that the requisite moment of inertia of the flywheel is

$$J = 3{\cdot}56 \times 10^4 \text{ kg.-m.}^2 \text{ units.}$$

If the flywheel is in the form of a solid disc D metres in diameter, the moment of inertia will be

$$J = M\left(\frac{D}{2\sqrt{2}}\right)^2 \text{ kg.-m.}^2 \text{ units} \quad . \quad . \quad . \quad (50)$$

where M kg. is the mass of the disc, that is

$$M = \frac{\pi}{4}(D \times 100)^2 \times t \times \sigma \times 10^{-3} \text{ kg.} \quad . \quad . \quad (51)$$

in which t cm. is the thickness of the disc and $\sigma = 7{\cdot}9$ gm. per c.cm. is the density of the steel. Since the mean speed of the set is 725 revs. per minute and assuming the maximum peripheral speed to be $v_m = 120$ m. per second, then

$$v_m = 120 = \pi D.n = \pi D \times 12{\cdot}8$$

and $$D = \frac{120}{\pi 12{\cdot}8} = 3{\cdot}1 \text{ metres,}$$

so that $$M = J\left(\frac{2\sqrt{2}}{3{\cdot}1}\right)^2 = 3{\cdot}56 \times 10^{-4}(0{\cdot}91)^2 = 30 \text{ metric-tons,}$$

and, by substitution in equation (51), the thickness of the disc will be

$$t = \frac{30 \times 10^3 \times 10^3}{\frac{\pi}{4} \times 310^2 \times 7{\cdot}9} = \frac{30 \times 10^6}{59{\cdot}5 \times 10^4} = 50{\cdot}5 \text{ cm.}$$

The Measurement of the Moment of Inertia of a Cylindrical Rotor

Several methods * are available for the measurement of the moment of inertia of a composite cylindrical structure and two of these will be considered briefly here.

(*i*) THE ROTOR IS ARRANGED WITH ITS SHAFT SUPPORTED ON HORIZONTAL KNIFE EDGES.—Fig. 28 shows diagrammatically the rotor R supported on a pair of horizontal knife-edges, one of which is shown by AB. Rigidly fixed to the shaft is a relatively small mass of m gm., the distance of the mass from the centre of the shaft being a cm. When this system is at rest, the mass m will, of course, set vertically under the axis of the shaft. If the mass is now displaced by a small angle θ from this equili-

* See *Engineering*, Feb. 24, 1928, p. 213.

brium position, and is then left free to move, the system will oscillate and the equation of motion can be obtained as follows:

The potential energy of the system will be

$$m.a - m.a \cos \theta \text{ gm.-cm.}$$

that is $$m.g.a(1 - \cos \theta) \text{ dyne-cm. or ergs} \qquad (52)$$

where $g = 981$ cm./sec.^{2} and is the acceleration due to gravity.

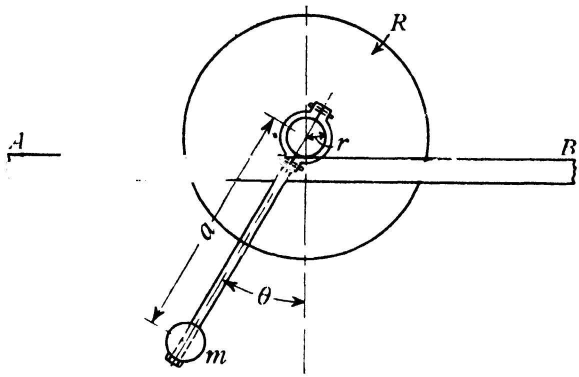


Fig. 28.

The kinetic energy of rotation of the system will be

$$\frac{1}{2}(J + ma^2)\left(\frac{d\theta}{dt}\right)^2 \text{ ergs} \qquad (53)$$

where J gm.-cm.^{2} units is the moment of inertia of the rotor about the shaft axis. It is to be observed that, if the radius of the shaft is small in comparison with the radius of the rotor, the kinetic energy of *translation* may be assumed to be negligibly small.

Since the total energy is constant it follows that

$$m.g.a(1 - \cos \theta) + \frac{1}{2}(J + m.a^2)\left(\frac{d\theta}{dt}\right)^2 = \text{constant} \qquad (54)$$

Differentiating this equation with respect to θ gives

$$m.g.a \sin \theta + (J + m.a^2)\frac{d^2\theta}{dt^2} = 0 \qquad (55)$$

When the angular displacement is kept small, then $\sin \theta \simeq \theta$, so that the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\frac{m.a.g}{J + m.a^2}\theta \qquad (56)$$

This is the standard form of the equation for simple harmonic motion and the time of one complete oscillation of the system will then be

$$\tau = 2\pi\sqrt{\frac{J + m.a^2}{m.a.g}} \text{ sec.} \quad . \quad . \quad . \quad . \quad (57)$$

or, if the product $m.a^2$ is small in comparison with J, then

$$\tau = 2\pi\sqrt{\frac{J}{m.a.g}} \text{ sec.} \quad . \quad . \quad . \quad . \quad (58)$$

from which it follows that

$$J = \frac{m.a.g}{4\pi^2}\tau^2 \text{ gm.-cm.}^2 \text{ units}$$

for m in grams and a in centimetres, so that

$$J = \tfrac{1}{4}Ma\tau^2 \text{ kg.-m.}^2 \text{ units} \quad . \quad . \quad . \quad (59)$$

where M is in kilograms, a in metres, and $9{\cdot}81 \simeq \pi^2$.

Example.—It was required to measure the moment of inertia of the rotor of a 20-b.h.p., 3-phase, 4-pole, 50-frequency induction motor. The machine was fitted with ball-bearings, and consequently the moment of inertia of the rotor could be measured by means of this oscillation method *in situ*, without having to remove it and support it on knife-edges as shown in Fig. 28.

The auxiliary pendulum which was secured to the shaft for the purpose of this measurement, comprised a wooden bath of effective length, 54 cm., that is, $a = 0{\cdot}54$ m., to the end of which was fixed a mass $M = 4{\cdot}5$ kg. The system was displaced by a small angle θ from its position of equilibrium and the mean of several measurements of the time of one oscillation was found to be $\tau = 1{\cdot}81$ seconds. The moment of inertia was thus found to be

$$J = \tfrac{1}{4}M.a.\tau^2 = \tfrac{1}{4} \times 4{\cdot}5 \times 0{\cdot}54 \times 1{\cdot}81^2 = 2{\cdot}0 \text{ kg.-m.}^2 \text{ units.}$$

(*ii*) The Rotor is Suspended by means of Two Ropes and Oscillated under Gravity Control.—The mode of suspension in this case is shown in Fig. 31, it being observed that the two hempen ropes A exert no torsional control on the movement of the rotor. Suppose, then, that the rotor is twisted through a small angle θ in the horizontal plane and let ϕ be the corresponding angle through which each of the suspension ropes will be turned. Then it follows that

$$\phi.l = a.\theta \quad \text{or} \quad \phi = \frac{l}{a}\theta \quad . \quad . \quad . \quad . \quad (60)$$

If l is large in comparison with a the angle ϕ will be small as compared with θ. When the rotor is turned through the angle θ in the horizontal plane it will rise through a vertical distance,

$$rs = l(1 - \cos\phi) \text{ cm.,}$$

so that the corresponding increase of potential energy due to twisting the rotor through the angle θ will then be

$$m.l(1 - \cos\phi) \text{ gm.-cm. or } m.g.l(1 - \cos\phi) \text{ ergs,}$$

where m gm. is the mass of the rotor and $g = 981$ as before.

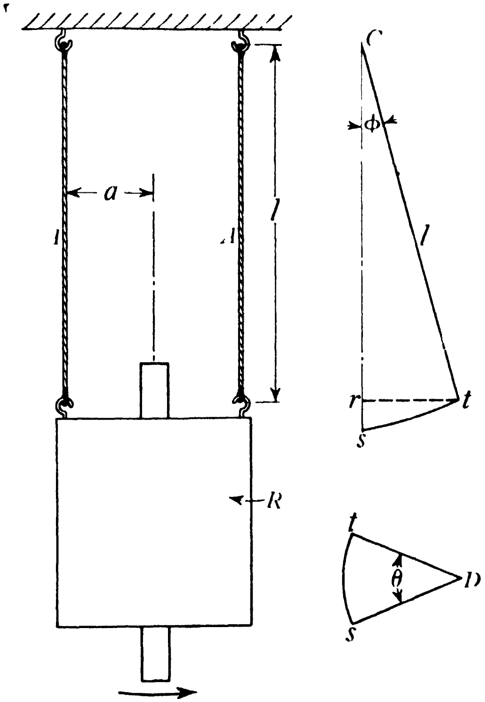


Fig. 29.

The kinetic energy of rotation is then

$$\tfrac{1}{2}J\left(\frac{d\theta}{dt}\right)^2 \text{ ergs,}$$

the kinetic energy due to the motion of the rotor in a vertical direction being negligible. The equation of motion will then be

$$m.g.l(1 - \cos\phi) + \tfrac{1}{2}J\left(\frac{d\theta}{dt}\right)^2 = \text{a constant,}$$

that is

$$m.g.l\left(1 - \cos\frac{a}{l}\theta\right) + \tfrac{1}{2}J\left(\frac{d\theta}{dt}\right)^2 = \text{a constant.}$$

Differentiating with respect to θ gives

$$\frac{a}{l}m.g.l\sin\frac{a}{l}\theta + J\frac{d^2\theta}{dt^2} = 0,$$

or noting that, θ is small and $\frac{a}{l}\theta$ is very much smaller, so that $\sin\frac{a}{l}\theta \simeq \frac{a}{l}\theta$, and consequently the equation of motion becomes

$$\left(\frac{a}{l}\right)^2 mgl\theta + J\frac{d^2\theta}{dt^2} = 0,$$

that is

$$\frac{d^2\theta}{dt^2} = -\frac{m.g.a^2}{J.l}\theta$$

from which

$$\tau = 2\pi\sqrt{\frac{J.l}{m.g.a^2}} \text{ sec.}$$

and

$$J = \frac{m.g.a^2}{4\pi^2 l}\tau^2 \text{ gm.-cm.}^2 \text{ units}$$

for m in grams: a and l in centimetres;

or

$$J = \frac{1}{4}\frac{Ma^2}{l}\tau^2 \text{ kg.-m.}^2 \text{ units}$$

for M in kilograms, l and a each in metres.

Example.—It was required to measure the moment of inertia of the rotor of a 25-b.h.p., 3-phase, 4-pole induction motor. As this machine was not fitted with roller- or ball bearings it was necessary to remove the rotor for the purpose of measuring its moment of inertia. The rotor was accordingly supported by a bi filar suspension as shown in Fig. 29, the data being as follows:

Free length of suspension cords . . .	$l = 2{\cdot}78$ m.
Distance between the two suspension cords .	$2a = 0{\cdot}18$ m.
Weight of rotor	$M = 107{\cdot}4$ kg.
Mean time of one complete oscillation . .	$\tau = 3{\cdot}6$ sec.

Hence, the moment of inertia was

$$J = \frac{1}{4}\frac{M.a^2.\tau^2}{l} = \frac{107{\cdot}4 \times 0{\cdot}09^2 \times 3{\cdot}6^2}{4 \times 2{\cdot}78}$$
$$= 1{\cdot}04 \text{ kg.-m.}^2 \text{ units.}$$

The Acceleration of a Direct Current Shunt Motor

In Fig. 30 is shown diagrammatically a shunt direct-current motor with a liquid (e.g. an aqueous soda solution) starting resistance. As the movable electrode is lowered into the liquid the resistance in series with the motor armature becomes gradually reduced until eventually all the resistance is cut out and the motor armature is connected directly across the supply mains.

The torque-speed characteristic for such a motor and starting resistance

is shown by the curve ABC in Fig. 31, of which the region B corresponds to the condition that all the starting resistance has been cut out and the remainder of the curve BC corresponds to the condition that the motor is operating under normal running conditions, that is to say, the speed is nearly constant for a wide range of loads, this being the characteristic feature of a direct-current type of motor.

Suppose, now, that this motor is being used to drive a centrifugal pump of which the starting torque is relatively large and the resistance-load torque is approximately proportional to the square of the speed, the total resisting torque being shown by the curve PQ in Fig. 31. The acceleration and speed of the motor during the starting period can be found graphically by means of the construction which is illustrated in

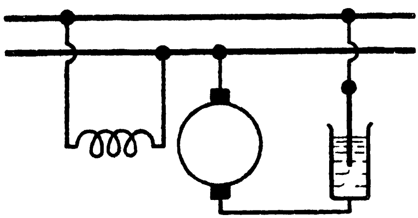


Fig. 30.

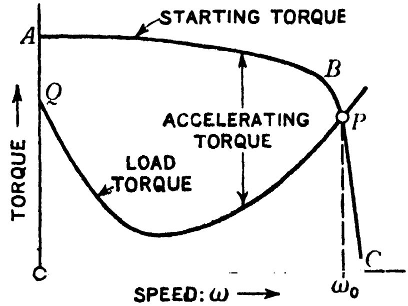


Fig. 31.

Fig. 32. For this purpose it is convenient to draw the torque-speed characteristic so that the scale of "relative torque", that is, $\frac{\tau}{\tau_0}$, is shown on the abscissa axis and the scale of "relative speed", that is, $\frac{\omega}{\omega_0} = s$, is shown on the ordinate axis. The normal full-load torque is τ_0 kg.-m., and the normal full-load angular velocity is $\omega_0 = 2\pi n_0$ radians per second.

At any speed ω during the starting period, the accelerating torque will be given by the intercept $\Delta = \frac{\tau}{\tau_0}$ in Fig. 32a. Since τ_0 kg.-m. is the normal full-load torque when the motor is running at the normal full-load speed of ω_0 radians per second, then the normal full-load power will be

$$W_0 = 9{\cdot}81\tau_0\omega_0 \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

But from the results obtained on page 387, equation (41),

$$\tau = \frac{J}{9{\cdot}81}\frac{d\omega}{dt} \text{ kg.-m.}$$

where τ kg.-m. is the accelerating torque, and J kg.-m.^{2} units is the moment of inertia of the rotating system, so that

$$\frac{d\omega}{dt} = 9{\cdot}81\frac{\tau}{J},$$

that is

$$d\omega = \left(\frac{9{\cdot}81\tau}{J}\right)dt \quad . \quad . \quad . \quad . \quad (62)$$

The moment of inertia J may be expressed in terms of the " mechanical time constant " T_m as already explained on page 388, that is

$$J = \frac{9{\cdot}81T_m\tau_0}{\omega_0},$$

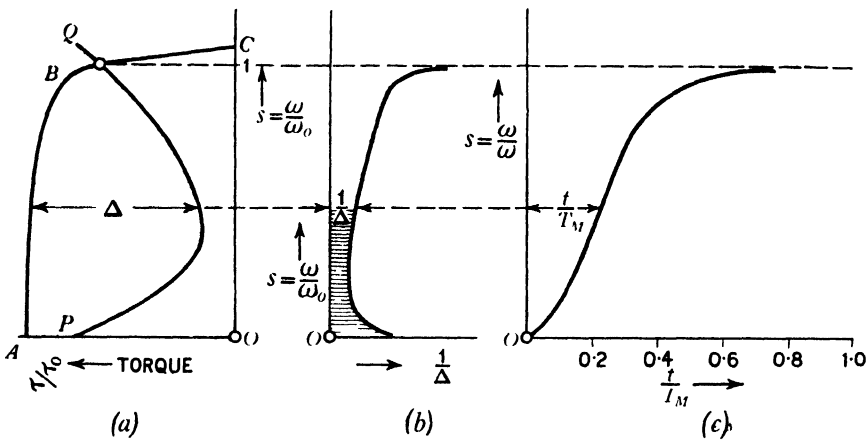


Fig. 32.

so that equation (62) may now be written $d\omega = \frac{\omega_0}{T_m}\frac{\tau}{\tau_0}dt$, that is

$$d\left(\frac{\omega}{\omega_0}\right) = \Delta d\left(\frac{t}{T_m}\right) \quad . \quad . \quad . \quad . \quad (63)$$

where $$\Delta = \frac{\tau}{\tau_0} = \frac{\text{accelerating torque}}{\text{normal full-load torque}},$$

or, writing $s = \frac{\omega}{\omega_0}$, then

$$\frac{t}{T_m} = \int\frac{ds}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

Referring now to the diagrams of Fig. 32. On the left-hand side (Fig. 32*a*) is shown the accelerating torque $\Delta = \frac{\tau}{\tau_0}$ as a function of the

relative speed $s = \frac{\omega}{\omega_0}$, and it will be seen that the shaded area shown in Fig. 32*b* will be given by the integral $\int_0^s \frac{ds}{1}$ and from the equation (64) this shaded area will be equal to $\frac{t}{T_m}$. In Fig. 32*c* the value of the shaded area is shown as a function of the ratio $\frac{t}{T_m}$. That is to say, the diagram of Fig. 32*c* gives the required time for the motor to run up to any speed s and eventually to the full-load speed.

The Magnetic Field due to a Symmetrical Three-Phase Current flowing in a Three-Phase Winding

In Fig. 33 is shown a star-connected three-phase winding 1 1′ : 2 2′ : 3 3′ : as arranged, for example, on the laminated stator core

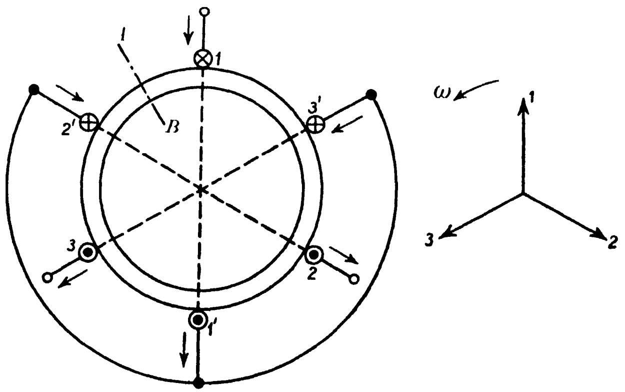


Fig. 33.

of an induction motor, and which is supplied with three-phase current defined by the expressions

$$i_1 = I_m \sin \omega t : \ i_2 = I_m \sin\left(\omega t - \frac{2\pi}{3}\right) : \ i_3 = I_m \sin\left(\omega t - \frac{4\pi}{3}\right).$$

In Fig. 33 is also shown the time vector diagram for the moment $t = \frac{\pi}{2\omega}$, so that at this moment the current in phase winding 1 1′ has its maximum positive value I_m, whilst the currents in the phase windings 2 2′ and 3 3′ are each negative and of magnitude equal to $\frac{1}{2}I_m$. In Fig. 33 these currents are shown diagrammatically in the respective windings, and it is assumed that when a current flows into the winding at the beginning end and out of the winding at the star-connected end, the direction of the current is taken to be positive.

The development of the winding of Fig. 33 is shown in Fig. 34 at the top of the series of diagrams, the positive direction of the current for

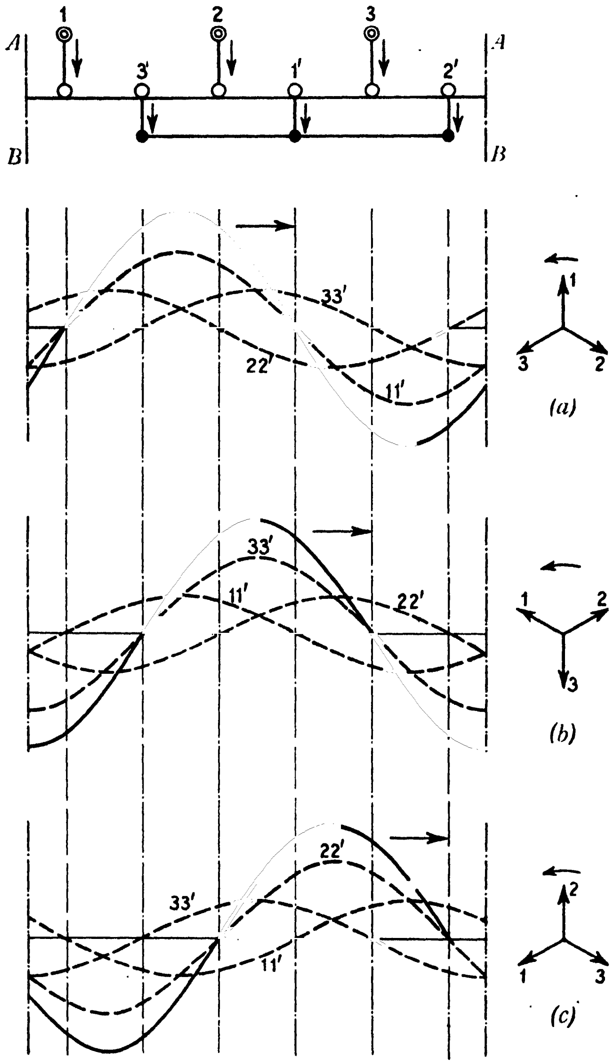


Fig. 34.

each winding being indicated by an arrow-head. The current in each phase-winding will give rise to a corresponding magnetic field across the air-gap of Fig. 34*a*, and it is assumed that this magnetic flux density for

each phase current is sinusoidally distributed in the air-gap. In Fig. 34*a* the conditions are shown for which the three-phase currents are defined by the adjacent time vector diagram, that is to say, the current in phase 1 1′ has its maximum positive value I_m and the currents in phases 2 2′ and 3 3′ are each negative and of magnitude $\frac{1}{2}I_m$. The sinusoidally distributed magnetic flux density due to the current in phase 1 1′, therefore, has its maximum positive value at this moment, and the magnetic fields due to the currents in the windings 2 2′ and 3 3′ are negative, and of peak value each equal to one-half that of the field due to the current in phase 1 1′. The three waves of magnetic field shown in Fig. 34*a* have been drawn in accordance with these conditions and are shown by the respective broken-line curves. The algebraic sum of corresponding ordinates of these three sine waves is shown by the full-line sine wave, and it is to be observed that the peak value of this resultant sine wave is 1·5 *times the peak value of the wave due to the current in phase* 1 1′, *that is,* 1·5 *times the peak value of the wave of magnetic flux density due to that phase-winding in which the current is a maximum.* It is also to be observed that the position of this resultant sine wave is coincident with that of the wave due to the phase in which the current is a maximum, that is, in this case, coincident in position with the field due to the current in phase 1 1′. In Fig. 34*b* the conditions are shown for the moment one-sxith of a cycle later than that of Fig. 34*a*, that is, such that $\omega t = \frac{\pi}{2} + \frac{\pi}{3}$: $t = \frac{5\pi}{6\omega}$ and the time vector diagram of Fig. 34*b* is consequently 60° ahead of that for Fig. 34*a*. At this moment the current in phase 3 3′ has reached its maximum negative value I_m, whilst the currents in phases 1 1′ and 2 2′ are each positive and of magnitude $\frac{1}{2}I_m$. The corresponding diagram of the magnetic flux density distribution for the respective phase-currents, as shown in Fig. 34*b*, *now gives a resultant sinusoidal magnetic flux density distribution of which the peak value is the same as that of Fig.* 34*a*. In Fig. 34*b*, however, the resultant magnetic wave has moved in the direction to the right of the resultant magnetic field of Fig. 34*a* and is now coincident in position with the field due to phase 3 3′, that is, *again coincident with the field due to that phase in which the current has its maximum value.*

In Fig. 34*c* the conditions are shown for the moment t such that $\omega t = \frac{\pi}{2} + \frac{2\pi}{3}$, that is, $t = \frac{7\pi}{6\omega}$, and the corresponding vector diagram is one-sixth of a cycle in advance of that of Fig. 34*b*. It will again be observed *that the resultant magnetic field has the same value as in Figs.* 34*a and* 34*b*, *and has moved further to the right so that it is again coincident in position with the field due to the phase winding in which the current is a maximum*, viz. at this moment, the current in phase 2 2′.

Inspection of these diagrams, therefore, shows that a symmetrical three-phase current supplied to a three-phase winding such as that shown, in Fig. 34*a*, gives rise to a resultant sinusoidal wave of magnetic flux

density distribution which, at every moment, is of constant magnitude and which rotates in the air-gap at a speed of n revs. per second where $n = f$ for a two-pole machine. If the machine has p pairs of poles, it is easily seen that the speed at which the resultant magnetic field rotates in the air-gap will be defined by the relationship $n . p = f$ hz. Thus, for a two-pole machine supplied with three-phase current at a frequency $f = 50$ hz., the speed of rotation of the resultant magnetic field will be 50 revs. per second, that is, 3,000 revs. per minute ; for a six-pole machine the speed will be 1,000 revs. per minute, and so on.

It is also to be observed that, if a series of diagrams similar to those of Fig. 34 be drawn for the condition that any two of the supply mains connected to the terminals of the three-phase winding of Fig. 33 are interchanged, *the direction of rotation of the resultant sine wave of magnetic flux density is reversed.*

Analysis of an Unsymmetrical Three-Phase System into Two Symmetrical and Oppositely Rotating Three-Phase Systems and a Symmetrical Non-Rotating System *

Any unsymmetrical polyphase current or pressure system can be analysed into a set of symmetrical components as follows :

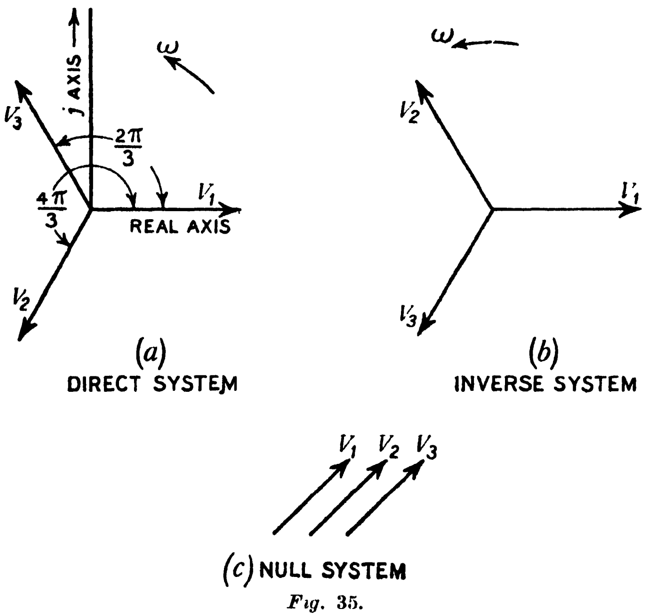


Fig. 35.

(*i*) A " direct " rotating symmetrical system of which the phase sequence is the same as that of the original unsymmetrical system.

* See also *Engineering*, May 10, 1940, p. 479.

(*ii*) An "inverse" rotating symmetrical system of which the phase sequence is opposite to that of the original unsymmetrical system.

(*iii*) A "null" system which is a non-rotating, that is, an alternating system.

Consider first a symmetrical three-phase pressure system rotating in the *counter-clockwise direction* as shown in Fig. 35*a*, in which the relationships are shown for the moment at which the vector $\mathfrak{B}_1$ is directed along the positive "real" axis. The vector $\mathfrak{B}_2$ is obtained by turning the vector $\mathfrak{B}_1$ through the angle $\frac{4\pi}{3}$ in the counter-clockwise direction, and the vector $\mathfrak{B}_3$ is obtained by turning the vector $\mathfrak{B}_1$ through the angle $\frac{2\pi}{3}$ also in the counter-clockwise direction, that is,

$$\mathfrak{B}_2 = \mathfrak{B}_1 e^{j\frac{4\pi}{3}} : \quad \mathfrak{B}_3 = \mathfrak{B}_1 e^{j\frac{2\pi}{3}}.$$

Denoting the vector quantity $e^{j\frac{2\pi}{3}}$ by the symbol a then the following relationships are easily obtained, viz.:

$$\left.\begin{aligned}
&a = e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2} : \quad a^2 = e^{j\frac{4\pi}{3}} = -\frac{1}{2} - j\frac{\sqrt{3}}{2}\\
&a^3 = e^{j2\pi} = 1 : \quad a^4 = e^{j\frac{8\pi}{3}} = a^3 \times a = a\\
&a^2 + a + 1 = 0 : \quad a + 1 = -a^2 : \quad \frac{1}{a} = a^2\\
&a - 1 = \sqrt{3}e^{j\frac{5\pi}{6}} = -j\sqrt{3}a^2 : \quad a^2 - 1 = j\sqrt{3}a
\end{aligned}\right\} \quad . \quad (65)$$

An inverse rotating symmetrical three-phase system, that is, one which rotates in the *clockwise direction*, is the same as one which rotates in the counter-clockwise direction but of which the phase numbers of two of its vectors have been interchanged as shown in Fig. 35*b*. This is easily seen by reference to the last paragraph in the preceding section (page 400), which states that the direction of rotation of the magnetic field due to a three-phase current in a three-phase winding will become reversed if any two of the leads from the mains to the terminals of the three-phase windings are interchanged.

A symmetrical "null" system is a non-rotating system such as is shown in Fig. 35*c*.

Now consider any unsymmetrical three-phase system $\mathfrak{S}_a : \mathfrak{S}_b : \mathfrak{S}_c$: as shown in Fig. 36. It is required to find the equivalent direct and inverse rotating three-phase systems and the symmetrical non-rotating system. Then, if the group of three symmetrical systems is identical with the original unsymmetrical three-phase system of Fig. 36, the following relationships must hold:

$$\left.\begin{aligned}
\mathfrak{S}_a &= \mathfrak{S}_{aD} + \mathfrak{S}_{ai} + \mathfrak{S}_{ao}\\
\mathfrak{S}_b &= \mathfrak{S}_{bD} + \mathfrak{S}_{bi} + \mathfrak{S}_{bo}\\
\mathfrak{S}_c &= \mathfrak{S}_{cD} + \mathfrak{S}_{ci} + \mathfrak{S}_{co}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (66)$$

that is,

$$\left.\begin{aligned} \mathfrak{S}_a &= \mathfrak{S}_{aD} + \mathfrak{S}_{ai} + \mathfrak{S}_{ao} \\ \mathfrak{S}_b &= a^2\mathfrak{S}_{aD} + a\mathfrak{S}_{ai} + \mathfrak{S}_{ao} \\ \mathfrak{S}_c &= a\mathfrak{S}_{aD} + a^2\mathfrak{S}_{ai} + \mathfrak{S}_{ao} \end{aligned}\right\} \quad . \quad . \quad . \quad (67)$$

By adding together the three equations (67), *the magnitude and direction of the "null" system of vectors is obtained*, that is,

$$\mathfrak{S}_{ao} = \tfrac{1}{3}[\mathfrak{S}_a + \mathfrak{S}_b + \mathfrak{S}_c] \quad . \quad . \quad . \quad . \quad (68)$$

and this is easily obtained by a simple graphical summation of the three original vectors.

Again, if the second equation of (67) is multiplied by the vector a [see

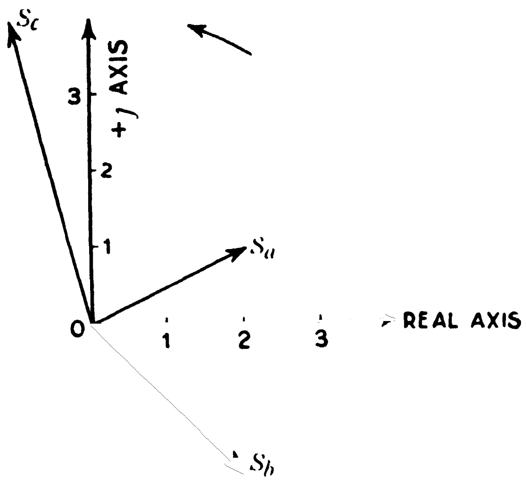


Fig. 36.

expressions (65)] and the third equation is multiplied by a^2, the following set of equations is obtained,

$$\left.\begin{aligned} \mathfrak{S}_a &= \mathfrak{S}_{aD} + \mathfrak{S}_{ai} + \mathfrak{S}_{ao} \\ a\mathfrak{S}_b &= a^3\mathfrak{S}_{aD} + a^2\mathfrak{S}_{ai} + a\mathfrak{S}_{ao} \\ a^2\mathfrak{S}_c &= a^3\mathfrak{S}_{aD} + a^4\mathfrak{S}_{ai} + a^2\mathfrak{S}_{ao} \end{aligned}\right\} \quad . \quad . \quad . \quad (69)$$

By adding together these three equations, *the direction and magnitude of the "direct" three-phase rotating system is obtained*, that is,

$$\mathfrak{S}_{aD} = \tfrac{1}{3}[\mathfrak{S}_a + a\mathfrak{S}_b + a^2\mathfrak{S}_c] \quad . \quad . \quad . \quad . \quad (70)$$

and $\mathfrak{S}_{aD}$ is thus obtained by a simple graphical construction.

Further, if the second equation of (67) is multiplied by a^2 and the third equation is multiplied by a, the following equations are obtained:

$$\left.\begin{aligned} \mathfrak{S}_a &= \mathfrak{S}_{aD} + \mathfrak{S}_{ao} + \mathfrak{S}_{ao} \\ a^2\mathfrak{S}_b &= a^4\mathfrak{S}_{aD} + a^3\mathfrak{S}_{ai} + a^2\mathfrak{S}_{ao} \\ a\mathfrak{S}_c &= a^2\mathfrak{S}_{aD} + a^3 S_{ai} + a\mathfrak{S}_{ao} \end{aligned}\right\} \quad . \quad . \quad . \quad (71)$$

If these three equations are added together, *the direction and magnitude of the "inverse" three-phase system is obtained*,

$$\mathfrak{S}_{ai} = \tfrac{1}{3}[\mathfrak{S}_a + a^2\mathfrak{S}_b + a\mathfrak{S}_c] \quad . \quad . \quad . \quad (72)$$

and $\mathfrak{S}_{ai}$ is thus derived by means of a simple graphical construction.

Example.—Let the unsymmetrical three-phase system be defined by

$$\mathfrak{S}_a = 2 + j \,;\ \mathfrak{S}_b = 2 - 2j \,:\ \mathfrak{S}_c = -1 + 4j,$$

as shown in Fig. 36*a*.

The "null" system is then given by

$$\mathfrak{S}_{10} = 1 + j.$$

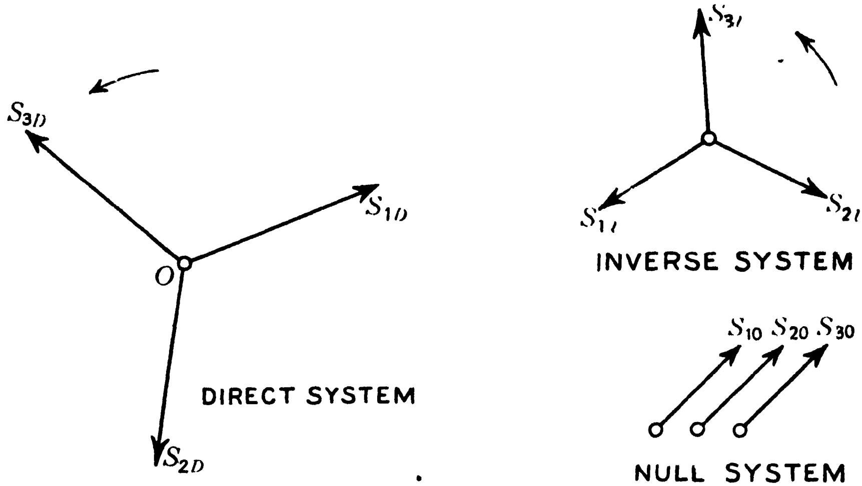


Fig. 37.

The "direct" system is given by

$$\mathfrak{S}_{1D} = 2{\cdot}23 + 0{\cdot}87j,$$

and the "inverse" system is given by

$$\mathfrak{S}_{1i} = -1{\cdot}23 - 0{\cdot}865j.$$

These three systems are shown in Fig. 37*b*, and it is easily seen that, by graphically superposing the three symmetrical systems, the original unsymmetrical three-phase system is obtained.

The measure of asymmetry of any given three-phase system is the "asymmetry factor" and is defined by the expression,*

$$u = |\mathfrak{U}| = \frac{|\mathfrak{S}_i|}{|\mathfrak{S}_D|} = \frac{\mathfrak{S}_a + a^2\mathfrak{S}_b + a\mathfrak{S}_c}{\mathfrak{S}_a + a\mathfrak{S}_b + a^2\mathfrak{S}_c} \quad . \quad . \quad . \quad (73)$$

and a three-phase system is said to be "symmetrical" when the value of u is less than 0·05.

* The symbol $|\mathfrak{S}|$ denotes the magnitude (i.e. the modulus) of the vector $\mathfrak{S}$.

It is useful to note that in a star-connected system the " null " component of an unsymmetrical three-phase current system is the current which will flow to the star point, that is to say, the current I_0 which will flow in the neutral line connected to the star point. If the star point is insulated, that is, if there is no neutral line, there can be no " null " component.

The " null " component of an unsymmetrical three-phase pressure system is the pressure V_0 which will appear at the star point. If the three pressure phases form a closed triangle, then in accordance with the expression (68) the null component must be zero.

Application of the Principle of Symmetrical Component Analysis to the Determination of the Rotating Magnetic Field due to a Three-Phase Current in a Three-Phase Winding

In the first place consider the magnetic field produced by an alternating current in a single-phase winding as shown in Fig. 38*a*. The current being defined by the expression

$$i = I_m \cos \omega t$$

and the vector of this current is shown in Fig. 40*a* for the moment

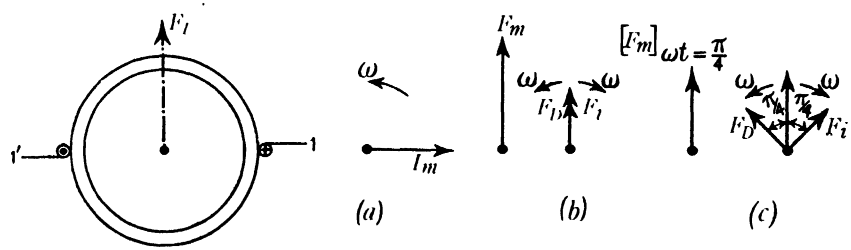


Fig. 38.

$t = o$, that is, $i = I_m$ for this moment. The direction of the current is marked in the single-phase winding, it being assumed that the positive direction is that for which the current enters the winding at the end marked with the undashed number 1. The positive direction of the magnetic field due to this current will be as shown by the chain-dotted vector F_l at right-angles to the plane of the winding 1 1′. In the case of the actual winding of a three-phase machine the turns of the winding would be distributed along the periphery of the gap so that the flux density distribution can be assumed to be sinusoidal throughout the gap and consequently this magnetic field may be represented by an alternating vector in the direction of F_l.

In Fig. 38*b* is shown the maximum value F_m of the flux density vector and corresponds to the position of the current vector as shown in Fig. 38*a*. This vector will alternate as a cosine function of the time and with a circular frequency ω. Such an alternating field vector can be resolved

into two equal vectors of constant magnitude and rotating at uniform angular velocity ω in opposite directions as shown in Fig. 38*b*, the magnitude of each of these component vectors being one half that of the maximum value F_m of the alternating vector.

For the moment illustrated in Fig. 38*b*, that is, for $t = 0$, the two component field vectors F_D and F_i, each of which is equal to $\frac{1}{2}F_m$, will be in coincidence in the direction of F_I, so that their resultant is equal to F_m. At a subsequent moment, for example, when $\omega t = \frac{\pi}{4}$, the magnitude of the alternating field vector will be $F_m \cos \omega t = \frac{2}{\sqrt{2}} F_m$, and at the same moment the two rotating vectors will each be inclined at 45° to the axis F_I, so that the resultant vector will be in the direction F_I, and will have the magnitude $2\frac{F_m}{2}\cos 45 = \frac{1}{\sqrt{2}}F_m$. It is therefore easily seen that at any moment the resultant of the two equal and oppositely rotating vectors F_D and F_i will be of the same magnitude and in the same direction as the vector of the alternating field, that is to say, the two equal and oppositely rotating vectors of magnetic fields are equivalent to the alternating field vector.

This equivalence can be applied to each phase of a three-phase current in a symmetrical three-phase winding, and in Fig. 39*a* is shown diagrammatically such a three-phase winding, $1\,1' : 2\,2' : 3\,3'$. This winding may be assumed to be connected as either a mesh or a star system since the results obtained in what follows will apply equally to each system of connections. The vector diagram for the three-phase currents is also shown in Fig. 39*a* for the moment at which the current in phase $1\,1'$ is a maximum, the respective currents for the three phase being as follows:

$$i_1 = I_m \cos \omega t$$

$$i_2 = I_m \cos\left(\omega t - \frac{2\pi}{3}\right)$$

$$i_3 = I_m \cos\left(\omega t - \frac{4\pi}{3}\right).$$

The positive directions for the magnetic fields due to the currents in the respective phase windings are shown by the vectors $F_{\mathrm{I}} : F_{\mathrm{II}} : F_{\mathrm{III}}$. For the moment $t = 0$ the two equal rotating fields for phase $1\,1'$ are in coincidence with the position of the datum vector F_{I} as is shown in Fig. 39*b*. As regards the field due to phase $2\,2'$ it is to be observed (see also the current vector diagram of Fig. 39*a*) that the current will reach its maximum value at a time $t = \frac{2\pi}{3\omega}$ sec. later than the moment $t = 0$, so that at the moment $t = 0$, the positions of the two component rotating

fields relatively to the datum vector F_{II} will be as shown in Fig. 39*b*. Similarly, it is easily seen that the two component rotating fields of phase 3 3′ at the moment $t = 0$ will occupy the position relatively to the datum vector F_{III} as shown in Fig. 39*b*. An inspection of the diagrams of Fig. 39*b* will show, therefore, that at the moment $t = 0$ the three component fields of all the three phases will coincide with the direction of F_{I}, whilst the three components F_i will be displaced relatively to each other by the angle 120° and consequently their sum at every moment will be zero. It follows, therefore, that the resultant magnetic field due to the three-phase currents in the three-phase winding as shown in Fig. 39*a* has a magnitude

$$3F_D = 1{\cdot}5F_m,$$

where F_m is the maximum peak value of the alternating field due to any one of the three-phase currents.

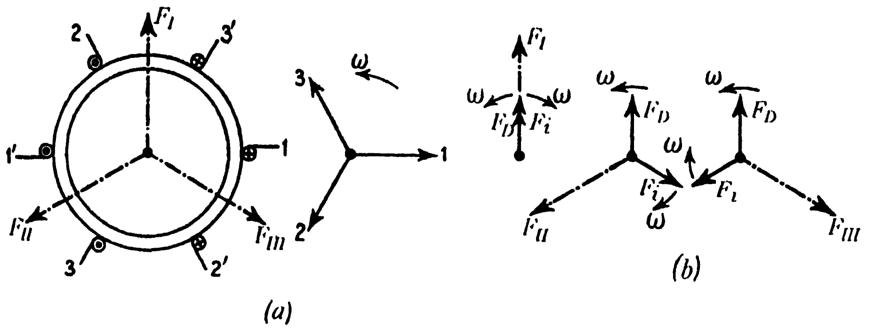


Fig. 39.

The results obtained in the foregoing may therefore be summarized as follows. A balanced three-phase current of circular frequency ω when flowing in a symmetrical three-phase winding will produce a rotating magnetic field of constant magnitude and rotating at the constant speed of ω electrical radians per second. The peak value of this rotating field will be 1·5 times the maximum peak value F_m of the alternating field due to the current in any one of the phases. The position in the air-gap of this rotating field at any moment will be coincident with the position of the alternating field due to that phase in which the current has its maximum value.

It is easily shown that if the connections to the supply main of any two of the phase windings are interchanged, then the direction of the rotating magnetic field will be reversed. These results are the same as those already obtained (Fig. 34 on page 398) by means of a different procedure.

Chapter XIII

NON-SINUSOIDAL WAVE FORMS: HARMONIC ANALYSIS: EFFECTS OF WAVE FORM ON ELECTRICAL MEASUREMENTS

ALTHOUGH sine wave forms of current and pressure are the standard in electrical engineering, it is seldom that these ideal wave-forms are actually realized in practice, although very close approximations are frequently obtained. On the other hand, wave forms which differ widely from the sinusoidal standard are of frequent occurrence and it is necessary to consider how such distorted waves are to be dealt with and how the methods of calculations and measurements which have been investigated in the foregoing chapters have to be modified when the wave-forms appreciably differ from the standard sine wave form.

Harmonic Analysis

In accordance with Fourier's theorem, it is known that any continuous, single-valued, periodic function, whatever its form, may be represented by a series of sinusoidal waves of different frequency, phase, and amplitude. Thus, if $f(x)$ is any periodic, continuous, single-valued, function of x, then:

$$f(x) = K + K_1 \cos (x - \phi_1) + K_2 \cos (2x - \phi_2) + \ldots K_n \cos (nx - \phi_n) + \ldots \quad (1)$$

where K; K_1; K_2 ... are constants.

If the axis of x divides a single complete period of the wave into a positive and negative half of equal area, the constant K becomes zero, and this is the condition commonly met with in practice. If, however (see, for example, Fig. 7), the curve is such that the constant K is not zero, then the value of K may be found thus,

$$K = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \quad (2)$$

and the horizontal axis is then moved parallel to itself by the appropriate amount as explained on page 414.

The term $K_1 \cos (x - \phi_1)$ in expression (1) is the component of lowest frequency and is called the *fundamental wave*. The remaining terms are called *harmonics* of the first term from the analogy to " overtones " in acoustics.

Characteristic Effects of Odd and Even Harmonics on the Wave-Form

In Fig. 1 is shown a sinusoidal wave of current of fundamental frequency, and a wave of triple frequency (i.e. a third harmonic). In Fig. 2 is shown the resultant wave-form obtained by plotting the algebraical sum of the instantaneous values of the two waves of Fig. 1.

If, now, the positive half-wave of Fig. 2 is moved along the abscissa axis by a half-period so as to occupy the position shown by the broken line in Fig. 2, this displaced half-wave will be seen to be the exact image in the abscissa axis of the negative half-wave. This result is a characteristic feature of all periodic wave forms which comprise only *odd harmonics* and is the type of wave which is most frequently met with in electrical engineering practice. By making use of this characteristic feature of the absence of any even harmonic in the wave-form, it is possible to say from an inspection of the wave-form whether or no, any

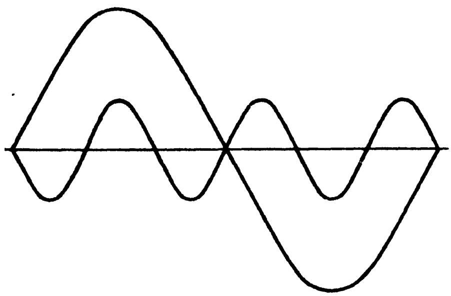


Fig. 1.

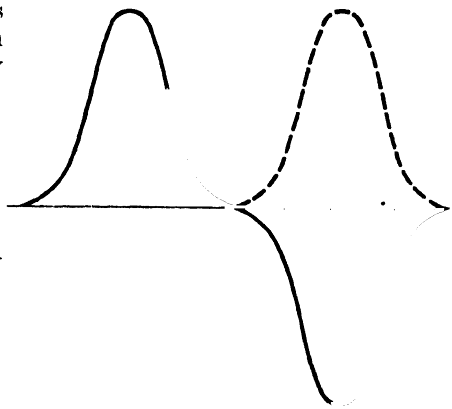


Fig. 2.

even harmonics are to be expected, and in this way the process of analysis of a given wave-form can often be expedited.

In Fig. 3 is shown a wave of current of the fundamental frequency and a second harmonic cosine term, and in Fig. 4 is shown the resultant wave-form which is obtained by taking the algebraical sum of corresponding instantaneous values of the two components shown in Fig. 3. If, now, the positive half-wave of Fig. 4 be moved along the abscissa axis so that it comes into the position shown by the broken line curve it will be seen that the negative half-wave is not now an image in the abscissa axis of the displaced positive half-wave and this is a characteristic feature of a wave-form which comprises one or more even harmonics.

Method of Harmonic Analysis

It has been seen in the foregoing how it is possible, by means of a simple inspection of the wave-form, to obtain information as to whether

even harmonics are to be expected from an analysis or whether only odd harmonics are present. Since the procedure for harmonics is somewhat different when even harmonics are to be found than when only odd harmonics are to be determined, the two cases will be dealt with separately.

CASE I.—*Only Odd Harmonics are Present.*—Since the constructional features of electrical machines are such as to preclude the development of odd harmonics in the generated e.m.f. wave-form, the characteristic feature of the great majority of distorted wave-forms which are met with in heavy current practice is that only odd harmonics are present in the wave-form. If, then, the wave-form has been examined in accordance with the principle referred to in connection with Figs. 2 and 4, and it has been found that only odd harmonics are to be looked for in the analysis, the method by which the amplitude of a harmonic of any required order, say the nth, can be determined, will be clear from the following considerations :

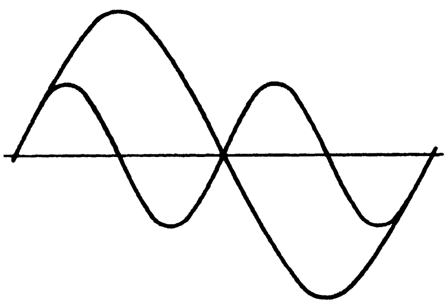


Fig. 3.

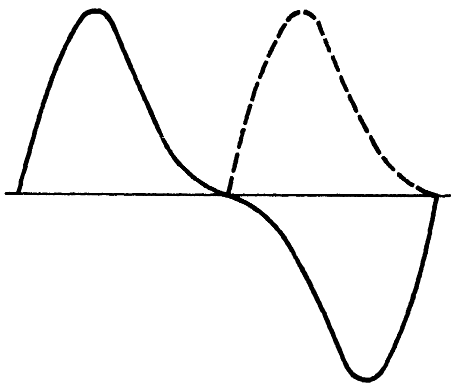


Fig. 4.

(i) In the first place consider the definite integral,

$$J_1 = \int_0^\pi \cos mx . \cos nx \, dx = \tfrac{1}{2}\int_0^\pi \{\cos(m-n)x + \cos(m+n)x\}dx \quad . \quad (3)$$

When m and n are both odd integers, the possible values of this integral are

$$J_1 = \begin{Bmatrix} 0, \text{ when } m \gtrless n \\ \frac{\pi}{2}, \text{ when } m = n > 0 \\ \pi, \text{ when } m = n = 0 \end{Bmatrix} \quad . \quad . \quad . \quad (4)$$

so that, for the particular case in which $m = n > 0$

$$J_1 = \int_0^\pi \cos^2 nx \, dx = \frac{\pi}{2} . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, it may be shown that, for all values of m and n which are both odd integers,

$$\int_0^\pi \sin mx . \cos nx \, dx = 0 \quad . \quad . \quad . \quad . \quad (6)$$

(*ii*) Next, consider the definite integral

$$J_2 = \int_0^\pi \sin mx . \sin nx \, dx = \tfrac{1}{2}\int_0^\pi \{\cos (m-n)x - \cos (m+n)x\} dx \quad . \quad (7)$$

when m and n are both odd integers, the possible values for this integral are

$$J_2 = \begin{cases} 0, \text{ when } m \lesseqgtr n \\ \dfrac{\pi}{2}, \text{ when } m = n > 0 \\ 0, \text{ when } m = n = 0 \end{cases} \quad . \quad . \quad . \quad (8)$$

so that, in the particular case in which $m = n > 0$,

$$J_2 = \int_0^\pi \sin^2 nx \, dx = \frac{\pi}{2} \quad . \quad . \quad . \quad . \quad (9)$$

The foregoing results may now be applied to the analysis of a periodic curve of which an examination of the wave-form in accordance with the principle illustrated in Fig. 2, has shown that only odd harmonics are present. If the wave-form is defined by the expression (1),

$$f(x) = \begin{Bmatrix} A_1 \cos x + A_3 \cos 3x + \ . \ . \ . + A_n \cos nx + \ . \ . \ . \\ + B_1 \sin x + B_3 \sin 3x + \ . \ . \ . + B_n \sin nx + \ . \ . \ . \end{Bmatrix} \quad . \quad (10)$$

then the coefficient of any cosine term $A_n \cos nx$ of this series can be found if the expression (10) is multiplied by $\cos mx$ and integrated with respect to x for the range from $x = 0$ to $x = \pi$, that is,

$$\int_0^\pi f(x) \cos nx \, dx$$

$$= \int_0^\pi \begin{Bmatrix} A_1 \cos x . \cos nx + \ . \ . \ . + A_n \cos^2 nx + \ . \ . \ . \\ + B_1 \sin x . \cos nx + \ . \ . \ . + B_n \sin nx . \cos nx + \ . \ . \ . \end{Bmatrix} dx \quad . \quad (11)$$

In accordance, however, with the results given in expressions (5) and (6), it will be seen that all the terms on the right-hand side of (11) will be zero with the single exception of $\int_0^\pi A_n \cos^2 nx \, dx$, and this particular term will have the value $A_n \frac{\pi}{2}$, so that,

$$\int_0^\pi f(x) \cos nx \, dx = \int_0^\pi A_n \cos^2 nx \, dx = A_n \frac{\pi}{2} \quad . \quad . \quad (12)$$

Hence, the required value of the coefficient A_n for the cosine term of the nth order will be,

$$A_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx \, dx \quad . \quad . \quad . \quad . \quad (13)$$

The coefficient B_n for the sine term of the nth order in the expression (10) is found by multiplying this expression by $\sin nx$, then,

$$\left.\begin{aligned}&\int_0^\pi f(x)\sin nx\,dx\\ &\quad=\int_0^\pi\left\{\begin{aligned}&A_1\cos x.\sin nx+\ldots+A_n\cos nx.\sin nx+\ldots\\ &+B_1\sin x.\sin nx+\ldots+B_n\sin^2 nx+\ldots\end{aligned}\right.\end{aligned}\right\}dx \quad . \quad (14)$$

and, in accordance with the results obtained in the foregoing treatment relative to the expressions (6) and (9), it will be clear that all the terms on the right-hand side of the expression (14) will be zero with the single exception of $\int_0^\pi B_n \sin^2 nx\,dx$, of which the value will be $\frac{\pi}{2}$, so that,

$$\int_0^\pi f(x)\sin nx\,dx=\int_0^\pi B_n\sin^2 nx\,dx=B_n\frac{\pi}{2} \quad . \quad . \quad (15)$$

that is,

$$B_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \quad . \quad . \quad . \quad (16)$$

The practical procedure, then, is as follows:

Take m equally spaced ordinates in the half-cycle of the curve, the distance between the ordinates being, therefore $\frac{\pi}{m}$.

Let $y_1, y_2, y_3 \ldots$ be the values of the respective ordinates. Then—

$$B_n=\frac{2}{\pi}\frac{\pi}{m}\left[y_1\sin n\frac{\pi}{m}+y_2\sin n\frac{2\pi}{m}+\ldots+y_{m-1}\sin\frac{n(m-1)\pi}{m}\right.$$
$$\left.+\,y_m\sin n\frac{m\pi}{m}\right];$$

that is, since in the last term of the above expression is $\sin n\pi=0$,

$$B_n=\frac{2}{m}\left[y_1\sin n\frac{\pi}{m}+y_2\sin n\frac{2\pi}{m}+\ldots y_{m-1}\sin\frac{n(m-1)\pi}{m}\right]$$

and $$A_n=\frac{2}{m}\left[y_1\cos n\frac{\pi}{m}+y_2\cos n\frac{2\pi}{m}+\ldots-y_m\right].$$

EXAMPLE 1.—Suppose it is required to analyse the current wave given in Fig. 5. It will be seen by inspection (see page 408) that there are *only odd* harmonics in this curve. The values of the abscissae of the curve are given in seconds and also in degrees, so that 360° corresponds to one cycle—viz. $\frac{2\pi}{\omega}$ seconds. Take nine equispaced ordinates—i.e. $m=9:\ \frac{\pi}{m}=20°$.

Write down a table of values as follows:

Ordinates	*Fundamental Wave*		*Third Harmonic*	
$i_1 = 19$	$i_1 \sin 20° = 6{\cdot}5$	$i_1 \cos 20° = 17{\cdot}9$	$i_1 \sin 60 = 16{\cdot}4$	$i_1 \cos 60° = 9{\cdot}5$
$i_2 = 31$	$i_2 \sin 40° = 20{\cdot}6$	$i_2 \cos 40° = 23{\cdot}9$	$i_2 \sin 120° = 26{\cdot}9$	$i_2 \cos 120° = -15{\cdot}5$
$i_3 = 38$	$i_3 \sin 60° = 33$	$i_3 \cos 60° = 19{\cdot}0$	$i_3 \sin 180° = 0$	$i_3 \cos 180° = -38$
$i_4 = 44$	$i_4 \sin 80° = 43{\cdot}2$	$i_4 \cos 80° = 7{\cdot}6$	$i_4 \sin 240° = -38{\cdot}1$	$i_4 \cos 240° = -22$
$i_5 = 49$	$i_5 \sin 100° = 48{\cdot}2$	$i_5 \cos 100° = -8{\cdot}5$	$i_5 \sin 300° = -42{\cdot}5$	$i_5 \cos 300° = 24{\cdot}5$
$i_6 = 53$	$i_6 \sin 120° = 45{\cdot}8$	$i_6 \cos 120° = -26{\cdot}5$	$i_6 \sin 360° = 0$	$i_6 \cos 360° = 53$
$i_7 = 55{\cdot}5$	$i_7 \sin 140° = 37{\cdot}0$	$i_7 \cos 140 = -42{\cdot}5$	$i_7 \sin 420° = 48$	$i_7 \cos 420 = 27{\cdot}7$
$i_8 = 31$	$i_8 \sin 160° = 10{\cdot}6$	$i_8 \cos 160° = -29$	$i_8 \sin 480° = 26{\cdot}9$	$i_8 \cos 480° = -15{\cdot}5$
$i_9 = 0$	$i_9 \sin 180° = 0$	$i_9 \cos 180° = 0$	$i_9 \sin 540° = 0$	$i_9 \cos 540° = 0$
	Sum = 244·9	Sum = −38·1	Sum = 37·6	Sum = 23·7

Hence
$$B_1 = \frac{244{\cdot}9}{4{\cdot}5} = 54{\cdot}2 : B_3 = \frac{37{\cdot}6}{4{\cdot}5} = 8{\cdot}25,$$
$$A_1 = -\frac{38{\cdot}1}{4{\cdot}5} = -8{\cdot}5 : A_3 = \frac{23{\cdot}7}{4{\cdot}5} = 5{\cdot}3.$$

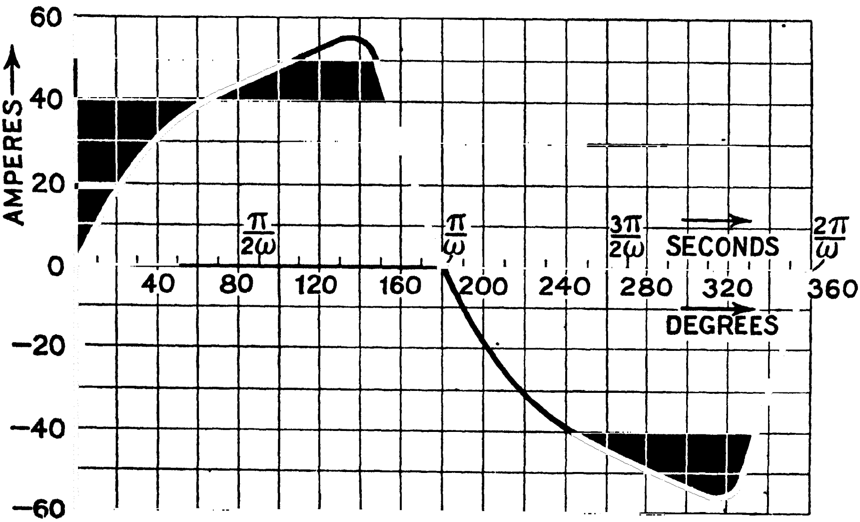


Fig. 5.

A similar analysis yields for the fifth harmonic—
$$B_5 = 0{\cdot}5 : \ A_5 = 3{\cdot}05,$$
and similarly for the coefficients of the higher harmonics.

A close approximation to the curve of Fig. 5 may therefore be represented by the series—

$$54{\cdot}2 \sin \omega t + 8{\cdot}25 \sin 3\omega t + 0{\cdot}5 \sin 5\omega t$$
$$- 8{\cdot}5 \cos \omega t + 5{\cdot}3 \cos 3\omega t + 3{\cdot}05 \cos 5\omega t.$$

A still closer approximation may be obtained by taking the harmonics higher than the fifth into account.

The above series may be rewritten—

$$54{\cdot}9 \sin (\omega t - 8^\circ\, 50') + 9{\cdot}82 \sin (3\omega t + 32^\circ\, 45')$$
$$+ 3{\cdot}1 \sin (5\omega t + 80^\circ\, 40'),$$

and this series is shown in Fig. 6.

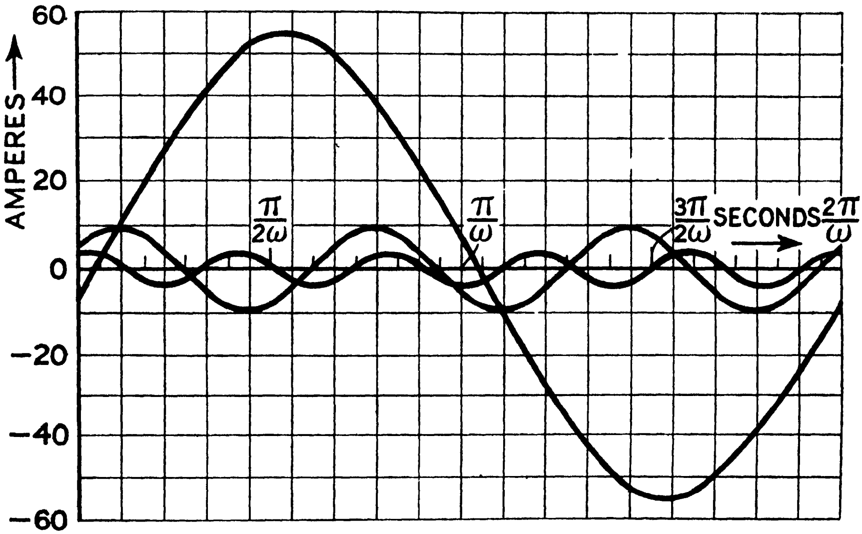


Fig. 6.

By adding corresponding ordinates of the waves in Fig. 6 and comparing with the ordinates of the original curve of Fig. 5, it will be seen what degree of approximation is made by neglecting harmonics higher than the fifth.

CASE II.—*When Even Harmonics are Present.*—If even harmonics are present in the wave-form the foregoing method cannot be applied to the analysis of the curve without some modification. Thus, in the previous Case I it has been seen that when m and n are both *odd integers*, the integral

$$\int_0^\pi \sin mx.\, \cos nx\, dx = 0 \quad . \quad . \quad . \quad . \quad (17)$$

If, however, either m or n is an *even integer*, the value of this integral

will not be zero. Suppose, however, that the limits of integration are taken from $-\pi$ to $+\pi$, then for all values of the integers m and n, whether odd or even. the integral,

$$\int_{-\pi}^{+\pi} \sin mx . \cos nx \, dx = 0 \quad . \quad . \quad . \quad (18)$$

Then :

(*i*) For all values of m and n which are not equal the integral

$$\int_{-\pi}^{+\pi} \cos mx . \cos nx \, dx = 0 \quad . \quad . \quad . \quad (19)$$

but for the single case in which $m = n$,

$$\int_{-\pi}^{+\pi} \cos^2 nx \, dx = \pi \quad . \quad . \quad . \quad . \quad (20)$$

(*ii*) For all values of the integers m and n which are not equal, the integral

$$\int_{-\pi}^{+\pi} \sin mx . \sin nx \, dx = 0 \, . \quad . \quad . \quad . \quad (21)$$

whilst for the single case in which $m = n$,

$$\int_{-\pi}^{+\pi} \sin mx . \sin nx \, dx = \int_{-\pi}^{+\pi} \sin^2 nx \, dx = \pi \quad . \quad . \quad (22)$$

Hence, in the case in which both even and odd harmonics are present, the coefficient of the nth term, whether odd or even, may be obtained as follows,

$$\int_{-\pi}^{+\pi} f(x) \cos nx \, dx = \int_{-\pi}^{+\pi} A_n \cos^2 nx \, dx = A_n \pi \quad . \quad . \quad (23)$$

so that

$$A_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos nx \, dx \quad . \quad . \quad . \quad . \quad (24)$$

and, similarly,

$$B_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin nx \, dx \quad . \quad . \quad . \quad . \quad (25)$$

EXAMPLE 2.—In Fig. 7 is shown a rectified sinusoidal wave-form of current so that for one half-cycle the current is zero and for the next half-cycle the current is of sine wave-form. An application of the rule referred to in connection with Fig. 4 shows at once that even harmonics are present in the wave of Fig. 7. Moreover, the Fourier series for the wave-form as defined in expression (1) shows that the constant K for the wave-form of Fig. 7 is not zero. The value of the constant is

$$K = \frac{1}{2\pi}\int_0^{\pi} I_m \sin \omega t \, dt = \frac{1}{\pi} I_m = 1{\cdot}6 \text{ amperes} \quad . \quad . \quad (26)$$

since $I_m = 5$ amperes in Fig. 7.

Taking the point O as origin and dividing each half of the wave by means of 12 equi-spaced ordinates, the distance between two successive ordinates will then be $\frac{180°}{12} = 15°$. Then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos nx \, dx \quad . \quad . \quad . \quad . \quad (27)$$

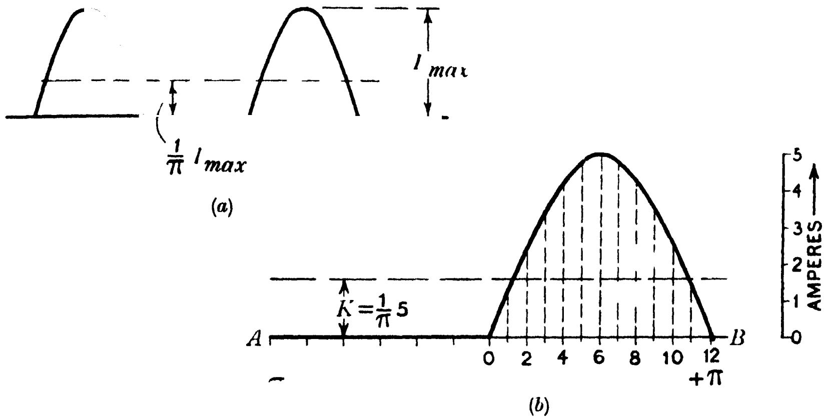


Fig. 7.

substituting $\frac{\pi}{12}$ for dx gives,

$$A_n = \frac{1}{12}\sum_{x=-\pi}^{x=+\pi} [f(x) \cos nx]$$

and since in the example of Fig. 7 the magnitude of each of the ordinates to the left of the origin O is zero it follows that,

$$A_n = \frac{1}{12}\sum_{x=0}^{x=+\pi} [f(x) \cos nx] \quad . \quad . \quad . \quad (28)$$

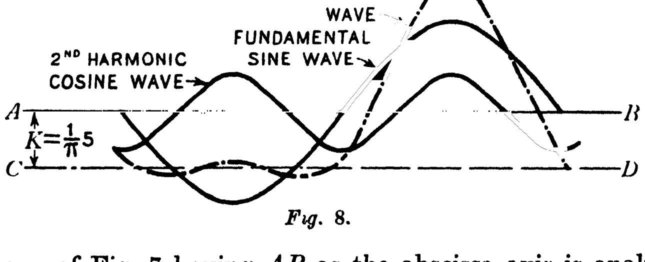


Fig. 8.

The wave of Fig. 7 having AB as the abscissa axis is analysed into its component harmonics as shown in Fig. 8 with AB as the abscissa

axis. This abscissa axis AB of Fig. 8 is then lowered by the amount $K = \frac{1}{\pi}5 = 1{\cdot}6$ amperes, the new abscissa axis so obtained being shown at CD in Fig. 8. The resultant wave referred to the abscissa axis CD is shown in Fig. 9. It will be seen that the fundamental wave and the second harmonic together give a remarkably close approximation to the original wave of Fig. 7.

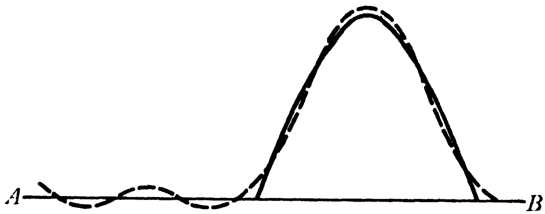


Fig. 9.

In the following table are shown the details for the analysis to determine the fundamental wave and the second harmonic.

Ordinates	*Fundamental Wave*		*Second Harmonic*	
$i_1 = 1{\cdot}3$	$i_1 \cos 15° = +1{\cdot}3$	$i_1 \sin 15° = 0{\cdot}34$	$i_1 \cos 30° = +1{\cdot}1$	$i_1 \sin 30° = 0{\cdot}6$
$i_2 = 2{\cdot}6$	$i_2 \cos 30° = +2{\cdot}1$	$i_2 \sin 30° = 1{\cdot}25$	$i_2 \cos 60° = +1{\cdot}2$	$i_2 \sin 60° = 2{\cdot}2$
$i_3 = 3{\cdot}5$	$i_3 \cos 45° = +2{\cdot}5$	$i_3 \sin 45° = 2{\cdot}50$	$i_3 \cos 90° = 0$	$i_3 \sin 90° = 3{\cdot}5$
$i_4 = 4{\cdot}4$	$i_4 \cos 60° = +2{\cdot}2$	$i_4 \sin 60° = 3{\cdot}80$	$i_4 \cos 120° = -2{\cdot}2$	$i_4 \sin 120° = 3{\cdot}8$
$i_5 = 4{\cdot}9$	$i_5 \cos 75° = +1{\cdot}3$	$i_5 \sin 75° = 4{\cdot}6$	$i_5 \cos 150° = -4{\cdot}2$	$i_5 \sin 150 = 2{\cdot}5$
$i_6 = 5{\cdot}0$	$i_6 \cos 90° = 0$	$i_6 \sin 90° = 5{\cdot}0$	$i_6 \cos 180° = -5{\cdot}0$	$i_6 \sin 180° = 0$
$i_7 = 4{\cdot}9$	$i_7 \cos 105 = -1{\cdot}3$	$i_7 \sin 105° = 4{\cdot}6$	$i_7 \cos 210° = -4{\cdot}2$	$i_7 \sin 210° = -2{\cdot}5$
$i_8 = 4{\cdot}4$	$i_8 \cos 120° = -2{\cdot}2$	$i_8 \sin 120° = 3{\cdot}8$	$i_8 \cos 240° = -2{\cdot}2$	$i_8 \sin 240 = -3{\cdot}8$
$i_9 = 3{\cdot}5$	$i_9 \cos 135° = -2{\cdot}5$	$i_9 \sin 135° = 2{\cdot}5$	$i_9 \cos 270° = 0$	$i_9 \sin 270° = -3{\cdot}5$
$i_{10} = 2{\cdot}6$	$i_{10} \cos 150° = -2{\cdot}1$	$i_{10} \sin 150° = 1{\cdot}25$	$i_{10} \cos 300° = +1{\cdot}2$	$i_{10} \sin 300° = -2{\cdot}2$
$i_{11} = 1{\cdot}3$	$i_{11} \cos 165° = -1{\cdot}3$	$i_{11} \sin 165° = 0{\cdot}34$	$i_{11} \cos 330° = +1{\cdot}1$	$i_{11} \sin 330° = -0{\cdot}6$
$i_{12} = 0$	$i_{12} \cos 180° = 0$	$i_{12} \sin 180° = 0$	$i_{12} \cos 360° = 0$	$i_{12} \sin 360° = 0$
	Sum $= 0$	Sum $= 30$	Sum $= -13{\cdot}2$	Sum $= 0$

Hence, the coefficient of the fundamental cosine wave is $A_1 = 0$, and the coefficient of the fundamental sine wave is $B_1 = \frac{30}{12} = +\,2{\cdot}5$.

Hence, the coefficient of the second harmonic cosine term is $A_2 = -\frac{13{\cdot}2}{12}$, i.e. $A_2 = -\,1{\cdot}1$, and the coefficient of the second harmonic sine term is $B_2 = 0$.

Some Special Cases of Harmonic Analysis

In most practical cases in which the harmonic analysis of wave-forms is required, a strictly accurate mathematical application of the principles

considered on page 410 is not possible, and the procedure to be followed is to approximate to the mathematical formulae by means of a graphical method as has been illustrated in the numerical examples which have been considered already on pages 412 and 415. There are, however, several special cases in which the analysis can be, mathematically, carried out and a few of the more important ones will now be considered.

(*i*) A RECTANGULAR WAVE-FORM.—This wave-form appears in many problems relating to the theory, design, and operation of electrical machines and lends itself to a very simple mathematical solution.

Suppose in Fig. 10 the rectangular-shaped wave represents a periodic current wave, such, for example, as is obtained in the commutation of a direct current, and let I amperes be the height of the rectangle. Then

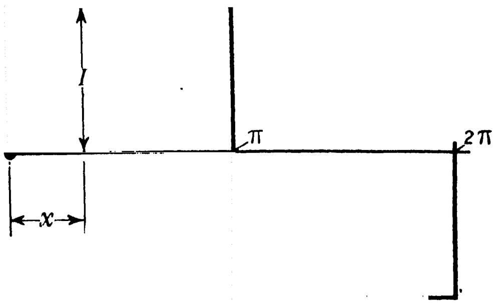


Fig. 10.

the coefficient of the nth harmonic cosine term is, by expression (13) on page 410, given by

$$A_n = \frac{2}{\pi} I \int_0^{\pi} \cos nx \, dx = \frac{2}{\pi} \frac{I}{n} \Big[\sin nx \Big]_0^{\pi} = 0 \quad . \quad . \quad (29)$$

whilst the coefficient of the nth harmonic sine term is given by

$$B_n = \frac{2}{\pi} I \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi} \frac{I}{n} \Big[-\cos nx \Big]_0^{\pi} \quad . \quad . \quad (30)$$

so that the complete analysis of this rectangular wave-form is

$$i = \frac{4}{\pi} I [\sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \ . \ . \ .] \quad . \quad . \quad (31)$$

In Fig. 11 is shown superposed on the positive half-wave rectangle, the fundamental, the third harmonic, and the fifth harmonic as derived from the expression (31). In Fig. 11 is also shown superposed on the negative half-wave rectangle the resultant of the three component sine waves, from which it will be seen how the harmonics do, in fact, build up the original wave, and by taking into account a sufficient number of

terms of the infinite series of the expression (31) the original wave-form can be approached to any desired degree of approximation.

(*ii*) A Triangular Wave-form.—This type of wave-form is shown in Fig. 12 and is frequently met with in connection with the theory of

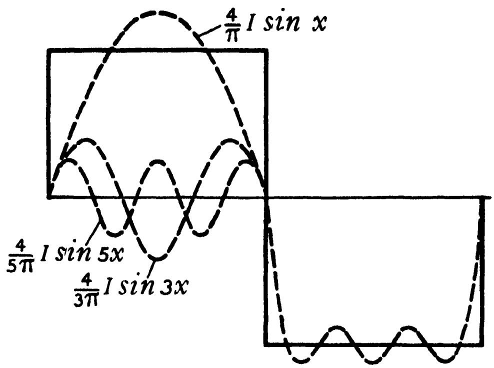


Fig. 11.

alternating-current machines, such, for example, as the armature reaction of a rotary converter, and an exact mathematical solution for the harmonic analysis can be obtained as follows :

A little consideration will show that if a wave-form has positive

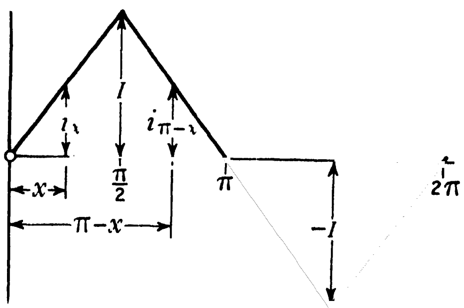


Fig. 12.

negative symmetry about the ordinate axis as shown in Fig. 13, there can be no cosine terms in the Fourier series (one example of this being the wave-form of Fig. 11). This condition can be mathematically expressed by the relationship,

$$f(x) = -f(-x).$$

Similarly, if the wave-form has positive-positive symmetry about the

ordinate axis as shown in Fig. 14, there can be no sine terms in the Fourier series, and this condition is mathematically expressed by the relationship,

$$f(x) = f(-x).$$

Hence, in the case of Fig. 12, the coefficient A_n for the nth harmonic

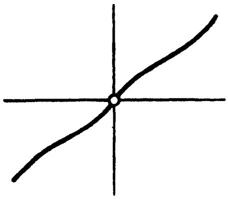


Fig. 13.

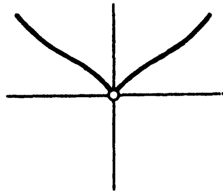


Fig. 14.

cosine term of the Fourier series is zero, and the analysis is reduced to the evaluation of the coefficients of the sine terms, viz.

$$B_n = \frac{2}{\pi}\int_0^{\pi}\{f(x)\sin(nx)\}dx \quad . \quad . \quad . \quad . \quad (32)$$

Reference to Fig. 12 will show that for the range $x = 0$ to $\frac{\pi}{2}$

$$f(x) = i_x = \frac{2}{\pi} I.x \quad . \quad . \quad . \quad . \quad (33)$$

so that

$$B_n = \frac{2 \times 2}{\pi}\int_0^{\pi/2}\left\{\frac{2}{\pi} Ix \sin nx\right\}dx$$

that is

$$= \frac{8}{\pi^2} I\int_0^{\pi/2}\{x \sin nx\}dx. \quad . \quad . \quad . \quad (34)$$

and hence

$$B_n = \frac{8}{\pi^2} I\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi/2} \quad . \quad . \quad (35)$$

The first term of the expression in brackets on the right-hand side of (35) will be zero for both limits of the integration, so that

$$B_n = \frac{8}{\pi^2} I\left[\frac{1}{n^2}\sin nx\right]_0^{\pi/2} = \frac{8}{\pi^2}\frac{I}{n^2} \text{ for } n = 1, 5, 9 \ldots$$

and

$$B_n = -\frac{8}{\pi^2}\frac{I}{n^2} \text{ for } n = 3, 7, 11, 15 \ldots$$

The complete analysis of the wave-form of Fig. 12 is therefore

$$i_x = \frac{8}{\pi^2} I[\sin x - \tfrac{1}{9}\sin 3x + \tfrac{1}{25}\sin 5x - \ldots] \quad . \quad . \quad (36)$$

In Fig. 15 are shown the first two terms of this series in the positive

half-period, whilst in the negative half-period the resultant of the two terms is shown superposed on the original triangular half-wave. It will

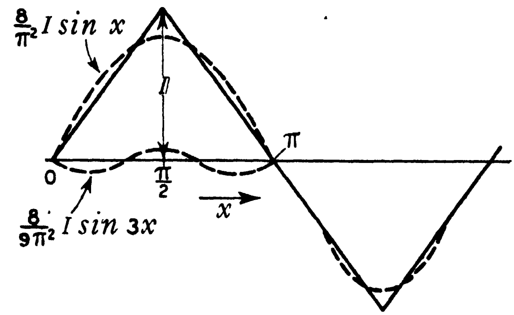


Fig. 15.

be seen how close an approximation is obtained in this case by means of only two terms of the analysed wave-form of expression (36).

(*iii*) A Trapezoidal Wave-form as shown in Fig. 16.—The Fourier series for the harmonic analysis of this wave-form is

$$i = f(x) = \frac{4}{\pi . \alpha} I[\sin\alpha . \sin x + \tfrac{1}{9}\sin 3\alpha . \sin 3x + \tfrac{1}{25}\sin 5\alpha . \sin 5x + \ . \ . \ .] \quad . \quad (37)$$

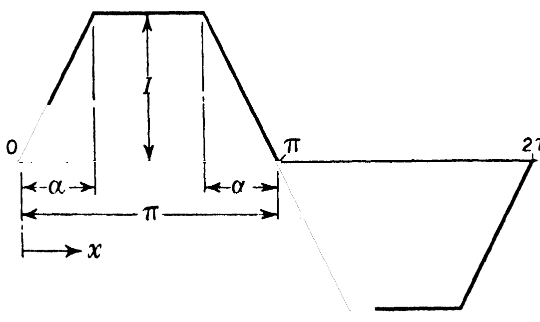


Fig. 16.

It is to be noted that the expression (37) for the analysis of the trapezoidal wave-form of Fig. 16 is a general case which includes both of the previous special cases which are illustrated in Figs. 10 and 12 respectively. Thus for the rectangular wave of Fig. 10 the quantity α of Fig. 16 is zero, and if this value for α is inserted in the expression (37), and noting that

$$\left[\frac{\sin n\alpha}{\alpha}\right]_{\alpha=0} = n$$

it is found that the expression (37) becomes identical with the expression (31).

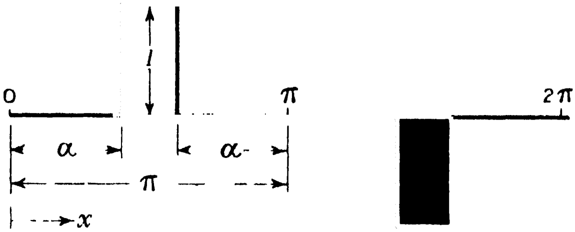


Fig. 17.

Similarly, the trapezoidal wave of Fig. 16 becomes the same as the triangular wave of Fig. 12 when $\alpha = \frac{\pi}{2}$ and, inserting this value for α in expression (37) the expression (36) is obtained

(*iv*) **A Periodic Pulse of Current or Pressure as shown in Fig. 17.**

$$i = f(x) = \frac{4}{\pi} I[\cos\alpha . \sin x + \tfrac{1}{3}\cos 3\alpha . \sin 3x + \tfrac{1}{5}\cos 5\alpha . \sin 5x + \ldots] \quad . \quad (38)$$

The Reactance of the Harmonics of a Non-Sinusoidal Wave-Form

It has been seen in the previous sections that a non-sinusoidal wave-

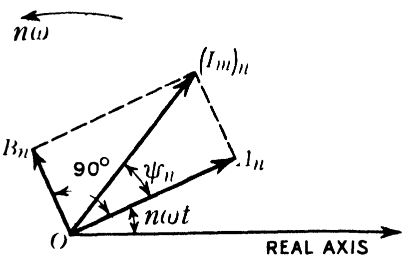


Fig. 18.

form may be analysed into a series of sinusoidal wave of different amplitudes and different frequencies, and it will be clear from a reference to Fig. 18 that the peak value of the nth harmonic will be given by

$$(I_m)_n = \sqrt{A_n^2 + B_n^2},$$

where A_n is the peak value of the nth cosine term, and
B_n ,, ,, ,, ,, ,, ,, ,, sine ,,

The instantaneous value of the nth harmonic of a current wave will then be given by

$$i_n = (I_m)_n \cos (n\omega t + \psi_n),$$

or

$$i_n = \sqrt{2} I_n \cos (n\omega t + \psi_n) \quad . \quad . \quad . \quad . \quad (39)$$

in which ω is the circular frequency of the fundamental wave and

$$\tan \psi_n = \frac{B_n}{A_n} :$$

$$I_n = \frac{1}{\sqrt{2}}(I_m)_n$$

and is the r.m.s. value of the nth harmonic of the current wave.

If, then, any given non-sinusoidal wave-form of current is defined by the expression

$$i = \sqrt{2}[I_1 \cos (\omega t + \psi_1) + I_3 \cos (3\omega t + \psi_3) + \ldots] \quad . \quad (40)$$

and if this current flows through a circuit as shown in Fig. 19, comprising a series connection of a resistance of R ohms, an inductance of L henry, and a capacitance of C farad, the p.d. at the supply terminals must have

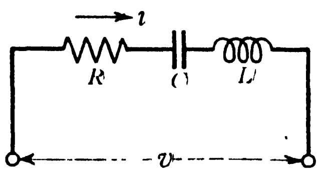


Fig. 19.

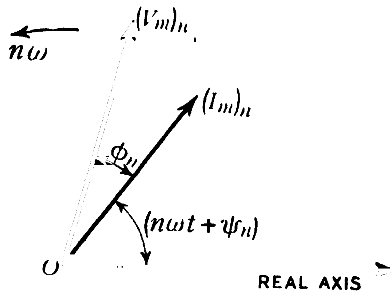


Fig. 20.

corresponding components to overcome the impedance of each harmonic of the current wave. That is to say, for the nth harmonic of current, the applied p.d. must have a component (Fig. 20)

$$\left.\begin{aligned} v_n &= \sqrt{2} V_n \cos (n\omega t + \psi_n + \phi_n) \\ \text{where} \quad V_n &= I_n \sqrt{R^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2} \\ \text{and} \quad \tan \phi_n &= \frac{n\omega L - \frac{1}{n\omega C}}{R}. \end{aligned}\right\} \quad . \quad . \quad . \quad (41)$$

Alternatively stated, each harmonic of the applied p.d. wave will produce a corresponding harmonic of the current wave. The quantity,

$$\left(n\omega L - \frac{1}{n\omega C}\right) \text{ ohms,}$$

is the *reactance* of the circuit for the nth harmonic of the current.

The relationship

$$\frac{V_n}{I_n} = \frac{(V_m)_n}{(I_m)_n} = \sqrt{R^2 + \left(n\omega L - \frac{1}{n\omega C}\right)^2} = Z_n \quad . \quad . \quad (42)$$

is the *impedance* of the circuit for the nth harmonic of the current.

Pressure resonance for the nth harmonic will occur when

$$n.\omega L = \frac{1}{n.\omega C}: \text{ that is, } n.\omega = \frac{1}{\sqrt{L.C}} \quad . \quad . \quad . \quad (43)$$

in which case $\frac{V_n}{I_n} = R$: and $\phi_n = 0$,

so that the consequent pressure across the inductance and across the capacitance will be

$$(V_n)_L = n.\omega L I_n = (n.\omega L)\frac{V_n}{R} = (V_n)_C.$$

Under such conditions, large excess pressures may be developed across the inductance and the capacitance of the circuit and may cause a breakdown of the insulation of the windings of machines and transformers and also of the dielectrics of cables.

Since the impedance of the circuit as defined by expression (42) is different for each of the harmonics, the current wave-form can only assume the same shape as that of the applied p.d. when the circuit comprises exclusively ohmic resistance R. The following special cases are of practical importance:

(*i*) If the circuit contains only a resistance R and an inductance L the impedance will be

$$Z_n = \sqrt{R^2 + (n\omega L)^2}.$$

That is to say, the impedance will increase as the order n of the harmonic increases, so that the ratio of the current amplitude to the pressure amplitude for the higher harmonics is less than for the lower harmonics. Consequently, inductance in an a.c. circuit tends to smooth out the higher harmonics of the current wave.

(*ii*) If the circuit contains only resistance R and capacitance C the impedance will be

$$Z_n = \sqrt{R_2 + \left(\frac{1}{n\omega C}\right)^2}.$$

In this case, the impedance decreases as the order n of the harmonic increases, and consequently capacitance reactance in a circuit tends to accentuate any departure of the pressure wave from the sinusoidal form and may therefore give rise to highly distorted wave forms of current and in certain circumstances, give rise to dangerous conditions.

R.M.S. Value, Power, and Power Factor of Non-Sinusoidal Wave Forms of Current and Pressure

From the definition on page 266, Chapter IX, it will be seen that the r.m.s. value of an alternating current is given by the expression

$$I = \sqrt{\frac{1}{\tau}\int_0^\tau i^2\,dt} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

where $\tau = \frac{1}{f}$ and is the periodic time of the alternating current wave.

When the current wave-form is non-sinusoidal as defined by the series,

$$\begin{aligned} i = {} & A_1 \cos \omega t + A_3 \cos 3\omega t + \ldots + A_n \cos n\omega t + \ldots \\ & + B_1 \sin \omega t + B_3 \sin 3\omega t + \ldots + B_n \sin n\omega t + \ldots \end{aligned}$$

then the expression (44) will involve a series of integrals which will all be zero with the exceptions that

$$(i) \qquad \frac{1}{\tau}A_n{}^2\int_0^\tau \cos^2 n\omega t\,dt = \tfrac{1}{2}A_n{}^2$$

and $$(ii) \qquad \frac{1}{\tau}B_n{}^2\int_0^\tau \sin^2 n\omega t\,dt = \tfrac{1}{2}B_n{}^2$$

so that $$I^2 = \tfrac{1}{2}\left\{\begin{array}{l} A_1{}^2 + A_3{}^2 + \ldots + A_n{}^2 + \ldots \\ + B_1{}^2 + B_3{}^2 + \ldots + B_n{}^2 + \ldots \end{array}\right\}$$

and $$I = \frac{1}{\sqrt{2}}\sqrt{I_{m1}{}^2 + I_{m3}{}^2 + I_{m5}{}^2 + \ldots}$$

that is

$$I = \sqrt{I_1{}^2 + I_3{}^2 + I_5{}^2 + \ldots} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

that is to say, the r.m.s. value I of the distorted wave-form of current, is equal to the square-root of the sum of the squares of the r.m.s. values of the component sinusoidal harmonics of the current wave.

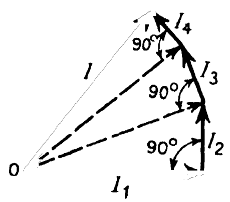


Fig. 21.

The graphical representation of the result given by expression (45) is shown in Fig. 21. In particular, if a direct current $I_{d.c.}$ (i.e. zero frequency) and an alternating current of r.m.s. value $I_{a.c.}$ are flowing in the same circuit, the r.m.s. effective value of the current in the circuit will be

$$I = \sqrt{I_{d.c.}{}^2 + I_{a.c.}{}^2}.$$

Similarly, the r.m.s. value of the applied p.d. will be

$$V = \sqrt{V_1{}^2 + V_3{}^2 + V_5{}^2 + \ldots} \quad . \quad . \quad . \quad (46)$$

The power of an alternating-current circuit is given by the integral

$$W = \frac{1}{\tau}\int_0^\tau v.i\,dt. \quad . \quad . \quad . \quad . \quad . \quad (47)$$

If the nth harmonic of the pressure wave is defined by

$$v_n = \sqrt{2} V_n \cos(n\omega t - \psi_n)$$

and the mth harmonic of the current wave is defined by

$$i_m = \sqrt{2} I_m \cos(m\omega t - \phi_m),$$

then the power of the nth harmonic of the pressure wave will be given by a series of integrals of the form

$$W_n = \frac{2V_n I_m}{\tau} \int_0^\tau \cos(n\omega t - \psi_n) \cos(m\omega t - \phi_m)\, dt \text{ watts} \quad . \quad . \quad (48)$$

where m may have the values 1, 2, 3, . . .

In consequence of the results which have already been obtained on pages 410 and 414, however, all such integrals as that in expression (48) will be zero, with the single exception of that for which $m = n$, in which case

$$W_n = \frac{2V_n I_m}{\tau} \int_0^\tau \cos(n\omega t - \psi_n) \cos(m\omega t - \phi\ \ )\, dt \text{ watts},$$

that is,

$$W_n = V_n I_n \cos \phi_n \text{ watts} \quad . \quad . \quad . \quad (49)$$

Hence, the total power for all the harmonics of the pressure and current waves will be

$$W = V_1 I_1 \cos \phi_1 + V_3 I_3 \cos \phi_3 + \ . \ . \ . \text{ watts} \quad . \quad (50)$$

where V and I are r.m.s. values.

That is to say, the total power supplied to a circuit when the pressure and current waves are non-sinusoidal will be equal to the sum of the powers due to the individual harmonics of the current pressure and current waves. Alternatively stated, the mth harmonic of the current wave is wattless with regard to the nth harmonic of the pressure wave. A consequence of this result is that alternating currents of different frequencies can be transmitted over one and the same line without any interference of the power due to the components of the different frequencies.

It will be seen from expression (41),

$$\tan \phi_n = \frac{n\omega L - \dfrac{1}{n\omega C}}{R},$$

that the phase displacement of the current and pressure is different for each harmonic, and consequently in expressing the total power in the circuit for non-sinusoidal wave in the form

$$W = V . I \cos \phi$$

the angle ϕ has no physical reality. In such cases, the power factor is defined as the ratio of the power W to the product of the r.m.s. current and the r.m.s. pressure, that is

$$\cos \phi = \frac{W}{V . I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The Effect of Wave-Form on the Measurement of Electrical Quantities

The normal types of ammeters and voltmeters used in practice are calibrated to read r.m.s. values, and this is satisfactory for most practical purposes. There are, however, cases in which this method of calibration is not suitable, as may be seen from the following examples.

(*i*) The Measurement of Self-induction by means of a Voltmeter and an Ammeter.—If a coil of self-inductance L henry has negligibly small resistance, then if a non-sinusoidal wave-form of pressure is applied to the terminals of the coil, the relationship between the r.m.s. pressure and the r.m.s. current for the individual harmonics will be

$$I_1 = \frac{V_1}{\omega L} : \ I_3 = \frac{V_3}{3\omega L} : \ I_5 = \frac{V_5}{5\omega L}.$$

The r.m.s. value of the total current will be

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots}$$

so that

$$I = \frac{1}{\omega L}\sqrt{V_1^2 + \left(\frac{V_3}{3}\right)^2 + \left(\frac{V_5}{5}\right)^2 + \ldots} \quad . \quad . \quad (52)$$

or

$$I = \frac{V_1}{\omega L}\sqrt{1 + \frac{1}{9}\left(\frac{V_3}{V_1}\right)^2 + \frac{1}{25}\left(\frac{V_5}{V_1}\right)^2 + \ldots} \quad . \quad . \quad (53)$$

The r.m.s. value of the total pressure of the supply will be

$$V = \sqrt{V_1^2 + V_3^2 + V_5^2 + \ldots}$$

that is

$$V = V_1\sqrt{1 + \left(\frac{V_3}{V_1}\right)^2 + \left(\frac{V_5}{V_1}\right)^2 + \ldots} \quad . \quad . \quad (54)$$

Eliminating V_1 between (53) and (54) gives

$$L = \frac{V}{\omega I}\sqrt{\frac{1 + \frac{1}{9}\left(\frac{V_3}{V_1}\right)^2 + \frac{1}{25}\left(\frac{V_5}{V_1}\right)^2 + \ldots}{1 + \left(\frac{V_3}{V_1}\right)^2 + \left(\frac{V_5}{V_1}\right)^2 + \ldots}} \quad . \quad . \quad (55)$$

and *this will be the correct expression for the inductance* L, when V is the r.m.s. value of the applied pressure and I is the r.m.s. value of the current. It is seen, therefore, that it is not sufficient to measure merely the r.m.s. value of the applied pressure V and the r.m.s. value of the current I, since the value of the inductance as deduced from these values will be

$$L_a = \frac{V}{\omega I} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and reference to the expression (55) shows that this value L_a would be too large. The percentage error in the value as derived from the expression (56), however, would not be serious unless the magnitude of

the pressure-wave harmonics were relatively very large. For example, suppose that there is only a third harmonic present in the pressure wave such that

$$V_3 = \tfrac{1}{4} V_1,$$

then
$$L = \frac{V}{\omega I}\sqrt{\frac{1 + \frac{1}{9} \times \frac{1}{16}}{1 + \frac{1}{16}}} = 0{\cdot}975 \frac{V}{\omega I}$$

so that
$$\frac{L}{L_a} = 0{\cdot}975,$$

and the error in the measurement of the inductance when the expression (56) is used would in this case only be about $2\frac{1}{2}$ per cent.

(*ii*) The Measurement of Capacitance by means of a Voltmeter and Ammeter.—If a circuit comprises only a resistance and a capacitance, and if the resistance is relatively small, then for non-sinusoidal waves of current and pressure the relationships for the individual harmonics will be

$$I_1 = \omega C V_1 : \; I_3 = 3\omega C V_3 : \; I_5 = 5\omega C V_5 :$$

also
$$I = \sqrt{I_1^2 + I_3^2 + I_5^2 + \ldots}$$

so that
$$I = \omega C V_1 \sqrt{1 + 9\left(\frac{V_3}{V_1}\right)^2 + 25\left(\frac{V_5}{V_1}\right)^2 + \ldots} \qquad (57)$$

The r.m.s. value of the applied pressure will be

$$V = \sqrt{V_1^2 + V_3^2 + V_5^2 + \ldots} = V_1\sqrt{1 + \left(\frac{V_3}{V_1}\right)^2 + \left(\frac{V_5}{V_1}\right)^2 + \ldots} \qquad (58)$$

or, eliminating V_1 between equations (57) and (58), gives

$$I = \omega C V \sqrt{\frac{1 + 9\left(\frac{V_3}{V_1}\right)^2 + 25\left(\frac{V_5}{V_1}\right)^2 + \ldots}{1 + \left(\frac{V_3}{V_1}\right)^2 + \left(\frac{V_5}{V_1}\right)^2 + \ldots}} \qquad (59)$$

and consequently, the correct value for the capacitance will be

$$C = \frac{I}{\omega V}\sqrt{\frac{1 + \left(\frac{V_3}{V_1}\right)^2 + \left(\frac{V_5}{V_1}\right)^2 + \ldots}{1 + 9\left(\frac{V_3}{V_1}\right)^2 + 25\left(\frac{V_5}{V_1}\right)^2 + \ldots}} \qquad (60)$$

If, however, the capacitance were to be deduced from knowledge of the r.m.s. value of the p.d. and the r.m.s. value of the current, that is,

$$C_a = \frac{I}{\omega V} \qquad (61)$$

reference to the expression (60) shows that the measured value C_a of the capacitance would be appreciably greater than the true value. Thus,

for example, if there is only a third harmonic in the pressure wave and if

$$V_3 = \tfrac{1}{4} V_1,$$

then the true capacitance would be given by

$$C = \frac{I}{\omega V}\sqrt{\frac{1 + \frac{1}{16}}{1 + \frac{9}{16}}} = 0{\cdot}825 \frac{I}{\omega V} = 0{\cdot}825 C_a,$$

so that the value measured by the expression (61) would be about 18 per cent. too large.

The Measurement of Resistance by Means of Alternating Current : Skin Effect

The total power supplied to a wire resistance when a non-sinusoidal current is flowing through the circuit is

$$W = I_1^2 R_1 + I_3^2 R_3 + I_5^2 R_5 + \dots \qquad (62)$$

where $R_1 : R_3 : R_5 : \dots$ are respectively the resistances of the circuit to the first, third, fifth . . . harmonics of the current wave.

When the diameter of the wire is small, the current will be distributed approximately uniformly throughout the cross-sectional area, in which case

$$R_1 = R_3 = R_5 = \dots R$$

and the power supplied in accordance with the expression (62) may then be stated as

$$W = (I_1^2 + I_3^2 + I_5^2 + \dots)R = I^2 R \qquad (63)$$

In general, however, this simple relationship will not hold. For example, if other conducting bodies are situated in the neighbourhood of the wire, mutual induction effects will give rise to " eddy currents " in these conducting bodies and the corresponding heat energy so produced must be supplied from the power in the resistance wire circuit so that the *effective* resistance of the wire will be increased correspondingly.

Again, for high-frequency currents, the electromagnetic field within the section of the conductor itself, and which is due to the current in the conductor, will give rise to eddy currents the resultant effect of which is to cause the current to concentrate near the surface or " skin " of the wire so that for a given current the effective cross-section of the conductor will be decreased and, consequently, the effective resistance will be increased. When the conductor is carrying a current of non-sinusoidal wave-form, the effective resistance for the higher harmonics will be greater than for the lower harmonics, and this phenomenon is generally known as the " skin effect ".

In the case of high-frequency radio circuits the increase of resistance due to the skin effect is of the greatest importance, and it is necessary to be able to calculate what the resistance will be in any particular case. Even in the case of circuits in which the current is of the normal heavy

current frequency of 50 hz., such current concentration near the surface of the conductor may develop under certain practical conditions of operation and may give rise to seriously large supplementary resistance losses resulting in a corresponding rise of temperature and loss of efficiency.

The following relationships enable the effective resistance to be calculated for a long straight cylindrical wire which is removed from the neighbourhood of neighbouring conductors.

Let a cm. be the radius of the wire,
ρ be the specific resistance of the material of the wire in electromagnetic units per centimetre cube,
f hz. be the frequency of the current,
μ be the permeability of the material of the wire,
R_0 the resistance to direct current,
R the resistance to alternating current of the frequency f.

Also let

$$y = \pi a\sqrt{\frac{f\cdot\mu}{\rho}} = \pi a\sqrt{\frac{\mu}{\rho}}\sqrt{f} \qquad (64)$$

Then

$$\left.\begin{aligned} \frac{R}{R_0} &= 1 + \frac{y^4}{3} - \frac{4}{45}y^8 : &&\text{when } y < 1 \\ \frac{R}{R_0} &= y + \frac{1}{4} + \frac{3}{64y} : &&\text{when } y > 1 \\ \frac{R}{R_0} &= 1\cdot26 : &&\text{when } y = 1. \end{aligned}\right\} \qquad (65)$$

Thus, for copper, $\rho = 1\cdot8 \times 10^3$ per cm. cube in electromagnetic units,

and $\mu = 1$, so that $\pi\sqrt{\frac{\mu}{\rho}} = 0\cdot074$

for aluminium, $\rho = 3 \times 10^3$ per cm. cube in electromagnetic units,

so that $\pi\sqrt{\frac{\mu}{\rho}} = 0\cdot057$

for iron, $\rho = 18 \times 10^3$ per cm. cube in electromagnetic units,

and assuming $\mu = 100$, then $\pi\sqrt{\frac{\mu}{\rho}} = 0\cdot20$.

Hence, at 50 hz. :

$$y = 0\cdot074\sqrt{50}\,a = 0\cdot52a \text{ for copper,}$$
$$y = 0\cdot057\sqrt{50}\,a = 0\cdot40a \text{ ,, aluminium,}$$
$$y = 0\cdot2\sqrt{50}\,a = 1\cdot42a \text{ ,, iron.}$$

A detailed investigation of " skin effect " will be found in Chapter XIV.

Chapter XIV

THE PENETRATION OF ALTERNATING MAGNETIC FLUX AND ALTERNATING CURRENT (SKIN-EFFECT)

WHEN an alternating magnetic flux is impressed on an iron wire in the axial direction, the eddy currents which will be generated will operate in accordance with Lenz's Law, so that they tend to weaken the magnetic flux which generates them. The effect will be such that the magnetic flux becomes non-uniformly distributed over the cross-section of the wire so that it becomes concentrated towards the surface, and for this reason the phenomenon is known as the "skin-effect". A precisely similar effect is produced when an alternating current flows along a wire, the magnetic field in the wire due to the current will act so as to produce a concentration of current near the surface of the wire. Such a non-uniform distribution of the current over the cross-section involves an increase of the effective resistance of the wire and particularly when high-frequency currents are used and when the wire is of low specific resistance material, this increase of the effective resistance may be extremely large and, in consequence, prohibitively large power losses may be produced. On the other hand however, the phenomenon of skin-effect is applied for important practical purposes such, for example, as to produce an automatic increase of the starting resistance of an induction motor. In what follows, the penetration of alternating magnetic flux and alternating current in long straight wires as well as in laminated conductor material will be considered, and the applications of the various formulae so derived will be illustrated by numerical examples.

Power Losses Due to Eddy Currents in Wires and Laminations in which an Alternating Magnetic Field is Uniformly distributed over the Cross Section

As a preliminary problem, the eddy current losses in iron wires and laminations will be investigated under the assumption that the flux is uniformly distributed throughout the cross-section. This assumption is sufficiently correct for those practical conditions for which the frequency is sufficiently low and the wires and laminations sufficiently thin.

(*i*) THE EDDY CURRENT LOSSES DUE TO AN ALTERNATING MAGNETIC FIELD IN AN IRON WIRE.—Fig. 1 shows diagrammatically the cross-section of an iron wire in which a sinusoidally varying magnetic flux is passing in a direction perpendicular to the plane of the paper. The consequent eddy currents will then flow in the direction of the plane of

the paper. The magnetic flux density is assumed to be uniform throughout the cross-section and the current which will be generated in the narrow concentric cylindrical band shown shaded in Fig. 1 will be considered. Then, for 1 cm. length of the wire, the r.m.s. value of the induced e.m.f. will be

$$e_x = 4f_f.f.\pi x^2.B.10^{-8} \text{ r.m.s. volts} \quad . \quad . \quad . \quad (1)$$

where f is the frequency, B is the peak value of the flux density wave and f_f is the form factor of this wave: for a sinusoidal wave $f_f = 1{\cdot}11$ (see Chapter IX, page 267).

The resistance of the cylindrical element will be $r = \dfrac{2\pi.x.\rho}{\delta x}$ ohms, where ρ is the specific resistance of the material in Ω/cm./cm.^{2} The power loss in the cylindrical element will then be

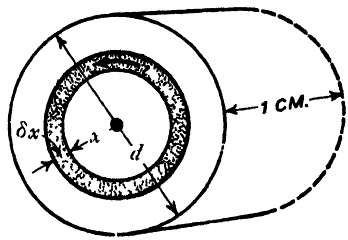


Fig. 1.

$$\frac{e_x^{\,2}}{r} = \frac{8\pi}{\rho}(f_f.f.B.10^{-8})^2x^3.\delta x \text{ watts,}$$

so that the eddy current loss for the whole cross-section of 1 cm. length of the wire will be given by

$$W = \int_0^{d/2}\frac{e_x^{\,2}}{r} = \frac{\pi}{8\rho}(f_f.f.B.10^{-8})^2d^4 \text{ watts} \quad . \quad . \quad (2)$$

where d cm. is the diameter of the wire. Since the volume for 1 cm. length is $\frac{\pi}{4}d^2$ c.cm., then if the right-hand side of the expression (2) is divided by $\frac{\pi}{4}d^2$, the result gives the eddy current loss in watts per cubic centimetre. It is, however, more convenient in practice to measure the diameter d in millimetres and the volume in cubic decimetres (1 c.dcm. (10^3 c.cm.), and if these units are introduced in the expression (2) it will be seen that

$$W = \frac{1}{2\rho}(f_f.f.B.d)^2 10^{-15} \text{ watts per c.dm.} \quad . \quad . \quad (3)$$

for *d in millimetres*, or

$$W = \sigma_e\left(d.f_f.\frac{f}{100}.\frac{B}{1{,}000}\right)^2 \text{ watts per c.dm.} \quad . \quad . \quad (4)$$

also for d in millimetres and in which $\sigma_e = \dfrac{10^{-5}}{2\rho}$ is the "eddy current constant". For iron wire the specific resistance may vary from about $1{\cdot}2 \times 10^{-5}$ to $4{\cdot}5 \times 10^{-5}$ Ω per centimetre cube, so that σ_e may vary from about 0·42 to 0·11.

(*ii*) Eddy Current Losses due to an Alternating Magnetic Flux in Thin Insulated Iron Laminations.—In Fig. 2 is shown a section of a thin insulated sheet of iron in which an alternating magnetic flux is assumed to be passing in the direction perpendicular to the plane of the paper and in consequence of which the eddy currents will flow in the direction of the plane of the paper. The shaded areas show elementary strips symmetrically placed with respect to the median plane, and represent a typical path of the eddy currents. For the development of the formulae defining the eddy current losses a length of 1 cm. in the plane of the paper and a length of 1 cm. perpendicular to the plane of the paper will be considered as shown in Fig. 2*a*. Since the typical strips are symmetrically placed with respect to the median plane of the lamina-

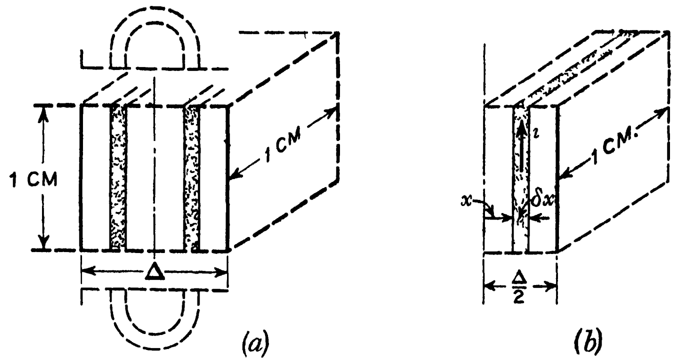


Fig. 2.

tion, it will be convenient to calculate the losses in that portion of the lamination which lies to the right of the median plane as shown in Fig. 2*b*. The e.m.f. induced in the elementary strip will then be

$$e_x = 4f_f.f.x.B.10^{-8} \text{ r.m.s. volts}$$

and the resistance of the strip will be $r = \frac{\rho}{\delta x} \Omega$, so that the eddy current loss will be

$$\frac{e_x^2}{r} = \frac{16.\delta x}{\rho}(f_f.f.B.x.10^{-8})^2 \text{ watts.}$$

The total loss for the width Δ cm. will then be

$$W = 2\int_0^{1/2} \frac{ex^2}{r} = 2\int_0^{1/2} \frac{16}{\rho}(f_f fB10^{-8}x)^2\, dx \text{ watts}$$

that is,

$$W = \frac{4}{3\rho}(f_f.f.B.\Delta)^2 10^{-16} \text{ watts per c.cm.}$$

If Δ is measured in millimetres and the volume measured in cubic decimetres, then

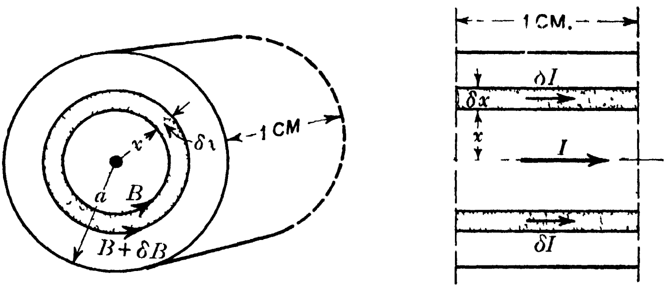
$$W = \sigma_e \left(\Delta . f_f \frac{f}{100} \frac{B}{1,000}\right)^2 \text{ watts per c.dcm.} \qquad . \qquad . \qquad (5)$$

for Δ in millimetres where the eddy current constant $\sigma_e = \frac{4}{3\rho} 10^{-5}$.

For values of ρ varying from about $1{\cdot}2 \times 10^5$ to $4{\cdot}5 \times 10^5$ the values of σ_e will vary from about 1·1 to 0·3.

Distribution of Alternating Current in a Long Straight Wire

The problem now to be considered is that of a long straight wire through which an alternating current is flowing. In Fig. 3a is shown a section of the wire of radius a cm., and a length of 1 cm. will be taken as the basis of the investigation. A concentric cylindrical element is



(a) (b)

Fig. 3

shown by the shaded area in the diagram, the inner radius of this element being x cm and the radial thickness δx cm. The current in the wire flows in the direction perpendicular to the plane of the paper so that the direction of the magnetic field which will be produced by this current will be in the plane of the paper. In Fig. 3b is shown a longitudinal section of the wire so that in Fig. 3b the direction of the current is in the plane of the paper and the direction of the magnetic flux at right angles to the plane of the paper.

The vector of current density in amperes per square centimetre in the cylindrical element is $\mathfrak{i}$ and $\mathfrak{J}$ is the vector of total current in amperes which flows through the section πx^2 sq. cm., that is, through the area which is enclosed by the inner cylindrical surface of radius x cm. The vector of the current which flows in the cylindrical band is $\delta\mathfrak{J}$ so that $\delta\mathfrak{J} = 2\pi . x . \delta x . \mathfrak{i}$ and, consequently

$$\frac{4\pi}{10}\mathfrak{J} = \mathfrak{H} 2\pi x = \frac{1}{\mu}\mathfrak{B} . 2\pi x \qquad . \qquad . \qquad . \qquad (6)$$

where $\mathfrak{H}$ oersted is the vector of magnetic force and $\mathfrak{B}$ the vector of magnetic induction at the inner periphery of the cylindrical band. The corresponding vector for the magnetic force at the outer periphery of the cylindrical band is given by the equation

$$\frac{4\pi}{10}(\mathfrak{I} + \delta\mathfrak{I}) = \frac{1}{\mu}(\mathfrak{B} + \delta\mathfrak{B})2\pi(x + \delta x) \quad . \quad . \quad . \quad (7)$$

Subtracting equations (6) and (7), gives

$$\frac{4\pi}{10}.\mu.\delta\mathfrak{I} = 2\pi(x\delta\mathfrak{B} + \mathfrak{B}\delta x)$$

or after substituting $\delta\mathfrak{I} = 2.\pi.x.\delta x.\mathfrak{i}$

$$\boxed{4\pi.\mathfrak{i}.\mu = 10\left(\frac{d\mathfrak{B}}{dx} + \frac{1}{x}\mathfrak{B}\right)} \quad . \quad . \quad . \quad (8)$$

Let $\mathfrak{v}$ volts be the component of the vector of the applied p.d. per centimetre length of the wire which drives the alternating current against the combined effect of the self-induction back e.m.f. induced in the cylindrical elementary band and also against the resistance of this cylindrical band. It is to be observed here that, whilst the current in the wire also produces a magnetic field which embraces the whole section of the wire, that is to say, a magnetic field in the surrounding space, this external field can have no effect on the distribution of the current density in the wire since the back e.m.f. which is induced by this external field will be the same for every such cylindrical band as that shown in Fig. 3*a*. The applied p.d., therefore, will have, in addition to the vector quantity $\mathfrak{v}$ per centimetre length, another component which will be equal and opposite to the back e.m.f. induced in the wire by the external field, and this additional component need not be considered here, as it will have no influence on the skin effect.

If $\mathfrak{e}$ volts per centimetre length of the wire is the back e.m.f. which is induced in the cylindrical band, then

$$\mathfrak{v} = \mathfrak{e} + r.\delta\mathfrak{I},$$

where r ohms is the resistance of the cylindrical band, so that

$$r.\delta\mathfrak{I} = (2\pi.x.\delta x.\mathfrak{i})\frac{\rho}{2\pi x\delta x} = \rho.\mathfrak{i}$$

where ρ in Ω/cm./cm.^{2} is the specific resistance of the material of the wire. Then

$$\mathfrak{v} = \mathfrak{e} + \rho.\mathfrak{i} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Since the applied p.d. will have the same value for all such elements, cylindrical bands as shown in Fig. 3*a*, the following equation is obtained by differentiating (9) with respect to x,

$$0 = \frac{d\mathfrak{e}}{dx} + \rho\frac{d\mathfrak{i}}{dx} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The increase δe of the induced e.m.f. in the cylindrical band when x increases by the amount δx is given by the equation

$$-\frac{1}{10^8}\frac{d}{dt}(\Delta\phi) = de: \text{ that is, } -\frac{1}{10^8}\frac{d}{dt}(\mathfrak{B}dx) = de,$$

since the element of magnetic flux passing through the band is $\Delta\phi = \mathfrak{B}\delta x$. Hence

$$-\frac{de}{dx} = \frac{1}{10^8}\frac{d\mathfrak{B}}{dt} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and, combining (10) and (11)

$$\boxed{\frac{d\mathfrak{B}}{dt} = 10^8\rho\frac{d\mathfrak{i}}{dx}} \quad . \quad . \quad . \quad . \quad (12)$$

Consider now the two principal equations (8) and (12). Differentiating (8) with respect to t gives

$$\frac{4\pi\mu}{10}\frac{d\mathfrak{i}}{dt} = \frac{d}{dx}\frac{d\mathfrak{B}}{dt} + \frac{1}{x}\frac{d\mathfrak{B}}{dt} \quad . \quad . \quad (13)$$

and eliminating $\frac{d\mathfrak{B}}{dt}$ between (12) and (13) gives

$$\frac{4\pi\mu}{10}\frac{d\mathfrak{i}}{dt} = 10^8\rho\frac{d^2\mathfrak{i}}{dx^2} + 10^8\rho\frac{1}{x}\frac{d\mathfrak{i}}{dx}$$

that is

$$\frac{d^2\mathfrak{i}}{dx^2} + \frac{1}{x}\frac{d\mathfrak{i}}{dx} - \frac{4\pi\mu}{\rho . 10^9}\frac{d\mathfrak{i}}{dt} = 0 \quad . \quad . \quad (14)$$

Now the current is a sine function of the time, so that

$$\mathfrak{i} = ie^{j\omega t} \text{ and consequently } \frac{d\mathfrak{i}}{dt} = j\omega\mathfrak{i}.$$

Substituting this expression for $\frac{di}{dt}$ in (14) gives

$$\boxed{\frac{d^2\mathfrak{i}}{dx^2} + \frac{1}{x}\frac{d\mathfrak{i}}{dx} - j\frac{4\pi . \mu . \omega}{\rho . 10^9}\mathfrak{i} = 0.} \quad . \quad . \quad . \quad (15)$$

Substituting $\qquad \mathfrak{Y}^2 = -j\left(\frac{4\pi\mu\omega}{\rho 10^9}\right) = \frac{4\pi\mu\omega}{\rho 10^9}e^{-j\frac{\pi}{2}}$

then $\qquad \mathfrak{Y} = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}}\Big/e^{-j\frac{\pi}{4}}$

or $\qquad \mathfrak{Y} = \sqrt{-j}\sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = (\sqrt{-j}|Y)$

and the solution of equation (15) may then be written

$$\mathrm{i}_x = CJ_0(\sqrt{-j}|Y.x) \quad . \quad . \quad . \quad . \quad (16)$$

where C is a constant and $J_0(\sqrt{-j}|Y.x)$ is the Bessel Function of *zero order.* This Bessel Function is the sum of the convergent infinite series,

$$J_0(\sqrt{-j}|Y.x) = 1 - \frac{(\frac{1}{2}\sqrt{-j}|Y.x)^2}{1!^2} + \frac{(\frac{1}{2}\sqrt{-j}|Y.x)^4}{2!^2}$$
$$- \frac{(\frac{1}{2}\sqrt{-j}|Y.x)^6}{3!^2} + \frac{(\frac{1}{2}\sqrt{-j}|Y.x)^8}{4!^2} \; . \; .$$

or, separating the real and the imaginary quantities,

$$J_0(\sqrt{-j}|Y.x) = \left\{1 - \frac{(\frac{1}{2}Yx)^4}{2!^2} + \frac{(\frac{1}{2}Yx)^6}{4!^2} - \ . \ . \ . \right\}$$
$$+ j\left\{\frac{(\frac{1}{2}Yx)^2}{1!^2} - \frac{(\frac{1}{2}Yx)^6}{3!^2} + \frac{(\frac{1}{2}Yx)^{10}}{5!^2} - \ . \ . \ . \right\}$$

If the current density at the surface of the wire is i_{max}, that is, for $x = a$, where a is the radius of the wire, then

$$i_{max} = CJ_0(\sqrt{-j}|Y.a),$$

so that

$$C = \frac{i_{max}}{J_0(\sqrt{-j}|Y.a)}$$

and hence, the solution of the equation (15) becomes

$$\boxed{\mathrm{i}_x = i_{max}\frac{J_0(\mathfrak{Y}x)}{J_0(\mathfrak{Y}a)}} \quad . \quad . \quad . \quad . \quad (17)$$

This equation may be written in any of the equivalent forms

$$\mathrm{i}_x = i_{max}\frac{J_0(\sqrt{-j}|Yx)}{J_0(\sqrt{-j}|Ya)} = i_{max}\frac{J_0(Yx \measuredangle -45^\circ)}{J_0(Ya \measuredangle -45^\circ)} = i_{max}\frac{J_0(y_x \measuredangle -45^\circ)}{J_0(y_a \measuredangle -45^\circ)}$$

where* y_x - Yx and is the modulus of the vector $\mathfrak{Y}x$, and observing that $-j = e^{-j\frac{\pi}{2}}$: $\sqrt{-j} = e^{-j\frac{\pi}{4}}$ (see also Fig. 4.)

The solution (17) may thus be written

$$\boxed{\begin{aligned}\mathrm{i}_x &= i_{max}\frac{J_0(y_x \measuredangle -45^\circ)}{J_0(y_a \measuredangle -45^\circ)} \\ &= \text{the vector } [q \measuredangle \phi].\end{aligned}} \quad . \quad (18)$$

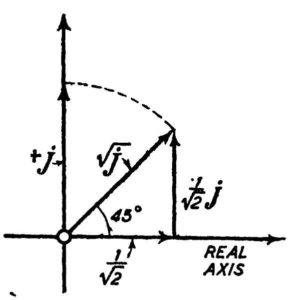


Fig. 4.

* The sign means "represents".

TABLE I.—BESSEL FUNCTIONS

y_x	$J_0(y_x \measuredangle -45^\circ)$		$J_1(y_x \measuredangle -45^\circ)$		y_x	$J_0(y_x \measuredangle -45^\circ)$		$J_1(y_x \measuredangle -45^\circ)$	
	q_0	ϕ_0°	q_1	ϕ_1°		q_0	ϕ_0°	q_1	ϕ_1°
0·1	1·0000	0·15	0·0500	− 44·931	5·1	6·6203	183·002	6·1793	97·533
0·2	1·0001	0·567	0·1000	− 44·714	5·2	7·0339	187·071	6·5745	101·518
0·3	1·0002	1·283	0·1500	− 44·350	5·3	7·4752	191·140	6·9960	105·504
0·4	1·0003	2·283	0·2000	43·851	5·4	7·9455	195·209	7·4456	109·492
0·5	1·0010	3·617	0·2500	− 43·213	5·5	8·4173	199·279	7·9253	113·482
0·6	1·0020	5·150	0·3001	42·422	5·6	8·9821	203·348	8·4370	117·473
0·7	1·0037	7·000	0·3502	− 41·489	5·7	9·5524	207·417	8·8830	121·465
0·8	1·0063	9·150	0·4010	− 40·358	5·8	10·160	211·487	9·5657	125·459
0·9	1·0102	11·550	0·4508	− 39·207	5·9	10·809	215·556	10·187	129·454
1·0	1·0155	14·217	0·5014	− 37·837	6·0	11·501	219·625	10·850	133·452
1·1	1·0226	17·167	0·5508	36·343	6·1	12·239	223·694	11·558	137·450
1·2	1·0319	20·333	0·6032	− 34·706	6·2	13·027	227·762	12·313	141·452
1·3	1·0436	23·750	0·6549	32·928	6·3	13·865	231·830	13·119	145·454
1·4	1·0584	27·367	0·7070	31·011	6·4	14·761	235·987	13·978	149·458
1·5	1·0768	31·183	0·7599	28·952	6·5	15·717	239·964	14·896	153·462
1·6	1·0983	35·167	0·8136	26·768	6·6	16·737	244·031	15·876	157·469
1·7	1·1243	39·300	0·8683	24·451	6·7	17·825	248·098	16·921	161·477
1·8	1·1545	43·550	0·9233	22·000	6·8	18·986	252·164	18·038	165·480
1·9	1·1890	47·883	0·9819	19·428	6·9	20·225	256·228	19·228	169·498
2·0	1·2286	52·283	1·0411	16·732	7·0	21·548	260·294	20·500	173·510
2·1	1·2743	56·750	1·1022	− 13·923	7·1	22·959	264·358	21·858	177·523
2·2	1·5250	61·233	1·1659	11·000	7·2	24·465	268·422	23·308	181·536
2·3	1·3810	65·717	1·2325	− 7·970	7·3	26·074	272·486	24·856	185·554
2·4	1·4421	70·183	1·3019	4·838	7·4	27·790	276·540	26·509	189·571
2·5	1·5111	74·650	1·3740	1·613	7·5	29·622	280·612	28·274	193·589
2·6	1·5830	79·114	1·4505	1·701	7·6	31·578	284·674	30·158	197·608
2·7	1·6665	83·499	1·5300	5·099	7·7	33·667	288·736	32·172	201·627
2·8	1·7542	87·873	1·6148	8·570	7·8	35·896	292·798	34·321	205·646
2·9	1·8486	92·215	1·7045	12·111	7·9	38·276	296·859	36·617	209·670
3·0	1·9502	96·518	1·7998	15·714	8·0	40·817	300·920	39·070	213·692
3·1	2·0592	100·789	1·9012	19·372	8·1	43·532	304·981	41·691	217·716
3·2	2·1761	105·032	2·0088	23·081	8·2	46·429	309·042	44·487	221·739
3·3	2·3000	109·252	2·1236	26·833	8·3	49·524	313·102	47·476	225·764
3·4	2·4342	113·433	2·2459	30·622	8·4	52·829	317·162	50·670	229·790
3·5	2·5759	117·605	2·3766	34·445	8·5	56·359	321·222	54·081	233·815
3·6	2·7285	121·760	2·5155	38·295	8·6	60·129	325·282	57·725	237·842
3·7	2·8895	125·875	2·6640	42·171	8·7	64·155	329·341	61·618	241·868
3·8	3·0613	129·943	2·8226	46·067	8·8	68·455	333·400	65·779	245·896
3·9	3·2443	134·096	2·9920	49·978	8·9	73·049	337·459	70·222	249·925
4·0	3·4391	138·191	3·1729	53·905	9·0	77·957	341·516	74·971	253·953
4·1	3·6463	142·279	3·3662	57·840	9·1	83·199	345·577	80·048	257·981
4·2	3·8671	146·361	3·5722	61·789	9·2	88·796	349·566	85·466	262·011
4·3	4·1015	150·444	3·7924	65·743	9·3	94·781	353·693	91·259	266·041
4·4	4·3518	154·513	4·0274	69·706	9·4	101·128	357·751	97·449	270·071
4·5	4·6179	158·586	4·2783	73·672	9·5	108·003	361·811	104·063	274·102
4·6	4·9012	162·657	4·5460	77·638	9·6	115·291	365·868	111·131	278·133
4·7	5·2015	166·726	4·8317	81·615	9·7	123·110	369·958	118·683	282·164
4·8	5·5244	170·795	5·1390	85·590	9·8	131·429	373·983	126·752	286·197
4·9	5·8696	174·865	5·4619	89·571	9·9	140·300	378·002	135·374	290·229
5·0	6·2312	178·933	5·8118	93·549	10	149·831	382·099	144·586	294·266

The values of the Bessel Functions J_0 are given in Table I and two numerical examples will make the procedure clear.

EXAMPLE 1.—For copper wire the specific resistance is

$$\rho = 1{\cdot}77 \times 10^{-6}\ \Omega/\text{cm.}/\text{cm.}^2$$

and the magnetic permeability is $\mu = 1$, so that

$$Y = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = \sqrt{\frac{8\pi^2 f}{1{\cdot}77 \times 10^3}} = 0{\cdot}211\sqrt{f},$$

and the following Table II shows the vector of current density at the centre of the wire, i.e. $[i]_{x=0}$ for a range of frequencies, when the radius of the wire is $a = 0{\cdot}2$ cm.

TABLE II

Frequency f *hz.*	Y	$Y.a$	$J_0(\sqrt{-j}\,Y.a)$	*Vector of Current Density at the Centre of the Wire, that is, for* $x = 0$
50	1·5	0·3	$1{\cdot}0002e^{j1{\cdot}28^\circ}$	$\frac{i_{max}}{1{\cdot}0002}e^{-j1{\cdot}28^\circ}$
250	3·23	0·646	$1{\cdot}0028e^{j6{\cdot}07^\circ}$	$\frac{i_{max.}}{1{\cdot}0028}e^{-j6{\cdot}07^\circ}$
500	4·75	0·95	$1{\cdot}013e^{j13{\cdot}9}$	$\frac{i_{max}}{1{\cdot}013}e^{-j13{\cdot}9^\circ}$
1,000	6·70	1·34	$1{\cdot}051e^{j25{\cdot}5^\circ}$	$\frac{i_{max}}{1{\cdot}051}e^{-j25{\cdot}5^\circ}$
5,000	15·0	3·0	$1{\cdot}950e^{j96{\cdot}52^\circ}$	$\frac{i_{max}}{1{\cdot}95}e^{-j96{\cdot}52^\circ}$
10,000	21·2	4·24	$3{\cdot}984e^{j148{\cdot}4^\circ}$	$\frac{i_{max}}{3{\cdot}984}e^{-j148{\cdot}4^\circ}$
14,000	25	5·0	$6{\cdot}231e^{j178{\cdot}9}$	$\frac{i_{max.}}{6{\cdot}231}e^{-j178{\cdot}9^\circ}$

In Fig. 5 is shown the vector spiral for the current density in this copper wire of 0·2 cm. radius at 8 equi-spaced distances along the radius of the wire from the surface to the centre for a frequency of 14,000 hz.

EXAMPLE 2.—An iron wire has a specific resistance of $\rho = 12 \times 10^{-6}$ Ω/cm./cm.^{2}, and a permeability $\mu = 2{,}500$. If the wire is carrying alternating current at 50 frequency, find the radius so that the vector of current density at the centre of the wire shall be in direction opposition of phase to that at the surface.

From the given data

$$Y = \sqrt{\frac{8\pi^2\mu f}{\rho . 10^9}} = \sqrt{\frac{8\pi^2 \times 2{,}500 \times 50}{12 \times 10^3}} = 28{\cdot}7.$$

Reference to equations (17) and (18) show that the current density vector at the centre, i.e. for $x = 0$, will be 180° out of phase with the current density at the surface when

$$[i]_{x=0} = i_{max}\frac{J_0(0)}{J_0(\mathcal{Y}a)} = \frac{i_{max}}{q_0 e^{j180°}},$$

that is, when

$$J_0(Ya \measuredangle -45°) = J_0(y_a \measuredangle -45°) = q_0 e^{j180°}$$

and from Table I it is seen that

$$J_0(5 \measuredangle -45°) = 6{\cdot}23 e^{j180°},$$

so that $Ya = 5$, and consequently the required radius of the wire will be

$$a = \frac{5}{Y} = \frac{5}{28{\cdot}7} = 0{\cdot}174 \text{ cm.}$$

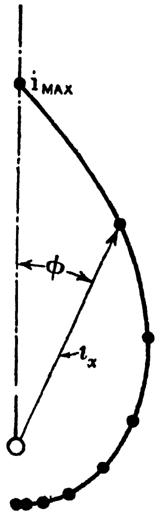
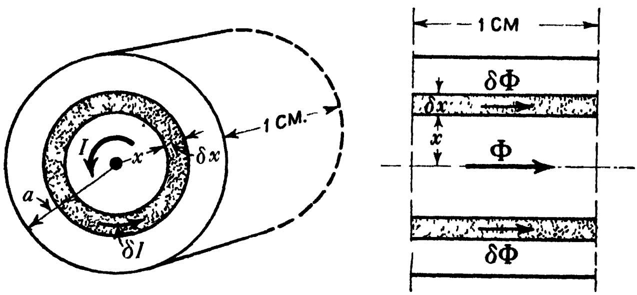


Fig. 5.

The Distribution of Alternating Magnetic Flux in a Long Straight Wire

If an alternating magnetic field is impressed on a long straight wire so that the direction of the flux is parallel to the axis of the wire, then eddy currents will be generated in a direction at right angles to the axis as indicated in Fig. 6. These eddy currents will give rise to a supplementary magnetic flux which will tend to suppress the original field to which they are due, and in consequence, there will be a non-uniform distribution of magnetic flux across the section of the wire so that at the surface the density of the magnetic flux will be a maximum and will progressively diminish towards the axis.



(*a*) (*b*)

Fig. 6.

In the following treatment the distribution of the magnetic flux

density will be found under the assumption that the magnitude of the impressed alternating field is a sine function of the time and that the permeability and the specific resistance of the material of the wire are constant and independent of the flux density, so that the magnitude of the eddy currents will also be a sine function of the time. In Fig. 6*a* is shown a cross-section of the wire and the conditions will be considered for 1 cm. length, the shaded portion representing a concentric narrow cylindrical band of radial thickness δx cm. and at a distance x cm. from the axis. The direction of the magnetic flux will be perpendicular to the plane of the paper as shown in Fig. 6*b* and the eddy currents will flow in the direction of the plane of the paper, the current density in the shaded cylindrical band being $\mathfrak{i}$ amperes per square centimetre.

The vector of the induced e.m.f. will be $\mathfrak{e}$ volts per centimetre at the inner periphery of the cylindrical band and $\mathfrak{e} + \delta\mathfrak{e}$ volts per centimetre at the outer periphery. If the vector of the total current flowing in the portion of cross-section corresponding to x cm. radius is $\mathfrak{J}$ amperes and the vector of the current which flows in the shaded cross-section is $\delta\mathfrak{J}$ amperes, then the intensity of the magnetic force at the inner periphery of the band will be defined by the vector $\mathfrak{H}$ oersted, so that the m.m.f. equation for the shaded cross section of Fig. 6*a* is

$$\frac{4\pi}{10}\mathfrak{J} = \mathfrak{H} \text{ oersted} \times 1 \text{ cm.}$$

since the axial length of the portion of wire shown in Fig. 6*a* is 1 cm. Similarly, for the outer periphery,

$$\frac{4\pi}{10}(\mathfrak{J} + \delta\mathfrak{J}) = \mathfrak{H} + \delta\mathfrak{H},$$

so that

$$\frac{4\pi}{10}\delta\mathfrak{J} = \delta\mathfrak{H} = \frac{1}{\mu}\delta\mathfrak{B},$$

where μ is the magnetic permeability.

But $\delta\mathfrak{J} = \mathfrak{i}\,\delta x$ and consequently,

$$\frac{4\pi}{10}\mathfrak{i}\,\delta x = \frac{1}{\mu}\delta\mathfrak{B}$$

that is

$$\frac{d\mathfrak{B}}{dx} = \frac{4\pi}{10}\mu.\mathfrak{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The magnetic flux which will pass along the cylindrical band, that is, in the direction perpendicular to the plane of the paper will be

$$\Delta\Phi = 2\pi x\,\delta x\,\mathfrak{B} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and the e.m.f. induced by this element of the alternating flux will be $\delta\mathfrak{e}$ where,

$$10^{-8}\frac{d}{dt}(\Delta\Phi) = -\left[2\pi(x + \delta x)(\mathfrak{e} + \delta\mathfrak{e}) - 2\pi x\mathfrak{e}\right]$$

or, substituting $\Delta\Phi$ from equation (20)

$$10^{-8}2\pi . x . \delta x \frac{d\mathfrak{B}}{dt} = -2\pi(x\delta\mathfrak{e} + \mathfrak{e}\delta x),$$

that is

$$x\frac{d\mathfrak{B}}{dt} = -10^8\left(x\frac{d\mathfrak{e}}{dx} + \mathfrak{e}\right) \quad . \quad . \quad . \quad . \quad (21)$$

Further, the e.m.f. which drives the current $\delta\mathfrak{i}$ round the cylindrical band is given by

$$\mathfrak{e} . 2\pi . x = \mathfrak{i} . \rho\left(\frac{2\pi x}{\delta x}\right)$$

so that

$$\mathfrak{e} = \mathfrak{i} . \rho \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Eliminating $\mathfrak{e}$ between (21) and (22) gives

$$x\frac{d\mathfrak{B}}{dt} = -10^8\left(x\frac{d\mathfrak{i}}{dx} + \mathfrak{i}\right)\rho \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and eliminating $\mathfrak{i}$ between (23) and (19) gives,

$$x\frac{d\mathfrak{B}}{dt} = -10^8\left[x\frac{d^2\mathfrak{B}}{dx^2} + \frac{d\mathfrak{B}}{dx}\right]\rho\frac{10}{4\pi\mu},$$

that is,

$$\frac{d^2\mathfrak{B}}{dx^2} + \frac{1}{x}\frac{d\mathfrak{B}}{dx} + \frac{4\pi\mu}{\rho 10^9}\frac{d\mathfrak{B}}{dt} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

Since $\mathfrak{B}$ is a sine function of the time, then

$$\mathfrak{B} = B_{max}e^{j\omega t}, \text{ so that } \frac{d\mathfrak{B}}{dt} = j\omega\mathfrak{B},$$

and by substituting this expression in equation (24) the general equation for the distribution of the alternating magnetic flux density throughout the cross-sectional area of the wire is obtained, viz.

$$\boxed{\frac{d^2\mathfrak{B}}{dx^2} + \frac{1}{x}\frac{d\mathfrak{B}}{dx} + j\frac{4\pi\mu\omega}{\rho 10^9}\mathfrak{B} = 0} \quad . \quad . \quad . \quad . \quad (25)$$

and substituting as before (see page 435),

$$\mathfrak{Y}^2 = j\left(\frac{4\pi\mu\omega}{\rho 10^9}\right) = \frac{4\pi\mu\omega}{\rho 10^9}\epsilon^{j\frac{\pi}{2}}$$

then

$$\mathfrak{Y} = \sqrt{j}\sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}}\Big|\epsilon^{j\frac{\pi}{4}} = \sqrt{j}\,|Y$$

and the solution of equation (25) may then be written

$$\mathfrak{B} = CJ_0(\mathfrak{Y}x) = CJ_0(\sqrt{j}\,|Yx) \quad . \quad . \quad . \quad . \quad (26)$$

where C is a constant and $J_0(\mathfrak{Y}x)$ is the Bessel Function of zero order.

If the flux density at the surface of the wire is $B_{max.}$, that is, for $x = a$ the radius of the wire, then

$$\boxed{\mathfrak{B} = B_{max.}\frac{J_0(\sqrt{j}|Yx)}{J_0(\sqrt{j}|Ya)}} \qquad . \quad . \quad . \quad (27)$$

Using the same notation as on page 436, this expression may be written

$$\boxed{\mathfrak{B} = B_{max.}\frac{J_0(y_x \measuredangle 45°)}{J_0(y_a \measuredangle 45°)} = B_{max.}(q \measuredangle \phi)} \qquad . \quad . \quad (28)$$

where $y_x = Yx : \; y_a = Ya$

The value of the Bessel Function $J_0(y_x \measuredangle 45°)$ may be obtained by reference to Table I if it be observed that the angle ϕ obtained from this table must be given the negative sign.

EXAMPLE.—For an iron wire having a magnetic permeability $\mu = 3{,}000$ and a specific resistance $\rho = 20 \times 10^{-6}\ \Omega/\text{cm.}/\text{cm.}^2$.

$$Y = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = \sqrt{\frac{8\pi^2 \times 3{,}000}{20 \times 10^3}}\sqrt{f} = 3{\cdot}43\sqrt{f}.$$

If the radius of the wire is $a = 0{\cdot}1$ cm., the flux density at the centre, i.e. for $x = 0$, will be given by the expression (28) as

$$[\mathfrak{B}]_{x=0} = B_{max.}\frac{1}{J_0(y_a \measuredangle 45°)}.$$

and for a frequency of 50,

$$[y_x]_{x=a} = Ya = 3{\cdot}43\sqrt{50} \times 0{\cdot}1 = 2{\cdot}42.$$

From Table I it is found that

$$J_0(y_a \measuredangle 45°) = 1{\cdot}45 \measuredangle - 70{\cdot}5°,$$

since, as already stated, the angle obtained from Table I must now be given the negative sign. Hence

$$[\mathfrak{B}]_{x=0} = \frac{B_{max.}}{1{\cdot}45e^{-j70\cdot5}} = \frac{B_{max.}}{1{\cdot}45}e^{j70\cdot5} = 0{\cdot}69B_{max.}e^{j70\cdot5}$$

Effective Resistance to Alternating Current of a Long Straight Wire

The non-uniform distribution of the current density as derived in the foregoing investigation involves an increase of the effective resistance of the wire as compared with the resistance when measured by means of direct current, and the value of the effective resistance may be calculated as follows.

At the surface of the wire there will be no back e.m.f. due to self-

induction, so that for a surface layer of radial thickness δ cm. and 1 cm. length of the wire the applied p.d. per centimetre will be given by

$$\mathfrak{v} = \dot{i}_{max}\,(2\pi a\delta)\frac{\rho}{2\pi a\delta} = \dot{i}_{max}\,\rho \text{ volts,}$$

where a cm. is the radius of the wire.

If $\dot{i}_{av}$ is the mean value of the current density for the non-uniform current distribution of alternating current, and if $\mathfrak{z}$ is the specific impedance of the conductor material and ρ the specific resistance, then

$$\mathfrak{v} = \dot{i}_{av}\,\mathfrak{z} = \dot{i}_{max}\,\rho,$$

so that

$$\frac{\mathfrak{z}}{\rho} = \frac{\dot{i}_{max}}{\dot{i}_{av}},$$

that is

$$\frac{\dfrac{\mathfrak{z}}{\pi a^2}}{\dfrac{\rho}{\pi a^2}} = \frac{\mathfrak{Z}}{R_{dc}} = \frac{\dot{i}_{max}}{\dot{i}_{av}} \qquad \text{(29)}$$

where $\mathfrak{Z}$ is the impedance vector of the wire and R_{dc} the resistance to direct current.

The mean value of the current density $\dot{i}_{av}$ may be found as follows. The total alternating current flowing in the wire (Fig. 3a) will be

$$I_T = \int_0^a \mathrm{i}_x 2\pi x\,dx = \int_0^a 2\pi x J_0(\sqrt{-j}\,|Yx)\,dx.$$

But

$$\int_0^x xJ_0(\sqrt{-j}\,|Yx)\,dx = \frac{x}{\sqrt{-j}}\,YJ_1(\sqrt{-j}\,|Yx),$$

where $J_1(\sqrt{-j}\,|Yx)$ is the Bessel Function of the *first order* and can be found from Table I. Hence the total alternating current flowing in the wire will be

$$I_T = \int_0^a 2\pi x\dot{i}_{max}\frac{J_0(\sqrt{-j}\,|Yx)}{J_0(\sqrt{-j}\,|Ya)}\,dx = \frac{2\pi\dot{i}_{max}}{\sqrt{-j}\,|YJ_0(\sqrt{-j}\,|Ya)}\Big\{xJ_1(\sqrt{-j}\,|Yx)\Big\}_0^a$$

that is

$$\boxed{I_T = \frac{2\pi\dot{i}_{max}\,a}{YJ_0(\sqrt{-j}\,|Ya)}\Big\{J_1(\sqrt{-j}\,|Ya)\Big\}e^{j45^\circ}} \qquad \text{(30)}$$

and the mean current density is $\dot{i}_{av} = \dfrac{I_T}{\pi a^2}$, so that from (29)

$$\frac{\text{Impedance to alternating current}}{\text{Resistance to direct current}} = \left[\frac{aY}{2}\right]\left[\frac{J_0(\sqrt{-j}\,|Ya)}{J_1(\sqrt{-j}\,|Ya)}\right]e^{-j45^\circ} \qquad \text{(31)}$$

The ratio $\dfrac{\text{Resistance to alternating current}}{\text{Resistance to direct current}}$ will then be given by the real component of the expression (31).

If, then, in Table I, page 437,*

ϕ_0 is the angle of the Bessel Function $J_0(\sqrt{-j}|Ya)$

ϕ_1 ,, ,, ,, ,, ,, ,, ,, $J_1(\sqrt{-j}|Ya)$

the value of the resistance to alternating current will be obtained from the following expression:

$$\frac{\text{Resistance to alternating current}}{\text{Resistance to direct current}} = \frac{YaJ_0(\sqrt{-j}|Ya)}{2J_1(\sqrt{-j}|Ya)} \cos(\phi_0 - \phi_1 - 45)^\circ. \quad (32)$$

EXAMPLE (*i*).—A copper wire of 0·6 cm. radius:

$\mu = 1 : \rho = 1{\cdot}77 \times 10^{-6}\ \Omega/\text{cm.}/\text{cm.}^2 : f = 500 : \omega = 2\pi f = 3{,}140 :$

$$Y = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = 4{\cdot}73 : \quad Ya = 2{\cdot}86 :$$

so that from Table I, page 437,

$$J_0(\sqrt{-j}|2{\cdot}86) = 1{\cdot}8e^{j90} : \quad J_1(\sqrt{-j}|2{\cdot}86) = 1{\cdot}67e^{j10{\cdot}3}.$$

and from expression (32)

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{2{\cdot}86}{2} \times \frac{1{\cdot}8}{1{\cdot}67} \cos(90^\circ - 10{\cdot}3^\circ - 45^\circ)$$

$$= 1{\cdot}43 \times 1{\cdot}08 \cos 34{\cdot}7^\circ = 1{\cdot}26.$$

EXAMPLE (*ii*).—An iron wire of 0·2 cm. radius:

$\mu = 1{,}000 : f = 50 : \omega = 2\pi f = 314 : \rho = 12 \times 10^{-6}\ \Omega/\text{cm.}/\text{cm.}^2$

$$Y = \sqrt{\frac{4\pi \times 1{,}000 \times 314}{12 \times 10^3}} = 18{\cdot}2 : \quad Ya = 3{\cdot}64 :$$

$$J_0(\sqrt{-j}\,3{\cdot}64) = 2{\cdot}8e^{j123{\cdot}8} : \quad J_1(\sqrt{-j}|3{\cdot}64) = 2{\cdot}63e^{j40{\cdot}23^\circ}.$$

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{3{\cdot}64}{2} \times \frac{2{\cdot}8}{2{\cdot}63} \cos(123{\cdot}8 - 40{\cdot}23 - 45^\circ)$$

$$= 1{\cdot}82 \times 1{\cdot}065 \cos 38{\cdot}57^\circ = 1{\cdot}51.$$

The Distribution of Alternating Magnetic Flux in Laminated Iron

In Fig. 7*a* is shown a cross-section of an iron lamination of thickness $2a$ cm. and for the purpose of this investigation a depth of 1 cm., i.e. perpendicular to the plane of the paper and a length of 1 cm. in the plane of the paper. The direction of the alternating magnetic flux is in the plane of the paper (Fig. 7*b*) and the direction of flow of the induced

* The expression $J_0(\sqrt{-j}\,|\,y_x)$ $J_0(\sqrt{-j}\,|\,Y.x)$ is, of course, equivalent to $J_0(y_x \measuredangle -45^\circ)$ of Table I, as has been explained already on page 436.

eddy currents will consequently be in the plane of the paper (Fig. 7*a*). The two shaded strips are symmetrically placed with respect to the median plane and represent a typical path of flow for the induced eddy currents. Since the conditions for each of the shaded strips are precisely the same, it will be convenient to confine attention to one such strip only as shown in the diagrams of Fig. 7.

The vector of current density at the distance x cm. from the median line is $\mathfrak{i}$ amperes per cm.^{2} and $\mathfrak{I}$ is the vector of the total current which flows between the median plane and the inner surface of the shaded strip. The induced e.m.f. along the strip at the distance x cm. from the median

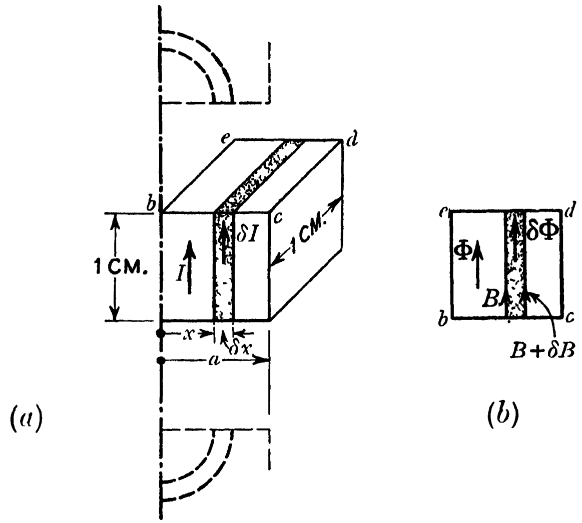


Fig. 7.

plane is $\mathfrak{e}$ per centimetre length in the plane of the paper. The vector of magnetic force is $\mathfrak{H}$ and the vector of magnetic induction is $\mathfrak{B}$, Fig. 7*b*. The following relationships may then be easily derived :

$$\frac{4\pi}{10}\mathfrak{I} = \mathfrak{H} \times 1 \text{ cm.} : \quad \frac{4\pi}{10}(\mathfrak{I} + \mathfrak{i}.\delta x) = \mathfrak{H} + \delta\mathfrak{H} : \quad \mathfrak{I} = \mathfrak{i}.\delta x$$

$$\frac{4\pi}{10}.\mathfrak{i}.\delta x = \delta\mathfrak{H} = \frac{1}{\mu}\,\delta\mathfrak{B},$$

so that

$$\frac{d\mathfrak{B}}{dx} = \frac{4\pi\mu}{10}.\mathfrak{i} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

The magnetic flux in the elementary strip is $\Delta\phi = \mathfrak{B}\delta x$, so that the induced e.m.f. in the strip will be given by the equation

$$10^{-8}\frac{d(\mathfrak{B}\delta x)}{dt} = -\,d\mathfrak{e}$$

that is, $$\frac{d\mathfrak{B}}{dt} = -10^{8}\frac{de}{dx}. \qquad (34)$$

The current which flows in the strip is driven by the induced e.m.f. $\mathfrak{e}$, so that $\mathfrak{e} = \delta\mathfrak{J}.r$: where $r = \frac{\rho}{\delta x}\ \Omega$ and is the resistance of the strip, and hence

$$\mathfrak{e} = \rho.\mathfrak{i} \qquad (35)$$

From (35) $\frac{d\mathfrak{e}}{dx} = \rho\frac{d\mathfrak{i}}{dx}$: and from (33) $\frac{d\mathfrak{i}}{dx} = \frac{10}{4\pi\mu}\frac{d^2\mathfrak{B}}{dx^2}$, hence

$$\frac{de}{dx} = \frac{10\rho}{4\pi\mu}\frac{d^2\mathfrak{B}}{dx^2},$$

and the combination of this expression with (34) gives

$$\frac{10\rho}{4\pi\mu}\frac{d^2\mathfrak{B}}{dx^2} = -10^{-8}\frac{d\mathfrak{B}}{dt},$$

that is

$$\boxed{\frac{d^2\mathfrak{B}}{dx^2} = -j\frac{4\pi.\omega.\mu}{10^9\rho}\mathfrak{B}} \qquad (36)$$

Similarly, from the three principal equations (33), (34) and (35) the differential equation for the eddy current density can be obtained, that is

$$\boxed{\frac{d^2\mathfrak{i}}{dx^2} = -j\left(\frac{4\pi\mu\omega}{\rho 10^9}\right).\mathfrak{i}} \qquad (37)$$

which is identical in form with equation (36).

The solution of equation (36) is

$$\mathfrak{B} = \mathfrak{A}\ \text{Cosh}\ \mathfrak{S}x,$$

where $\mathfrak{S}^2 = -j\frac{4\pi\mu\omega}{\rho 10^9}$: that is $\mathfrak{S} = (1-j)\sqrt{2\pi\frac{\mu\omega}{\rho 10^9}}$, since $\sqrt{-j} = \epsilon^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1-j)$. If B_{max} is the induction density at the surface of the lamination, that is, when $x = a$, then the arbitrary constant is $\mathfrak{A} = \frac{B_{max}}{\text{Cosh}\ \mathfrak{S}a}$, so that the solution of equation (37) then becomes

$$\boxed{\mathfrak{B}_x = B_{max}\frac{\text{Cosh}\ \mathfrak{S}x}{\text{Cosh}\ \mathfrak{S}a}} \qquad (38)$$

Writing $\alpha = \sqrt{2\pi\frac{\mu\omega}{\rho 10^9}}$, then $\mathfrak{S} = \alpha - j\alpha$,

so that

$$\text{Cosh } \mathfrak{S}x = \text{Cosh } (\alpha x - j\alpha x) = (\text{Cosh } \alpha x \cos \alpha x - j \text{ Sinh } \alpha x \sin \alpha x),$$

or

$$\text{Cosh } \mathfrak{S}x = s_x e^{-j\phi_x}$$

where

$$s_x = \sqrt{\text{Cosh}^2 \alpha x \cos^2 \alpha x + \text{Sinh}^2 \alpha x \sin^2 \alpha x} = \sqrt{\tfrac{1}{2}(\text{Cosh } 2\alpha x + \cos 2\alpha x)}$$

$$\tan \phi_x = \text{Tanh } \alpha x \tan \alpha x,$$

and hence

$$\boxed{\mathfrak{B}_x = B_{max.}\frac{s_x}{s_a}.e^{-j(\phi_x - \phi_a)}} \qquad . \quad . \quad . \ (39)$$

The symbol α in the foregoing is related to the symbol Y of page 438, viz.

$$Y = \sqrt{2}\alpha.$$

EXAMPLE.—For an iron lamination $\mu = 3{,}000$: $f = 10{,}000$: $\omega = 62{,}800$: $\rho = 12 \times 10^{-6}$ Ω/cm./cm.^{2} : $a = 0{\cdot}01$ cm.

$$\alpha = \sqrt{\frac{2\pi\mu\omega}{\rho 10^9}} = 314 : \ 2\alpha.a = 6{\cdot}28 = 360^\circ : \ \alpha.a = 3{\cdot}14 = 180^\circ$$

$$s_a = \sqrt{\tfrac{1}{2}(\text{Cosh } 2\alpha.a + \cos 2\alpha.a)} = \sqrt{\tfrac{1}{2}} \times 281 = 11{\cdot}85$$

$$\tan \phi_a = \text{Tanh } \alpha.a \times \tan \alpha.a = \text{Tanh } 3{\cdot}18 \times \tan 180^\circ : \ \phi_a = 180^\circ.$$

For $x = 0$: $\phi_x = 0$: $s_x = 1$, so that

$$\mathfrak{B}_{x=0} = B_{max.}\frac{1}{11{\cdot}85}e^{+j180} = 0{\cdot}085 B_{max.}e^{j180^\circ}.$$

That is to say, the vector of the induction density at the median plane of the lamination is exactly opposite in phase to the vector of induction density at the surface.

At the distance $x = \frac{1}{4}a$ from the median plane :

$$2\alpha.a = 1{\cdot}57 = 90^\circ : \ \alpha.a = 0{\cdot}785 = 45^\circ :$$

$$s_x = \sqrt{\tfrac{1}{2}(\text{Cosh } 1{\cdot}57 + \cos 90^\circ)} = 1{\cdot}08 :$$

$$\tan \phi_x = \text{Tanh} \times 0{\cdot}785 \times \tan 45^\circ = 0{\cdot}625,$$

so that

$$\phi_x = 33^\circ.$$

therefore

$$\mathfrak{B}_{x=\frac{1}{4}a} = B_{max.}\frac{1{\cdot}08}{11{\cdot}85}e^{-j(33^\circ - 180^\circ)} = 0{\cdot}091\ B_{max.}e^{j147^\circ}$$

The Penetration of Alternating Current in Laminated Conductors

In Fig. 8 is shown a view of the lamination which corresponds to that of Fig. 6, that is to say, the thickness of the lamination is $2a$ cm. and, as before, the conditions are symmetrical each side of the median plane *hbf*. An elementary cube of the material *bfegch* of 1-cm. sides

is shown and the alternating current is assumed to be flowing in a direction at right angles to the plane of the paper so that the magnetic flux in this case will be in the direction of the plane of the paper. The following

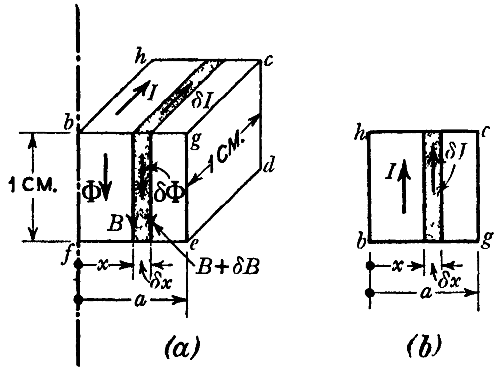


Fig 8.

relationships can then be easily derived in a precisely similar way to those on page 440.

$$\frac{4\pi}{10}\mathfrak{I} = \mathfrak{H} \times 1 \text{ cm.} : \quad \frac{4\pi}{10}(\mathfrak{I} + \mathfrak{i}.\delta x) = (\mathfrak{H} + \delta\mathfrak{H}) : \quad \frac{4\pi}{10}.\, \mathfrak{i}.\delta x = \delta\mathfrak{H} :$$

so that, $$\frac{4\pi.\mathfrak{i}.\delta x}{10} = \delta\mathfrak{H} : \quad \text{and } \mathfrak{B} = \mu.\mathfrak{H},$$

therefore $$\frac{d\mathfrak{B}}{dx} = \frac{4\pi\mu}{10}.\mathfrak{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Similarly, if $\mathfrak{v}$ is the applied p.d. per centimetre length of the current path, then (see also expression (36))

$$\mathfrak{v} = \mathfrak{e} + r\delta\mathfrak{I} = \mathfrak{e} + \rho.\mathfrak{i}$$

so that $$\frac{d\mathfrak{e}}{dx} = -\rho\frac{d\mathfrak{i}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The magnetic flux in the shaded strip is $\delta\phi = \mathfrak{B}\delta x$, so that

$$\mathfrak{e} = -10^{-8}\frac{d}{dt}(\delta\phi) = -10^{-8}\delta x\frac{d\mathfrak{B}}{dt}$$

or $$\frac{d\mathfrak{e}}{dx} = -\frac{1}{10^8}\frac{d\mathfrak{B}}{dt} \quad . \quad . \quad . \quad . \quad (42)$$

From (40), (41), and (42) it easily follows that,

$$\frac{d^2\mathfrak{i}}{dx^2} = \frac{4\pi\mu}{\rho 10^9}\frac{d\mathfrak{i}}{dt}$$

that is,
$$\frac{d^2\mathfrak{i}}{dx^2} = j\frac{4\pi\mu\omega}{\rho.10^9}\mathfrak{i} \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The solution of this equation is
$$\mathfrak{i} = \mathfrak{A}\ \text{Cosh}\ \mathfrak{S}x,$$

where
$$\mathfrak{S}^2 = j\frac{4\pi\mu\omega}{\rho 10^9} : \quad \mathfrak{S} = \sqrt{j}\sqrt{\frac{4\pi\mu\omega}{\rho 10^9}} = (1+j)\sqrt{\frac{2\pi\mu\omega}{\rho 10^9}}$$

or
$$\mathfrak{S} = \alpha + j\alpha : \quad \text{where } \alpha = \sqrt{\frac{2\pi\mu\omega}{\rho 10^9}}.$$

If i_{max} is the current density at the surface of the lamination, then
$$\mathfrak{A} = i_{max}/\text{Cosh}\ \mathfrak{S}a$$
so that
$$\mathfrak{i} = i_{max}\frac{\text{Cosh}\ \mathfrak{S}x}{\text{Cosh}\ \mathfrak{S}a} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

But
$$\text{Cosh}\ \mathfrak{S}x = \text{Cosh}\ (\alpha x + j\alpha x) = \text{Cosh}\ \alpha x \cos \alpha x + j\ \text{Sinh}\ \alpha x \sin \alpha x,$$
that is
$$\text{Cosh}\ \mathfrak{S}x = s_x e^{j\phi_x},$$
where
$$s_x = \sqrt{\tfrac{1}{2}(\text{Cosh}\ 2\alpha x + \cos 2\alpha x} : \quad \text{and } \tan\phi_x = \text{Tanh}\ \alpha x \tan \alpha x :$$
so that
$$\boxed{\mathfrak{i} = i_{max}\frac{s_x}{s_a}e^{j(\phi_x - \phi_a)}} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

EXAMPLE.—Take the same numerical as has been given in the previous section on page 447 for the iron lamination :
$$\mu = 3{,}000 : \quad f = 10{,}000 : \quad \rho = 12 \times 10^{-6}\ \Omega/\text{cm./cm.}^2$$
$$a = 0{\cdot}01 \text{ cm.} : \quad \alpha = 314.$$

It will be seen that the current density at the median plane $\mathfrak{i}_{x=0}$ is related to the current density at the surface by the equation
$$\mathfrak{i}_{x=0} = \frac{i_{max}}{11{\cdot}85}e^{-j180} = 0{\cdot}085 i_{max} e^{-j180^\circ},$$
that is to say, the current at the median plane lags by 180° on the current at the surface.

It is to be observed that equation (45) may be written in the form
$$\mathfrak{i} = i_{max}\frac{\text{Cosh}\ \mathfrak{S}x}{\text{Cosh}\ \mathfrak{S}a} = \frac{i_{max}}{2\mathfrak{K}}\{e^{\alpha x}e^{j\alpha x} + e^{-\alpha x}e^{-j\alpha x}\}$$

and at any moment t
$$\mathfrak{i}_t = \mathfrak{i}.e^{j\omega t} = \frac{i_{max}}{2\mathfrak{K}}\{e^{\alpha x}e^{j(\omega t + \alpha x)} + e^{-\alpha x}e^{j(\omega t - \alpha x)}\}$$

The expression on the right-hand side comprises *two travelling waves* (Fig. 8).

(*i*) The incident wave $\frac{\dot{i}_{max}}{2\Re} e^{\alpha x} e^{j(\omega t + \alpha x)}$

of which the *amplitude decreases as x decreases.* That is to say, *this is a damped forward wave travelling from the surface of the lamination towards the median plane.*

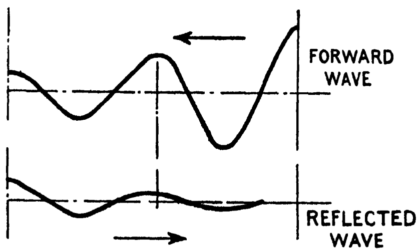


Fig. 9.

(*ii*) The reflected wave (Fig. 9)

$$\frac{\dot{i}_{max}}{2\Re} e^{\alpha x} e^{+j(\omega t - \alpha x)},$$

of which *the amplitude decreases as x increases.* That is to say, *this is a damped reflected wave travelling from the median plane towards the surface of the lamination.*

In each case the wave length λ cm. is given by the equation

$$\alpha . \lambda = 2\pi : \text{ that is, } \lambda = \frac{2\pi}{\alpha} \text{ cm.,}$$

so that in the foregoing numerical example for iron laminations in which $f = 10{,}000$: $\alpha = 314$,

$$\lambda = \frac{2\pi}{314} = 0{\cdot}02 \text{ cm.} = 0{\cdot}2 \text{ mm.}$$

The speed of travel of the waves may be found by the following considerations. In the time of one cycle of the applied p.d. the wave will have travelled a distance equal to one wave-length, that is, the wave will travel λ cm. in $\frac{1}{f}$ second, so that the velocity of travel will be

$$\lambda . f = \frac{2\pi f}{\alpha} = \frac{\omega}{\alpha} \text{ cm. per second.}$$

In the foregoing numerical example referring to iron laminations

$f = 10{,}000$: $\omega = 62{,}800$: $\alpha = 314$: so that the speed of travel will be

$$\frac{62{,}800}{314} = 200 \text{ cm. per second.}$$

For copper laminations, if $f = 500$: $\rho = 1{\cdot}77 \times 10^{-6}$: $\mu = 1$, then

$$\alpha = \sqrt{\frac{2\pi\mu\omega}{\rho 10^9}} = \sqrt{10{\cdot}6} = 3{\cdot}35,$$

and the speed of travel will be

$$\frac{3{,}140}{3{\cdot}35} = 935 \text{ cm. per second.}$$

Equivalent Depth of Penetration of Alternating Current

It has been seen in the foregoing equation (44) that the vector of current density at a distance x cm. from the median plane of a laminated conductor is given by the expression

$$\dot{\mathrm{i}}_x = \dot{i}_{max} \frac{\text{Cosh } \mathfrak{S}x}{\text{Cosh } \mathfrak{S}a},$$

where i_{max} is the current density in amperes per square centimetre at the surface of the lamination, also the vector $\mathfrak{S} = \alpha + j\alpha$, the factor $\alpha = \sqrt{\frac{2\pi\mu\omega}{\rho 10^9}}$ and $2a$ cm. is the thickness of the lamination, so that

$$\dot{\mathrm{i}}_x = \dot{i}_{max} \frac{\text{Cosh } (\alpha x + j\alpha x)}{\text{Cosh } (\alpha a + j\alpha a)},$$

that is

$$\boxed{\dot{\mathrm{i}}_x = \dot{i}_{max} \left[\frac{e^{\alpha x} e^{j\alpha x} + e^{-\alpha x} e^{-j\alpha x}}{e^{\alpha a} e^{j\alpha a} + e^{-\alpha a} e^{-j\alpha a}}\right]} \quad . \quad . \quad . \quad (46)$$

In this expression the terms involving the negative powers of e refer to the current wave which is reflected from the median plane as explained on page 450. If the factor α is large, e.g. if the frequency is large, or if the thickness $2a$ of the lamination is large, then the reflected current wave becomes negligibly small, so that the terms which involve the negative powers of e may be neglected. The expression (46) will then become

$$\dot{\mathrm{i}}_x = \dot{i}_{max} \left(\frac{e^{\alpha x} e^{j\alpha x}}{e^{\alpha a} e^{j\alpha a}}\right) = \dot{i}_{max} e^{\alpha(x-a)} e^{j\alpha(x-a)}$$

and the magnitude of the current density vector is therefore given by

$$|\,\dot{\mathrm{i}}_x\,| = \dot{i}_{max} e^{-\alpha(a-x)} = \dot{i}_{max} e^{-\alpha s} \quad . \quad . \quad . \quad (47)$$

where s cm. is the distance measured from the surface of the lamination.

If this expression is plotted as a function of the distance s the logarithmic curve shown in Fig. 10 will be obtained.* It is a well known property of such an exponential curve that a tangent drawn to the curve at the point A will cut the abscissa at a point B where $GB = \frac{1}{\alpha}$. Further, the area enclosed between this curve and the co-ordinate axes will be given by the expression

$$\text{Area} = \int_0^\infty |\,i\,|\, ds = i_{max} \int_0^\infty e^{-\alpha s} ds = i_{max} \cdot \frac{1}{\alpha} \quad . \quad . \quad (48)$$

That is to say, the area of the shaded rectangle in Fig. 10 will be equal to the area enclosed between the logarithmic curve and the co-ordinate

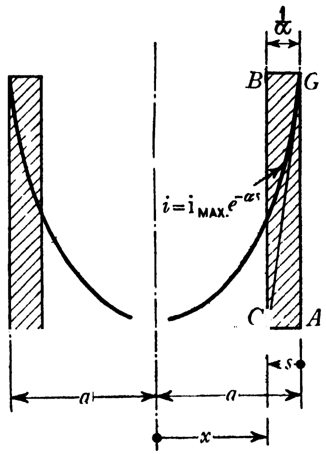


Fig. 10.

axes. The total current, therefore, which flows across each square centimetre of cross section of the lamination, e.g. in Fig. 8, where $a \times (bf) = 1$ square centimetre, will be the same as if the current density were uniform for a depth of σ cm. and equal to the actual current density at the surface, that is

$$\sigma = \frac{1}{\alpha} = \sqrt{\frac{\rho 10^9}{2\pi\mu 2\pi f}} = 5{,}030\sqrt{\frac{\rho}{\mu . f}} \text{ cm.} \quad . \quad . \quad (49)$$

and σ is termed the "equivalent depth of penetration". This quantity σ provides a valuable basis for comparative data on the penetration of alternating currents as dependent upon frequency, permeability, and specific resistance, it being observed that it is only applicable when the conditions assumed in the foregoing are reasonably closely fulfilled. For example, the following table shows the values of σ at various frequencies for copper at a temperature of 18° C. and iron at a temperature of

* See also *The Engineer*, March 6, 1942, p. 209.

1,500° C., the latter condition, for example, being such as would be found in the case of a high-frequency furnace. For the data in this table the specific resistance for copper at 18° C. has been taken to be

$$\rho = 1{\cdot}72 \times 10^{-6}\ \Omega/\text{cm./cm.}^2$$

and for iron at 1,500° C. the value for ρ has been taken as

$$150 \times 10^{-6}\ \Omega/\text{cm./cm.}^2$$

Frequency f *hertz*	*Copper at* 18° C. σ *cm.*	*Iron at* 1,500° C. σ *cm*
10,000	0·065	0·61
1,000	0·21	1·90
500	0·30	2·70

With regard to the expression (49) for the depth of penetration σ, it is to be observed that the angle of phase displacement has been neglected : this is a sufficiently good approximation when σ is sufficiently small.

Some Useful Practical Formulae for Skin Effect

The equivalent depth of penetration of alternating current has been for a long straight wire, and is given by the expression (49), viz.,

$$\sigma = \frac{1}{2\pi}\sqrt{\frac{\rho . 10^9}{\mu . f}}\ \text{cm.}$$

that is,

$$\sigma = 5{,}030\sqrt{\frac{\rho}{\mu . f}}$$

where ρ is the specific resistance in Ω/cm./cm.^{2} and f is the frequency in hertz. This expression may be re-written in one or other of the following forms :

$$\sigma = \frac{50}{\pi}\sqrt{\frac{\rho'}{f'\mu}}\ \text{mm. : } \left.\begin{array}{l}\text{for } \rho' \text{ in } \Omega/\text{m./mm.}^2 \\ \text{,, } f' \text{ ,, kHz.}\end{array}\right\} \quad . \quad . \quad \text{(i)}$$

$$\sigma = \frac{50}{\pi}\sqrt{\frac{\rho''}{f\mu}}\ \text{mm. } \left.\begin{array}{l}\text{for } \rho'' \text{ in } \Omega/\text{km./mm.}^2 \\ \text{,, } f \text{ ,, hz.}\end{array}\right\} . \quad . \quad \text{(ii)}$$

$$\sigma = \frac{5}{\pi^2}\sqrt{\frac{\rho''}{f'\mu}}\ \text{mm. : } \left.\begin{array}{l}\text{for } \rho'' \text{ in } \Omega/\text{km./mm.}^2 \\ \text{,, } f' \text{ ,, kHz.}\end{array}\right\} \quad . \quad \text{(iii)}$$

For copper wire the respective expressions for the specific resistance are :

$$\rho = \frac{1{\cdot}76}{10^6} = \frac{1}{570{,}000}\ \Omega/\text{cm./cm.}^2$$

$$\rho' = \frac{1{\cdot}76}{10^2} = \frac{1}{57}\ \Omega/\text{m./mm.}^2$$

$$\rho'' = 17{\cdot}6 = \frac{10^3}{57}\ \Omega/\text{km./mm.}^2$$

See also Appendix II, Table I, page 529.

EXAMPLE.—A single-phase overhead line comprises two copper conductors each of $2a$ mm. diameter. The resistance to d.c. for the "go" and "return" lines in series will be $\frac{2\rho''}{\pi a^2}$. The resistance to alternating current is (see Fig. 11)

$$\frac{2\rho''}{2\pi a\sigma},$$

so that the ratio

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \frac{a}{2\sigma}.$$

Since the permeability in this case is $\mu = 1$, then from expression (ii) above

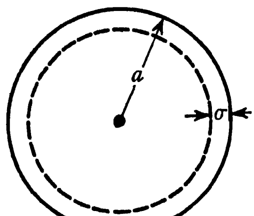


Fig. 11.

$$\sigma = \frac{50}{\pi}\sqrt{\frac{\rho''}{f}} = \frac{67}{\sqrt{f}},$$

so that

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 0{\cdot}0075a\sqrt{f}$$

and the resistance to a.c. $= 0{\cdot}0075a\frac{2\rho''}{\pi a^2}\sqrt{f}$

$$= \frac{0{\cdot}085}{a}\sqrt{f}\ \Omega/\text{km. run.}$$

If $a = 2$ mm. : $f = 10^6$ hz.,

$$\text{resistance to a.c.} = 0{\cdot}042\sqrt{f} = 42\ \Omega/\text{km. run,}$$

$$\text{,,}\quad \text{,, d.c.} = \frac{42}{0{\cdot}0075 \times 2\sqrt{10^6}} = 2{\cdot}8\ \Omega/\text{km. run.}$$

Some Further Useful Practical Formulae

The following formulae cover all practical cases with a high degree of accuracy when applied within the specified limits :

Denoting as before (page 444) $Y = \sqrt{\frac{4\pi\mu\omega}{\rho 10^9}}$ and a cm. the radius of the wire then for :

$Ya > 2{\cdot}82$:

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 0{\cdot}325\ Ya + 0{\cdot}25 + \frac{0{\cdot}134}{Ya}$$

$Ya = 2{\cdot}82$:

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 1{\cdot}26$$

$Ya < 2{\cdot}82$:

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 1 + \frac{(Ya)^4}{190} - \frac{(Ya)^8}{46{,}100}.$$

The following formulae also give accurate results within the limits specified :

(*i*) *For Large Skin Effects.* That is for

$$Ya > 50 : \text{ or } a\sqrt{\frac{f\mu}{\rho}} > 1{\cdot}8 \times 10^5.$$

$$\text{Resistance to a.c.} = \frac{1}{a}\sqrt{f\mu\rho 10^{-9}}\ \Omega/\text{cm.}$$

$$\text{Resistance to d.c.} = \frac{\rho}{\pi a^2}\ \Omega/\text{cm.}$$

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \pi a\sqrt{\frac{f\mu}{\rho 10^9}} = \frac{a}{2\sigma}\ \text{(Fig. 11)}$$

(*ii*) *For Small Skin Effects.* That is for $Ya < 2{\cdot}82$

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \left\{1 + \frac{1}{3}\left[\pi a\sqrt{\frac{\mu f}{\rho 10^9}}\right]^4\right\}.$$

The test for all these practical formulae is the application of the mathematical results derived in the foregoing from the Table I for the Bessel Functions.

A comprehensive survey of comparative results obtained by means of the foregoing approximate formulae and by the exact equating will be found in Appendix II, page 529.

Example for the Comparison of the Approximate and Exact Formulae for the Ratio $\frac{R_{a.c.}}{R_{d.c.}}$.

$a = 2$ mm. : $\mu = 500$: $f = 50$ hz. : $\rho = 12 \times 10^{-6}\ \Omega$/cm./cm.^{2}
$\sigma - 0{\cdot}083 \times 10^6$.

$$Y = \sqrt{\frac{4\pi.\mu.\omega}{\rho.10^9}} - \sqrt{\frac{4\pi \times 500 \times 314}{12 \times 10^3}}\quad 12{\cdot}4.$$

$$Y.a = 2{\cdot}48 : \ (Y.a)^2 = 0{\cdot}134 : \ (Y.a)^4 = 0{\cdot}0179.$$

From the formula :

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 1 + \frac{(Y.a)^2}{190} - \frac{(Y.a)^8}{46{,}100}$$

$$= 1 + \frac{(2{\cdot}48)^4}{190} - \frac{(2{\cdot}48)^4}{46{,}100}$$

$$= 1{\cdot}185.$$

From the formula :

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = \left\{1 + \frac{1}{3}\left[\pi a \sqrt{\frac{\mu . f}{\rho \times 10^9}}\right]^4\right\}$$

$$= \left\{1 + \frac{1}{3}\left[\pi \times 0{\cdot}2\sqrt{\frac{500 \times 50}{12 \times 10^3}}\right]^4\right\}$$

$$= \underline{1{\cdot}223.}$$

From the Bessel Functions, Table I, page 437.

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} \quad \frac{Y.a}{2} \times \frac{1{\cdot}49}{1.36}\cos(\phi_0 - \phi_1 - 45^\circ): \text{(see also page 444)}$$

$$= 1{\cdot}24 \times 1{\cdot}1 \times \cos 30{\cdot}7^\circ.$$

$$= \underline{1{\cdot}17.}$$

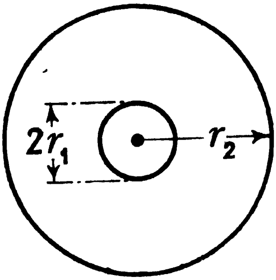


Fig. 12.

The Co-axial Cable.

In Fig. 12 is shown the arrangement of the core and sheath of one type of co-axial cable.

Russell's formula for the high-frequency resistance of such a cable is as follows :

$$R = 4{\cdot}12 \times 10^{-4}\left\{\frac{1}{r_1} + \frac{1}{r_2}\right\}\sqrt{f} \text{ ohms per km.} \quad . \quad . \quad (50)$$

in which r_1 and r_2 are in centimetres and f in hz.

The inductance of this cable is,

$$L = \left(2 \log_e \frac{r_2}{r_1}\right)10^{-4} \text{ henry/km.} \quad . \quad . \quad . \quad . \quad (51)$$

The capacitance of this cable is,

$$C = \frac{1}{1{\cdot}8}\,\frac{1}{\log_e \dfrac{r_2}{r_1}}\,10^{-7} \text{ farad/km.} \quad . \quad . \quad . \quad (52)$$

EXAMPLE.—

$$r_1 = 0{\cdot}161 \text{ cm.} : \; r_2 = 0{\cdot}95 \text{ cm.}$$

$$R = 41{\cdot}2 \times 10^{-4}\sqrt{f}\left\{\frac{1}{0{\cdot}161} + \frac{1}{0{\cdot}95}\right\}$$

that is, $$R = 298 \times 10^{-4}\sqrt{f} \text{ ohms per km.}$$

If
$$f = 40\text{M hz.} = 40{,}000 \text{ k hz.}$$
$$R = 188 \text{ ohms per km.}$$
$$L = (2 \log_e 5\cdot9) \times 10^{-4} = 3\cdot54 \times 10^{-4} \text{ henry/km.}$$
$$C = \frac{1}{1\cdot8} \times \frac{1}{\log_e 5\cdot9} \times 10^{-7} = 0\cdot031\ \mu\text{F/km.}$$

The surge impedance (see Chapter XV, page 459) is

$$Z_0 = \sqrt{\frac{L}{C}} \times \sqrt{\frac{3\cdot54 \times 10^8}{10^4 \times 3\cdot1}} = 107 \text{ ohms}$$

Chapter XV

PROPAGATION OF ELECTRIC ENERGY ALONG TRANSMISSION LINES AND CABLES

Velocity of an Electric Surge of Pressure and Current

CONSIDER an isolated bound charge of electricity which gives rise to a pressure hump as shown in Fig. 1. Such a pressure distribution may be produced on a transmission line, for example, by induction from a thunder-cloud A in the neighbourhood. If a discharge

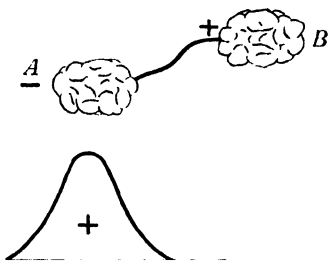


Fig. 1.

now takes place between the cloud A and another cloud B the induced charge on the line will be released and will then divide and travel in each direction with a speed c shown in Fig. $2a$ for the two component

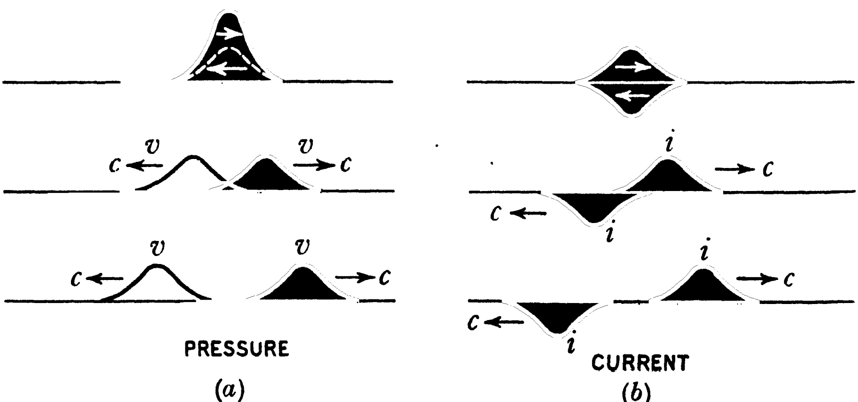


Fig. 2.

pressure waves, and in Fig. 2*b* for the corresponding current components.

Now consider an elementary length δx of a transmission line system

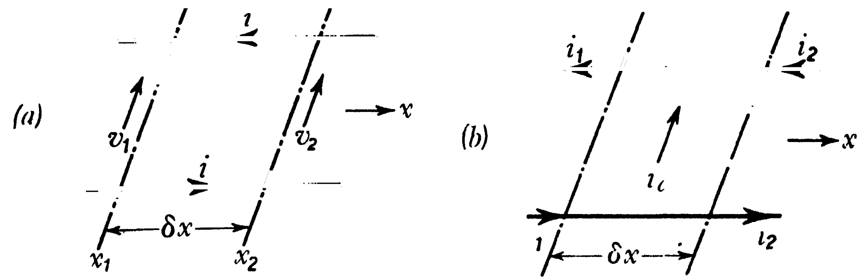


Fig. 3.

of negligible resistance and comprising two overhead conductors as shown in Fig. 3*a*. The pressure drop across this element will be

$$-\delta v = (v_1 - v_2) = L\delta x\frac{di}{dt},$$

where L henry is the inductance per unit length of the double line so that

$$\frac{dv}{dx} = -L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The current drop in the same element δx as shown in Fig. 3*b* will be

$$-\delta i = (i_1 - i_2) = C\delta x\frac{dv}{dt}$$

where C farad is the capacitance per unit length between the two lines, so that

$$\frac{di}{dt} = \quad C\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The two simultaneous equations (1) and (2) may be solved by assuming that $v = Z_0 i$ where Z_0 is a constant. Then, by substitution in equation (1) and (2) respectively, it is seen that

$$\left.\begin{aligned} Z_0\frac{di}{dx} &= -L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad (a) \\ \frac{di}{dx} &= -CZ_0\frac{di}{dt} \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

and, dividing (3*a*) by (3*b*) it is found that

$$Z_0^2 = \frac{L}{C}: \text{ that is, } Z_0 = \pm\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad (4)$$

and after substituting the value for Z_0 in the initially assumed relationship $v = Z_0 i$ it is found that

$$v = \pm \sqrt{\frac{L}{C}} . i \qquad . \quad . \quad . \quad . \quad . \qquad (5)$$

It is to be observed that Z_0 has a double sign and from what follows it will be seen that these two signs correspond to the two directions of movement of the surge as shown in Fig. 2. That is to say, for a surge moving in the positive direction of x

$$Z_0 = + \sqrt{\frac{L}{C}},$$

whilst for the surge moving in the negative direction of x, that is, for example, for a " reflected wave ", the relationship is,

$$Z_0 = - \sqrt{\frac{L}{C}}.$$

Now consider a point P on the travelling wave of current as shown in Fig. 4, the speed of travel being c cm./sec. in the direction from left

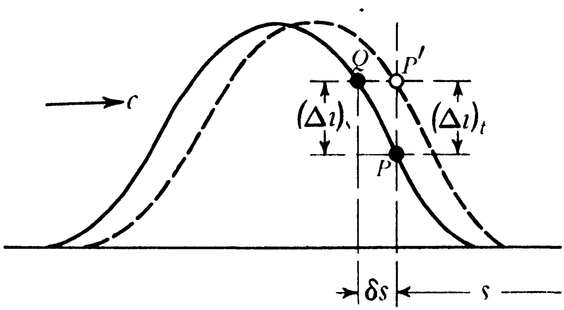


Fig. 4.

to right. The position of the point P is defined by s, the distance from that end of the line towards which the surge is travelling, and t is the time at which P is at the position s. For a small increment δs the increase of the current ordinate is from P to Q, viz. $(\Delta i)_s$. Now, inspection of the diagram of Fig. 4 shows that if the increment of current at the position s on the line is $(\Delta i)_t$ in the element of time δt, then these current increments will be equal, $(\Delta i)_s = (\Delta i)_t$ *if* $\frac{\delta s}{\delta t} = c$ the speed of travel of the surge, that is, if

$$\frac{(\Delta i)_s}{\delta s} = \frac{1}{c} \frac{(\Delta i)_t}{\delta t}.$$

But from equations (3*b*) and (4)

$$\frac{\Delta i_s}{\delta s} = \sqrt{LC}\frac{\Delta i_t}{\delta t},$$

that is to say, $$c = \frac{1}{\sqrt{LC}}.$$

If the surge is travelling in the opposite direction to that shown in Fig. 3, then $\frac{(\Delta i)_s}{\delta s} = -\frac{1}{c}\frac{(\Delta i)_t}{\delta t}$, so that

$$c = -\frac{1}{\sqrt{LC}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It has already been shown that $Z_0 = \pm\sqrt{\frac{L}{C}}$, and since $c = \pm\frac{1}{\sqrt{LC}}$ it follows that

$$Z_0C = \frac{1}{c}: \quad Z_0 = \frac{1}{C.c}$$

and $$(L \text{ in henry per cm.}) \times (C \text{ in farad per cm.}) = \frac{1}{c^2},$$

where Z_0 is the "surge impedance" of the line

$$\left.\begin{array}{l} c = 3 \times 10^{10} \text{ cm./sec. for an overhead line} \\ \text{and } c \simeq 1{\cdot}5 \times 10^{10} \text{ cm./sec. for a normal high tension cable} \end{array}\right\} \quad . \quad (7)$$

otherwise, for an overhead line,

$$(L \text{ in millihenry per km.}) \times (C \text{ in } \mu\text{F per km.}) = 0{\cdot}0111,$$

and for an underground cable,

$$(L \text{ in millihenry per km.}) \times (C \text{ in } \mu\text{F per km.}) = 0{\cdot}0444.$$

For a single overhead line of 50 mm.² cross-section and about the usual height above the earth's surface

$$L \simeq 1{\cdot}67 \text{ millihenry/km.}: \quad C = 0{\cdot}0067 \ \mu\text{F/km.},$$

so that $$Z_0 = \sqrt{\frac{1{\cdot}67}{10^3} \times \frac{10^6}{0{\cdot}0067}} = 500 \ \Omega,$$

that is, under the assumption that there is no mutual inductance with neighbouring lines.

For an underground cable, $L \simeq 0{\cdot}33$mH/km.: $C \simeq 0{\cdot}133$ μF/km.

$$Z_0 \simeq \sqrt{\frac{0{\cdot}33}{10^3} \times \frac{10^6}{0{\cdot}133}} \simeq 50 \ \Omega.$$

(*i*) A Three-phase Overhead Transmission Line.—Assuming the symmetrical arrangement shown in Fig. 5:

(See page 105; also Test Papers, Chapter IV, Example 5).

The star capacitance per phase is

$$C = \frac{1}{18 \times 10^6 \log_e \left(\frac{2a}{d}\right)} \text{ F/km.}$$

so that the inductance per phase is

$$L = \frac{1}{c^2 C} = \frac{1}{(3 \times 10^5)^2 C} \text{ henry/km.}$$

EXAMPLES.—(1) Pressure between the lines is 132 kV.; distance between lines is $a = 3{\cdot}2$ metres; diameter of each conductor is $d = 16$ mm., then

$$C = \frac{1}{18 \times 10^6 \log_e \left(\frac{640}{1{\cdot}6}\right)} = \frac{0{\cdot}00925}{10^6} \text{ F/km.}$$

$$L = \frac{10^6}{9 \times 10^{10} \times 0{\cdot}00925} = \frac{1{\cdot}19}{10^3} \text{ H/km.}$$

$$Z_0 = \sqrt{\frac{L}{C}} = 360\ \Omega \text{ per phase.}$$

Resistance to d.c. $R = 0{\cdot}09\ \Omega$/km.

(2) Pressure between lines is 200 kV.; distance between lines $a = 4{\cdot}6$ metres; diameter of each conductor is $d = 25$ mm.

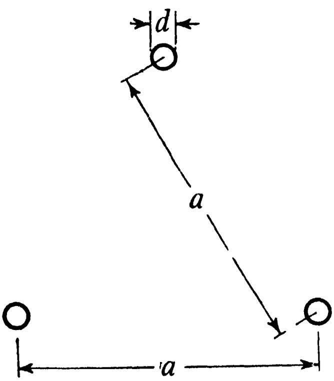


Fig. 5.

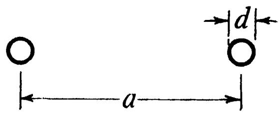


Fig. 6.

$$C = 0{\cdot}0096\ \mu\text{F/km.}: \ L = 0{\cdot}00127 \text{ H/km.}:$$
Resistance to d.c. $= 0{\cdot}045\ \Omega$/km.,

so that $$Z_0 = 360\ \Omega \text{ per phase.}$$

(*ii*) A SINGLE-PHASE OVERHEAD LINE (Fig. 6) (see also Chapter IV, page 103, Example 4):

$$C = \frac{1}{\left[82 \log_{10} \frac{2a}{d}\right] 10^6} \text{ F/km.}: \ Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{c.C}:$$

For $a = 2$ metres: $d = 8$ mm.

$$Z_0 = [82 \log_{10} 500]3{\cdot}3\ \Omega = 730\ \Omega: \ L = \frac{Z_0}{c} = 2{\cdot}4 \text{ mH/km.}$$

(*iii*) Three-phase Cables.—(1) "Belted" Cables (see Fig. 12*b*, page 106, Chapter IV). Line pressure = 10 kV; section of each conductor (core) = 10 mm.^{2}:

$$C = 0{\cdot}18\ \mu\text{F/km.}: \quad L = 0{\cdot}00042\ \text{H/km.}: \quad Z_0 = 49\ \Omega \text{ per phase.}$$

(2) "Screened" Cables (see Fig. 13*b*, page 106, Chapter IV).

Line pressure = 60 kV.; section of each core = 120 mm.^{2}:

$$C = 0{\cdot}19\ \mu\text{F/km.}: \quad L = 0{\cdot}00051\ \text{H/km.}: \quad Z_0 = 52\ \Omega \text{ per phase.}$$

It is useful to note that the inductance L for cables is from $\dfrac{1}{3{\cdot}5}$ (for high pressure systems) to $\dfrac{1}{5}$ (for low pressure systems) of the value for overhead lines.

The capacitance C for cables is from 13 times (for high pressure cables) to 20 times (low pressure systems) the value for overhead lines.

The Energy of an Electric Surge

An electric surge (Fig. 7) involves a definite amount of energy which appears partly as electrostatic strain in the dielectric and partly as an electromagnetic field. The electrostatic energy per kilometre of the line is

$$u_e = \tfrac{1}{2}Cv^2 \text{ joules} \qquad . \quad . \quad (8)$$

where C farad is the capacitance of the line per kilometre and e volts is the pressure at that part of the surge which is under consideration. The electromagnetic energy per kilometre is

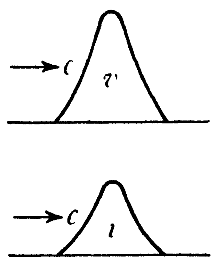


Fig. 7.

$$u_m = \tfrac{1}{2}Li^2 \text{ joules} \qquad . \quad . \quad . \quad . \quad (9)$$

where L henry per kilometre is the inductance and i amperes is the current at that part of the line considered.

Now it has been shown on page 459, expression (4), that

$$\left(\frac{v}{i}\right)^2 = \frac{L}{C} = Z_0^2,$$

that is $$v^2C = i^2L,$$

so that the electrostatic energy u_e defined by (8) is equal to the electromagnetic energy u_m defined by (9), and consequently the total energy of the surge per kilometre, is

$$u = 2u_e = 2u_m = Cv^2 = Li^2 \text{ joules} \qquad . \quad . \quad . \quad (10)$$

The total energy for the complete wave of the surge will therefore be

$$U = \int u\,dx = C\int v^2\,dx = L\int i^2\,dx \text{ joules,}$$

where the integration is extended over the whole span of the surge (Fig. 7).

Since it has been shown in the foregoing that the surge impedance of an overhead line is much greater than for an underground cable, it will be clear that, for a given pressure wave, the *power* of a surge in a cable will be much greater than on an overhead line. Suppose, for example, $Z_0 = 500$ ohms for an overhead line and $Z_0 = 50$ ohms for an underground cable, and let the peak value of the surge wave be $v_{max} = 100{,}000$ volts, then the power corresponding to this peak value of the surge will be

(*i*) For an overhead line,

$$w = v_{max} \cdot i_{max} = \frac{v^2_{max}}{Z_0} = \frac{(10^5)^2}{Z_0} = \frac{10^{10}}{500} \text{ watts} = 20{,}000 \text{ kW}.$$

(*ii*) For an underground cable,

$$w = \frac{v^2_{max}}{Z_0} = 200{,}000 \text{ kW}.$$

These enormous values of the power associated with such surges are heavily damped by the losses in the resistance of the conductor and in the dielectric leakage resistance, and the general effect of this damping action may be seen as follows:

Let the resistance of the conductor be r ohm per kilometre and the insulation leakage resistance a ohms per kilometre. Then, in an elementary length δx of the conductor, the power loss due to the heating effects will be

$$(\delta w)_h = i^2 r \,\delta x + \frac{v^2}{a}\,\delta x = i^2\left(r + \frac{Z_0^2}{a}\right)\delta x \tag{11}$$

The power which enters the elementary length δx of the conductor as given by the foregoing results will be

$$w_s = i^2 Z_0,$$

so that the decay of energy in this element of the conductor will be

$$-(\delta w)_s = -2iZ_0\delta i,$$

but

$$-(\delta w)_s = -2iZ_0\delta i,$$

so that

$$i^2\left(r + \frac{Z_0^2}{a}\right)\delta x = -2iZ_0\delta i,$$

that is

$$-\frac{\delta i}{i} = \left(\frac{r + \frac{Z_0^2}{a}}{2Z_0}\right)\delta x$$

or

$$\frac{di}{i} = -\frac{1}{2}\left(\frac{r + \frac{Z_0^2}{a}}{Z_0}\right)dx.$$

The solution of this differential equation is

$$i = i_0 e^{-\frac{1}{2}\left(\frac{r}{Z_0}+\frac{Z_0}{a}\right)x} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $e = 2{\cdot}718$—that is, the base of natural logarithms, and i_0 is the value of i when $x = 0$. Similarly

$$v = v_0 e^{-\frac{1}{2}\left(\frac{r}{Z_0}+\frac{Z_0}{a}\right)x} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The expressions (12) and (13) show that the surge wave of current and the surge wave of pressure remain of constant shape, but decay logarithmically as shown in Fig. 8.

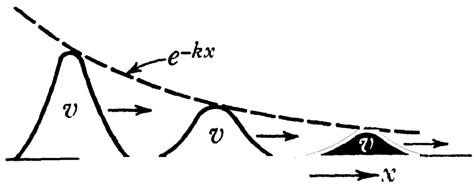


Fig. 8.

When the insulation resistance of the line is very high so that $\frac{1}{a} \simeq 0$, then

$$\frac{v}{v_0} = \frac{i}{i_0} = e^{-\frac{1}{2}\frac{r}{Z_0}x} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Suppose, for example, that a single-phase overhead line has a resistance of $r = 0{\cdot}6\ \Omega$ per kilometre run and a surge impedance of $Z_0 = 500\ \Omega$, and it is required to find how far a surge will travel along the line before its magnitude has become reduced to one-third of its original value, then

$$\frac{v}{v_0} = \frac{i}{i_0} = e^{-\frac{1}{2}\frac{r}{Z_0}x} = \tfrac{1}{3},$$

but $\qquad e^{-1\cdot 1} = \tfrac{1}{3}$: so that $\frac{1}{2}\frac{r}{Z_0}x = 1{\cdot}1$

or $\qquad x = 2{\cdot}2\frac{Z_0}{r} = 2{\cdot}2\frac{500}{0{\cdot}6} = 1{,}840$ km.,

whilst for an underground cable for which $Z_0 = 50\ \Omega$ and $r = 0{\cdot}6\ \Omega$/km.

$$x = 184 \text{ km.}$$

The value which has been assumed for the resistance r in the foregoing example is the value as measured by direct current. For high-speed travelling waves, such as surges on transmission lines and in cables, however, the resistance will be very much greater than the d.c. value (see also Chapter XIV). For non-sinusoidal surge waves the effective resistance is not known and cannot be calculated. If, for example, the

resistance of the travelling surge is taken to be 50 times the d.c. value, then the distance travelled before the magnitude of the surge in an overhead line has become reduced to one-half its original value will be 36·8 km., whilst for an underground cable it will only be 3·68 km.

The time taken for a surge to become practically extinguished can be found by writing

$$\frac{v}{v_0} = \frac{i}{i_0} = 3 \text{ per cent.} = 0{\cdot}03,$$

in which case the magnitude of v can be assumed to have become practically negligibly small. Then since

$$x = c.t : \ Z_0 = \sqrt{\frac{L}{C}} : \ c = \frac{1}{\sqrt{LC}}$$

$$\frac{v}{v_0} = \frac{i}{i_0} = e^{-\frac{1}{2}\frac{r}{Z_0}x} = e^{-\frac{1}{2}\frac{c}{Z_0}t} = e^{-\frac{1}{2}\frac{R}{L}t},$$

so that the time taken for the surge to decay to 3 per cent. of its original value will be given by

$$e^{-\frac{1}{2}\frac{R}{L}t} = 0{\cdot}03 = e^{-3{\cdot}5}.$$

If the conductor resistance is taken to be 10 times the d.c. value, that is, 6 Ω/km., and the line inductance $L = 0{\cdot}0017$ henry/km. for an overhead line, then

$$\frac{1}{2}\frac{R}{L}.t = 3{\cdot}5 : \text{ or } t = 7 \times \frac{L}{R} = 0{\cdot}002 \text{ sec.},$$

whilst if $L = 0{\cdot}00033$ henry/km. for an underground cable, then

$$t = 0{\cdot}00038 \text{ sec.}$$

From these data it is clear that such surge phenomena practically disappear within a few ten-thousandths of a second.

The Effect on Surge Phenomena of a Junction of Overhead Line and Underground Cable

When an electric surge reaches a junction of an overhead line and underground cable, reflexion of both current and pressure waves will occur.

(i) Suppose in the first place that the pressure surge is travelling in the direction from the cable to the overhead line and suppose a rectangular wave of pressure originates in the cable as shown in Fig. 9 by v_{f1} moving with the velocity c towards the junction. When this pressure surge reaches the junction a *reflected* wave v_{r1} will develop and will travel back along the cable, whilst a wave of pressure v_{f2} will travel forward on the overhead line as shown in Fig. 9. The pressure conditions at the junction will then be defined by

$$v_{f1} + v_{r1} = v_{f2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and since (see page 460, equation (5))

$$\frac{v_f}{i_f} = +Z_0 \text{ for the forward wave : } \frac{v_r}{i_r} = -Z_0 \text{ for the reflected wave :}$$

the current conditions at the junction will be defined by the equation

$$i_{f1} + i_{r1} = i_{f2} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

also
$$\frac{v_{f1}}{i_{f1}} = Z_{01} : \frac{v_{r1}}{i_{r1}} = -Z_{01} : \frac{v_{f2}}{i_{f2}} = Z_{02} : \frac{v_{r2}}{i_{r2}} = -Z_{02}$$

Then by substituting in (16) and combining with (15) it is found that

$$v_{f2} = \left(\frac{2Z_{02}}{Z_{01} + Z_{02}}\right)v_{f1} \quad . \quad . \quad . \quad . \quad (17)$$

and similarly for the current relationships

$$i_{f2} = \left(\frac{2Z_{01}}{Z_{01} + Z_{02}}\right)i_{f1} \quad . \quad . \quad . \quad . \quad (18)$$

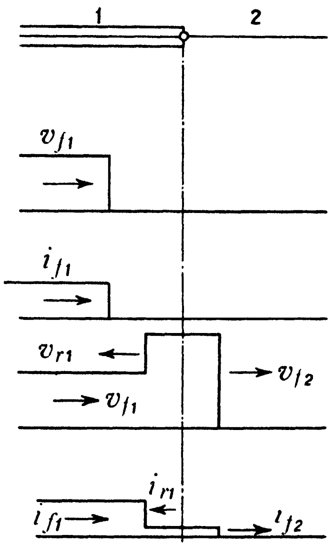


Fig. 9.

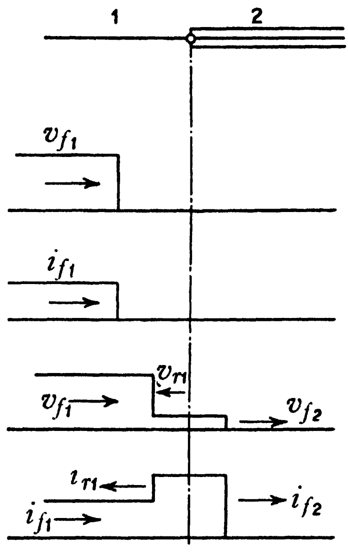


Fig. 10.

These two equations thus define the *forward* travelling waves of pressure and current, respectively.

The equations for the reflected waves are similarly found,

$$v_{r1} = \left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right)v_{f1} : \; i_{r1} = \left(\frac{Z_{01} - Z_{02}}{Z_{01} + Z_{02}}\right)i_{f1}.$$

When $\dfrac{Z_{02}}{Z_{01}}$ is very large, e.g. when the overhead line in Fig. 9 is disconnected from the cable, then

$$v_{f2} = 2v_{f1} : \ i_{f2} = 0.$$

As a general example, suppose $Z_{02} = 500\ \Omega$ and $Z_{01} = 50\ \Omega$, then

$$v_{f2} = 1\cdot 8v_{f1} : \ v_{r1} = 0\cdot 82v_{f1} : \ i_{r1} = -\ 0\cdot 82i_{f1} : \ i_{f2} = 0\cdot 18i_{f1}.$$

(*ii*) If the surge is travelling in the direction from the overhead line to the cable, the conditions will be as shown in Fig. 10. The same general expressions as have been obtained in the previous problem can be applied immediately to the present one. Thus, taking the previously assumed numerical data in which $Z_{02} = 50\ \Omega$ for the cable and $Z_{01} = 500\ \Omega$ for the overhead line, and substituting these values in the general expressions obtained for the previous problem, then

$$v_{f2} = \left(\frac{2Z_{02}}{Z_{01} + Z_{02}}\right)v_{f1} = \frac{100}{550}v_{f1} = 0\cdot 18v_{f1}$$

$$v_{r1} = \left(\frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}\right)v_{f1} = -\ \frac{450}{550}v_{f1} = -\ 0\cdot 82v_{f1}$$

$$i_{r1} = \left(\frac{Z_{01} - Z_{02}}{Z_{01} + Z_{02}}\right)i_{f1} = \frac{450}{550}i_{f1} = 0\cdot 82i_{f1}$$

$$i_{f2} = \frac{2Z_{01}}{Z_{01} + Z_{02}}i_{f1} = \frac{1{,}000}{550}i_{f1} = 1\cdot 8i_{f1}.$$

(*iii*) The energy conditions when the surge arrives at the junction of an underground cable and an overhead line may be obtained as follows : On arrival at the junction the power is

$$w_1 = \frac{v_{f1}{}^2}{Z_{01}} \text{ watts.}$$

The power which is transmitted across the junction to the line 2 is

$$w_2 = \frac{v_{f2}{}^2}{Z_{02}} \text{ watts,}$$

so that
$$\frac{w_2}{w_1} = \frac{Z_{01}}{Z_{02}}\left(\frac{v_{f2}}{v_{f1}}\right)^2 = \frac{Z_{01}}{Z_{02}}\left(\frac{2Z_{02}}{Z_{01} + Z_{02}}\right)^2$$

that is

$$\frac{w_2}{w_1} = \left(\frac{2}{\sqrt{\dfrac{Z_{01}}{Z_{02}}} + \sqrt{\dfrac{Z_{02}}{Z_{01}}}}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Since this ratio is unaltered if Z_{01} and Z_{02} are interchanged, it follows that the ratio of the transmitted energy to the incident energy is the same whether the surge reaches the junction from the underground cable

or from the overhead line. Further, since the right-hand side of this expression appears as a square term, the ratio $\frac{w_2}{w_1}$ is always positive.

Definition of the Wave Form of a Surge

The shape of the waves of pressure surges met with in practice generally comprises a steep wave front which rises rapidly to a crest value and then tails off at a comparatively slow rate as shown * in Fig. 11. The shape of such a wave may be defined conveniently as follows :

If a straight line is drawn through the points A and B on the wave front such that the pressure at A is $0{\cdot}1\,V_m$ and the pressure at B is $0{\cdot}9\,V_m$, where V_m is the crest value of the pressure wave, and if this line is produced to meet the abscissa axis at G and to cut the horizontal

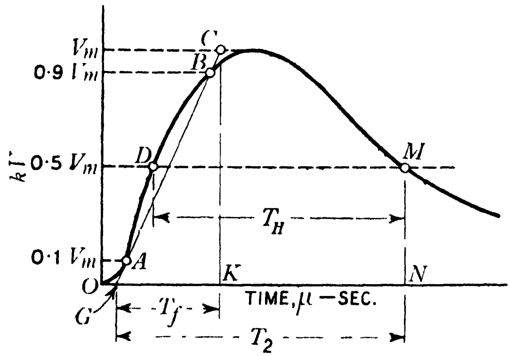


Fig. 11.

line through the crest value at C, then the projection of CG on the abscissa axis (that is, the time axis), viz. KG is termed the "duration of the wave front", T_f. The time for which the wave has a value greater than $\frac{1}{2}\,V_m$ is termed the "duration of the half-value", T_h. The ratio $\frac{V_m}{T_f}$ is termed the "steepness of the wave-front" and is measured in kV. per micro-seconds.

The "duration of the wave front", T_f, may be from $0{\cdot}5$ to 5μsec., and the "duration of the half value" T_H may be from 5 to 500μsec. where $\mu = 1$ micro-second $= 10^{-6}$ second.

The General Equations for Long-distance Transmission Lines

In Fig. 12 is shown a single-phase transmission line, that is, the "go" and "return" lines, the generator terminals being AB and the consumer's terminals being C and D. If, now, a sinusoidal wave of pressure be applied by the generator to the terminals A and B of the

* *Engineering*, August 14, 1942, p. 121.

transmission line, the pressure and current conditions will vary from point to point along the line. In order to investigate the manner in which the current and pressure will vary throughout the line it is necessary to consider the effect of the distributed inductance and the distributed capacitance of the two lines. To simplify the treatment as much as possible, it will be assumed that the line has negligibly small resistance

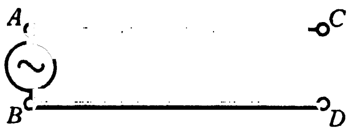


Fig. 12.

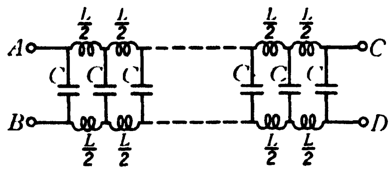


Fig. 13.

and that the insulation is perfect, that is to say, the line will be assumed to be a " no loss " line.

In Fig. 13 is shown diagrammatically the distributed inductance of L henry per kilometre length of the line, that is, for both the " go " and " return " conductors, the distributed capacitance between the two lines being C farad per kilometre run. Suppose, now, that the conditions be considered for a point P on the line distant s km. from the consumer's

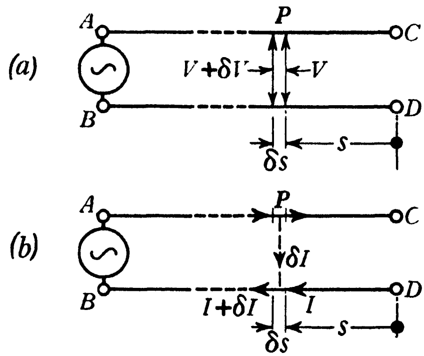


Fig. 14.

end (Fig. 14), and let δx be an element of length at this point of the line. The impedance vector per unit length of the line will then be

$$\mathfrak{Z} = j\omega L = jx,$$

where ω is the circular frequency of the supply pressure of the generator, that is, $\omega = 2\pi f$ and x Ω is the inductive reactance per kilometre length. Similarly, the admittance vector per kilometre length of the line is

$$\mathfrak{Y} = j\omega C = jb,$$

where b siemens is the susceptance per kilometre length of the line.

If, now, in Fig. 14, $\mathfrak{V}$ is the vector of pressure and $\mathfrak{J}$ is the vector of current at any point in the line distant s km. from the consumer's end, then for the element of length δs the capacitance current which will pass between the two lines will be

$$\delta\mathfrak{J} = \mathfrak{V}.\mathfrak{Y}\delta s = \mathfrak{V}j\omega C\delta s$$

so that

$$\frac{d\mathfrak{J}}{ds} = \mathfrak{V}j\omega C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Similarly, the pressure rise in the element δs will be

$$\delta\mathfrak{V} = \mathfrak{J}.\mathfrak{Z}\delta s = \mathfrak{J}j\omega L\delta s,$$

so that
$$\frac{d}{ds}\mathfrak{V} = \mathfrak{J}j\omega L.$$

Differentiating equation (20) with respect to s gives

$$\frac{d^2\mathfrak{J}}{ds^2} = \frac{d}{ds}\mathfrak{V}j\omega C = \mathfrak{J}\,(j^2\omega^2 LC),$$

that is

$$\frac{d^2\mathfrak{J}}{ds^2} = -\,\mathfrak{J}\,(\omega^2 LC) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In the same way it is easy to deduce that

$$\frac{d^2\mathfrak{V}}{ds^2} = -\,\mathfrak{V}\,(\omega^2 LC) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Substituting α^2 for $-\,\omega^2 LC$, that is, $\alpha = j\omega\sqrt{LC} = j\frac{\omega}{c}$, where c is the velocity of propagation, viz. $c = 3 \times 10^5$ km./sec. for overhead lines and $c \doteqdot 1{\cdot}5$ km./sec. for underground cables (see expression (7), page 461). The solution of the simultaneous differential equations (21) and (22) is

$$\left.\begin{array}{ll} \mathfrak{J}_s = \mathfrak{A}e^{j\alpha s} + \mathfrak{B}e^{-j\alpha s} \quad . \quad . \quad . \quad . & (a) \\ \mathfrak{V}_s = Z_0\{\mathfrak{A}e^{j\alpha s} - \mathfrak{B}e^{-j\alpha s}\} \quad . \quad . & (b) \end{array}\right\} \quad . \quad . \quad . \quad (23)$$

where $Z_0 = \sqrt{\frac{L}{C}}$ is the "surge impedance" or "initial sending-end impedance" and the vectors $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary constants which may be found when the load conditions at the consumer's terminals are known. If the consumer's load is defined by the vectors $\mathfrak{V}_2$ and $\mathfrak{J}_2$, then the arbitrary constants as found by substituting these values in equations (23) and also putting $s = 0$, then

$$\mathfrak{A} = \frac{1}{2}\left\{\mathfrak{J}_2 + \frac{1}{Z_0}\mathfrak{V}_2\right\} : \quad \mathfrak{B} = \frac{1}{2}\left\{\mathfrak{J}_2 - \frac{1}{Z_0}\mathfrak{V}_2\right\}.$$

If these values for the arbitrary constants are now inserted in the general

equations (23), the current and pressure vectors for any point on the line will be given by the simultaneous equations

$$\left.\begin{aligned} \mathfrak{J}_s &= \mathfrak{J}_2 \cdot \tfrac{1}{2}(e^{j\alpha s} + e^{-j\alpha s}) + \mathfrak{V}_2 \frac{1}{Z_0} \tfrac{1}{2}(e^{j\alpha s} - e^{-j\alpha s}) \quad . \quad . \quad . \quad (a) \\ \mathfrak{V}_s &= \mathfrak{V}_2 \cdot \tfrac{1}{2}(e^{j\alpha s} + e^{-j\alpha s}) + \mathfrak{J}_2 Z_0 \tfrac{1}{2}(e^{j\alpha s} - e^{-j\alpha s}) \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad (24)$$

These equations may be re-written in the following equivalent form,

$$\left.\begin{aligned} \mathfrak{J}_s &= \mathfrak{J}_2 \cos \alpha s + \mathfrak{V}_2 \frac{1}{Z_0} j \sin \alpha s \quad . \quad . \quad . \quad (a) \\ \mathfrak{V}_s &= \mathfrak{V}_2 \cos \alpha s + \mathfrak{J}_2 Z_0 j \sin \alpha s \quad . \quad . \quad . \quad (b) \end{aligned}\right\} \quad . \quad . \quad (25)$$

in which the angle $\alpha = \omega Z_0 C$: so that, since $Z_0 = \sqrt{\frac{L}{C}}$

$$\alpha = \omega Z_0 C : \frac{1}{c} = \sqrt{LC} : \alpha = \frac{\omega}{c}.$$

For overhead lines operating at 50 frequency

$c = 3 \times 10^5$ km./sec. : $\omega = 2\pi f = 314$: $\alpha = 1{\cdot}05 \times 10^{-3}$ radian/km.

If the line is open-circuited at the consumer's end, then $I_2 = 0$:

$$\mathfrak{V}_2 = V_2, \text{ so that } \mathfrak{J}_s = j\frac{V_2}{Z_0} \sin \alpha s : \quad \mathfrak{V}_s = V_2 \cos \alpha s.$$

These expressions show that the current is 90° ahead of the pressure in

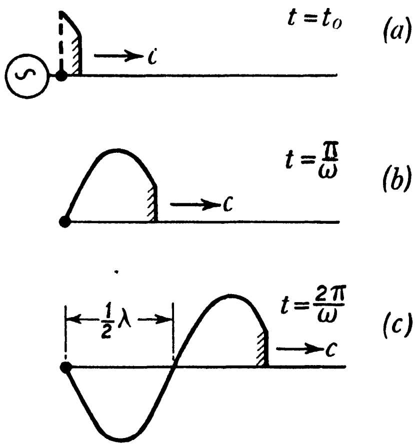


Fig. 15.

time, that is to say, the current is a capacitance current through the line. This current is termed the " charging current " of the line.

Suppose in Fig. 15 that a sine wave of p.d. $v = V_{max} \sin \omega t$ is applied to the line at the moment t_0, so that $v = V_{max.} \sin \omega t_0$ as shown in

Fig. 15*a*. The applied pressure wave will then travel along the line at the speed of c km./sec. and at the time $t = \frac{\pi}{\omega}$ the p.d. at the generator terminals will be zero and the pressure wave will be distributed along the line as shown in Fig. 15*b*. At the time $t = \frac{2\pi}{\omega}$ the p.d. at the generator terminals will again be zero and the pressure wave will then be distributed along the line as shown in Fig. 15*c*. Since the applied pressure wave is travelling along the line at the velocity of c km./sec. it is easily seen by reference to Fig. 15*c* that

$$\tfrac{1}{2}\lambda = \frac{\pi}{\omega}c \text{ km. or } \lambda = \frac{2\pi}{\omega}c = \frac{c}{f} \text{ km.}$$

so that $$\lambda . f = c$$

also, since $$\alpha = \frac{\omega}{c}$$

$$\lambda = \frac{2\pi}{\alpha} = \frac{c}{f} \text{ km.} \qquad (26)$$

EXAMPLE.—Consider a single-phase overhead transmission line consisting of two parallel conductors for which the surge impedance is

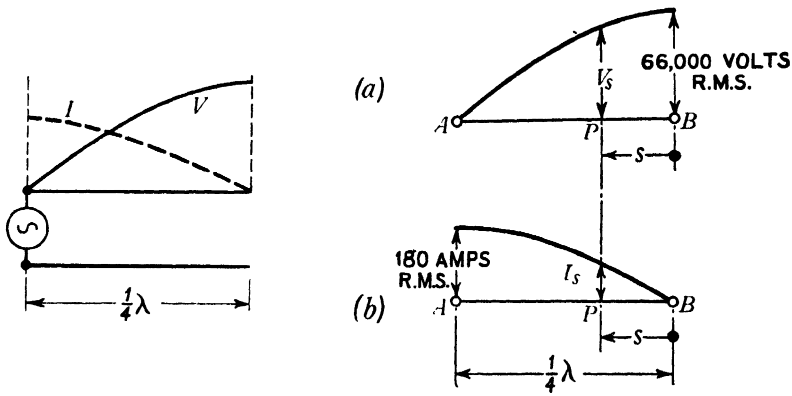


Fig. 16.

750 Ω. The pressure at the consumer's end is $V_2 = 66{,}000$ r.m.s. volts at 50 frequency, so that $\omega = 314$ and the wave length will then be

$$\lambda = \frac{2\pi}{\alpha} = \frac{c}{f}: \text{ that is, } \lambda = 5{,}980 \text{ km.},$$

and the quarter wave-length is 1,495 km. It will be assumed that the line is open-circuited at the consumer's end, so that $I_2 = 0$.

In Fig. 16*b* is shown the wave of the charging current of the line, and in Fig. 16*a* is shown the pressure wave for a length of line equal to

$\frac{1}{4}\lambda$. It will be seen that an indefinitely small pressure applied at the generator and A will produce an indefinitely large pressure at the consumer's end when the consumer's terminals are open-circuited. This is an important case of line "resonance" which will only appear in this extreme form when there are no losses in the line as has been assumed to be the case in the foregoing investigation. Even when the losses are taken into account, however, there will be immense pressure rises at the consumer's terminals when the consumer's load is zero and the length of line is $\frac{1}{4}\lambda$. In practice, however, a transmission line is not likely to approach a length equal to one-quarter wavelength, so that the rise of pressure at the consumer's end due to the inherent resonance effect is not likely to reach excessively high values. For instance, it will be seen by inserting the foregoing data of the line in the general equations that for a line 500 km. long when the consumer's terminals are open-circuited and the pressure at those terminals is 66,000 volts, the pressure at the generator end of the line will be $66{,}000 \times \cos 60° = 57{,}000$ volts.

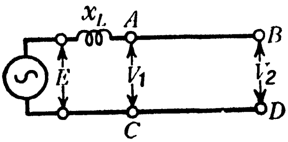


Fig. 17.

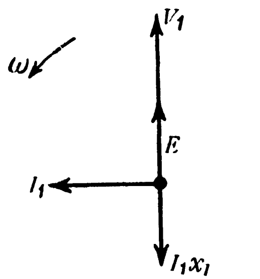


Fig. 18.

Dangerously large resonance effects, however, may be produced in relatively short lines when on no-load, and these are due to the leakage reactance of the generator and transformer. Thus, suppose in Fig. 17 that the generator and transformer combined leakage reactance is x_L when reduced to a transformation ratio of unity, the pressure at the generator terminals being V_1 and the pressure at the consumer's terminals V_2. The vector diagram relating the induced e.m.f. of the generator E and the pressure $\mathfrak{V}_1$ at the generator terminals, will be as shown in Fig. 18, so that

$$E = V_1 - x_L I_1.$$

If it is assumed that, as a reasonable practical condition $x_L = Z_0$, then since equation (25) establishes the following relationships, when the line is open at the consumer's end, i.e. when $I_2 = 0$,

$$\frac{V_s}{I_s} = \frac{V_1}{I_1} = \frac{Z_0}{\tan \alpha s},$$

where s is the length of the transmission line, and consequently,

$$E = V_1 - V_1\frac{x_L}{Z_0}\tan \alpha s = V_1(1 - \tan \alpha s),$$

so that $$\frac{V_1}{E} = \frac{1}{1 - \tan \alpha s} : \text{ and since } V_2 = \frac{V_1}{\cos \alpha s},$$

then $$\frac{V_2}{E} = \frac{1}{\cos \alpha s - \sin \alpha s}.$$

Thus for a line 500 km. long this relationship shows that

$$\frac{V_1}{E} = 2{\cdot}36 : \quad \frac{V_2}{E} = 2{\cdot}73 : \quad \frac{V_2}{V_1} = 1{\cdot}16.$$

For a three-phase overhead line the corresponding data can easily be found by taking the value of $Z_0 = 360\ \Omega$ per phase. For a three-phase underground cable the corresponding value of the surge impedance would be $Z_0 \simeq 50\ \Omega$ per phase.

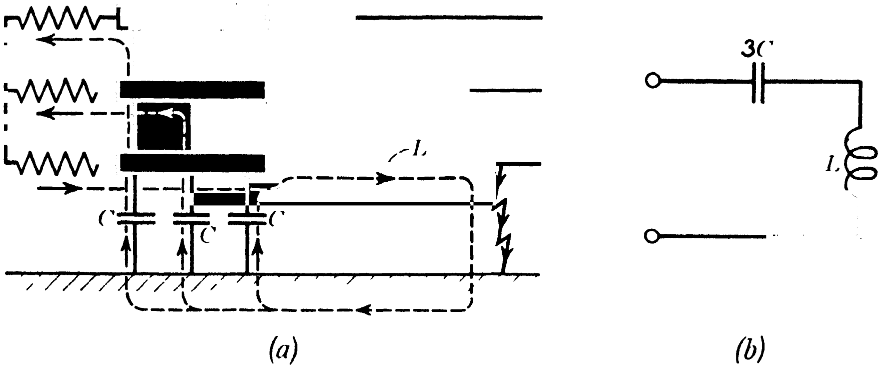


Fig. 19.

With regard to what has been said in the foregoing it is to be observed that, if a transmission line should approach in length the quarter wave-length, viz. $\frac{1}{4}\lambda = \frac{1}{4}\frac{c}{f}$, dangerous pressure rises must be expected when the line is unloaded. It has also been pointed out that for a supply frequency of 50, a quarter-wave length is much greater than the length of any normal transmission likely to be met with in practice. If, however, the supply pressure is not a pure sine wave, then one or other of the harmonics in the wave may give rise to dangerous possibilities of pressure rises. If, for example, there is a pronounced thirteenth harmonic, the quarter-wave length for this frequency will be

$$\frac{1}{4}\lambda_{13} = \frac{c}{4 \times 13 \times 50} = \frac{300{,}000}{260} = 116 \text{ km.},$$

and this length of line will be met with very frequently in practice.

Another practical case in which serious pressure rises may develop in a portion of the transmission system is that in which underground cables are connected to overhead lines and a short-circuit to earth develops on one of the overhead lines, so that a considerable length of the line is in series with the short-circuit current as shown in Fig. 19*a* for a three-phase system. For example, suppose the total capacitance to earth of the three lengths of cable is 150 μF, and suppose the inductance of the short-circuited length of overhead line is 60 millihenry (mH.), that is $L = 0{\cdot}06$ henry: $3C = 150 \times 10^{-6}$ farad, the capacitance reactance will then be $\frac{1}{\omega 3C} = \frac{10^6}{314 \times 150} = 21{\cdot}2\ \Omega$ and the inductive reactance will be $\omega L = 314 \times 0{\cdot}06 = 18{\cdot}84\ \Omega$. If the line pressure is 10,000 volts, so that the phase pressure is 5,760 volts then, referring to the equivalent circuit of Fig. 19*b*, the resultant reactance of the circuit will be

$$\left(\frac{1}{3\omega C} - \omega L\right) = (21{\cdot}2 - 18{\cdot}84) = 2{\cdot}36\ \Omega,$$

so that the current will be 2,440 amperes. The pressure across the cable will consequently rise to the value

$$V_c = \frac{I}{\omega 3C} = 2{,}440 \times 21{\cdot}2 = 52{,}000 \text{ volts},$$

and this will almost certainly destroy the cable.

The Natural Power of a Transmission Line

The general simultaneous equations (25), as given on page 472 for a single-phase system, are applicable also to a three-phase system if the pressures are taken to be the phase pressures, that is $\frac{1}{\sqrt{3}} \times$ (line pressure). Referring now to the diagrammatical representation in Fig. 13, which shows the distributed inductance of L henry per kilometre and the distributed capacitance of C farad per unit length of one phase of a three-phase transmission line. In the element of length δs the pressure rise due to the inductance $L\delta s$ will be (see Fig. 20*a*)

$$\delta\mathfrak{V} = j\mathfrak{J}(\omega L\delta s),$$

and the current increase in the element δs due to the capacitance will be (see Fig. 20*b*)

$$\delta\mathfrak{J} = j\mathfrak{V}(\omega C\delta s).$$

Now, if the conditions are such that the angle $\delta\alpha$ in Fig. 20*a* is equal to the angle $\delta\alpha$ in Fig. 20*b*, then

$$\frac{\delta\mathfrak{V}}{\mathfrak{V}} = \frac{\delta\mathfrak{J}}{\mathfrak{J}}, \text{ that is } \frac{\mathfrak{J}\omega L\delta s}{\mathfrak{V}} = \frac{\mathfrak{V}\omega C\delta s}{\mathfrak{J}}$$

or

$$\left(\frac{\mathfrak{V}}{\mathfrak{J}}\right)^2 = \frac{L}{C} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If this relationship holds for every elementary length of the line it will be seen that *the pressure and current will each be of constant magnitude throughout the line, and if the pressure vector and current vector are in phase at the consumer's end, they will also be in phase at the generator end of the line, so that*

$$\frac{\mathfrak{V}_1}{\mathfrak{I}_1} = \frac{\mathfrak{V}_2}{\mathfrak{I}_2} = \sqrt{\frac{L}{C}} = Z_0 \quad . \quad . \quad . \quad . \quad (28)$$

where $Z_0\ \Omega$ is the surge impedance per phase of the line.

The phase displacement between the pressure vector at the generator end of the line and the pressure vector at the receiver's end (see Fig. 20*c*) is given in radial measure, viz.

$$\frac{\text{arc } \alpha . s}{V_1} = \frac{\omega L s I_1}{V_1} = \frac{\omega L s}{Z_0} = \omega L s \sqrt{\frac{C}{L}} = \omega . s\sqrt{LC} = \frac{\omega}{c} . s = \alpha . s,$$

where $\alpha = \dfrac{\omega}{c}$, as already derived on page 473.

The same result will obviously be obtained by means of similar considerations with regard to Fig. 20*d*.

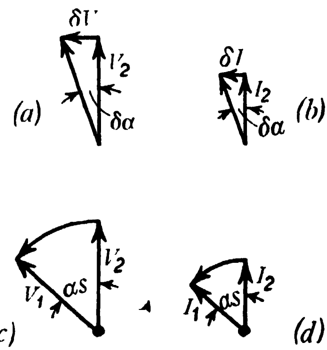


Fig. 20.

Suppose, now, that a three-phase transmission line is symmetrically loaded at the consumer's end with a current of I_2 amperes per phase, such that

$$I_2 = \frac{V_p}{Z_0},$$

where V_p is the phase pressure and Z_0 is the surge impedance of the line per phase. For a three-phase overhead line operating at 220 kV. line pressure, the following characteristic constants are representative of practical conditions,

$L = 0{\cdot}0013$ henry per phase per km. : $C = 0{\cdot}0096\ \mu$F per phase per km.,

so that
$$Z_0 = \sqrt{\frac{L}{C}} = 360\ \Omega.$$

For underground three-phase cables the value of Z_0 may be taken to be about 50 per phase.

It will be seen that when the conditions as shown in Fig. 20*c* and 20*d* [and which are also defined by the expression (28)] are satisfied, the power supplied to the line by the generator being equal to the power received by the consumer per phase (the line being assumed to be a " no loss "

line), then, the pressure and current vectors will be in phase at every point along the line, that is

$$V_1 I_1 = V_2 I_2 = \frac{V_1^2}{Z_0} = \frac{V_2^2}{Z_0} \text{ per phase,}$$

where V_1 and V_2 are phase pressures at the generator end and the consumer's end of the line, respectively. The total three-phase power which is transmitted will then be

$$W_n = 3\frac{V_1^2}{Z_0} = 3\left(\frac{V_l}{\sqrt{3}}\right)^2 \frac{1}{Z_0} = \frac{V_l^2}{Z_0} \text{ watts.} \qquad (29)$$

where V_l is the line pressure.

The power defined by these relationships is termed the "Natural Power" of the transmission line and represents an ideal condition for power transmission. The natural power W_n is usually taken as the standard of reference in the investigations of transmission line problems. The following table gives the magnitude of the natural power in megawatts (MW.), that is, 1 MW. = 1,000 kW. The natural power is stated for three-phase overhead lines and for three-phase underground cables. The surge impedances are taken to be $Z_0 = 360\ \Omega$ per phase for overhead lines and $Z_0 = 50\ \Omega$ per phase for underground cables.

TABLE

Line Pressure kV	Natural Power in MW	
	Three Phase Overhead Lines	Three Phase Underground Cables
15	0 62	4 5
33	3 0	22 0
66	12 0	87
132	48	345
220	135	970

Travelling Waves of Current and Pressure on Transmission Lines

It is of interest and also of practical importance to examine the general equations for the pressure and current of a transmission system from a different point of view to the foregoing, and this can be most clearly understood if, in the first place, it is assumed that the consumer's terminals are open-circuited so that the current $I_2 = 0$. Consider first the equation 24*b* for the pressure, viz.

$$\mathfrak{V}_s = \mathfrak{V}_2 . \tfrac{1}{2}(e^{j\alpha s} + e^{-j\alpha s}) \quad . \quad . \quad . \quad . \quad (30)$$

where $\mathfrak{V}_2$ is the time vector of pressure and may be written $\mathfrak{V}_2 = V_{2m}e^{j\omega t}$, so that equation (30) may now be expanded into the form

$$\mathfrak{V}_s = \tfrac{1}{2} V_{2m} e^{j(\omega t + \alpha s)} + \tfrac{1}{2} V_{2m} e^{j(\omega t - \alpha s)} \quad . \quad . \quad . \quad (31)$$

At the point on the line for which $s = 0$, that is, at the consumer's terminals $\mathfrak{V}_{s\ 0} = V_{2m}e^{j\omega t}$ and the pressure is alternating sinusoidally and of peak value V_{2m} and r.m.s. value $\frac{1}{\sqrt{2}}V_{2m}$. At any point distant s km. from the consumer's terminals the pressure at any moment will be given by the expression

$$\mathfrak{V}_s = V_{2m}\tfrac{1}{2}(e^{j\alpha s} + e^{-j\alpha s})e^{j\omega t},$$

so that at any point distant s km. from the consumer's terminals the pressure will have the peak value $V_{2m} \cos \alpha s$, as will be seen from Fig. 21 (see also page 473). When $\alpha s = \frac{\pi}{2}$ this peak value will be zero. All

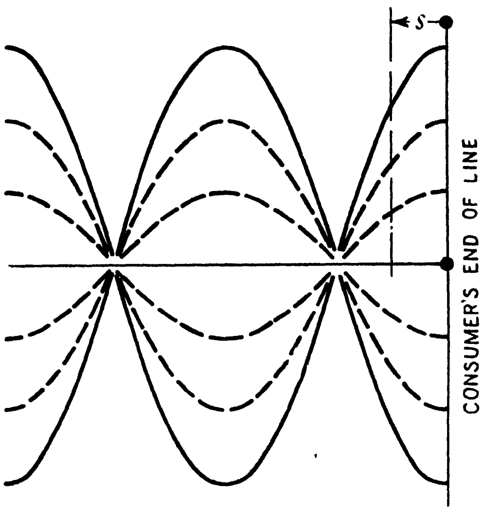


Fig. 21.

these results can be represented by a *standing wave* of pressure so situated on the line that the anti-node is at the consumer's terminals, as is shown in Fig. 21.

In a precisely similar way it can be shown that the current conditions for every point in the line can be represented by a *standing* wave of current of which the node, i.e. zero current, is at the consumer's terminals.

The same results can be arrived at by means of a somewhat different interpretation of the expression (30).

(*i*) *The component* $\frac{1}{2}\mathfrak{V}_2 e^{j\alpha s} = \frac{1}{2}V_{2m}e^{j(\omega t + \alpha s)}$ represents a pressure which is varying sinusoidally in space, that is, along the line, and also varying sinusoidally in time so that a change in time of $t = \frac{\pi}{2\omega}$ sec. will produce

exactly the same alteration of pressure as a change in the distance $s = \frac{\quad}{2\alpha}$.
Consider now a sine wave of constant peak value $\frac{1}{2}V_{2m}$ moving at a uniform speed of c km./sec. along the transmission line in the direction from the generator towards the consumer's terminals as shown in Fig. 22*a*, which represents the wave in the position such that its positive peak is at the

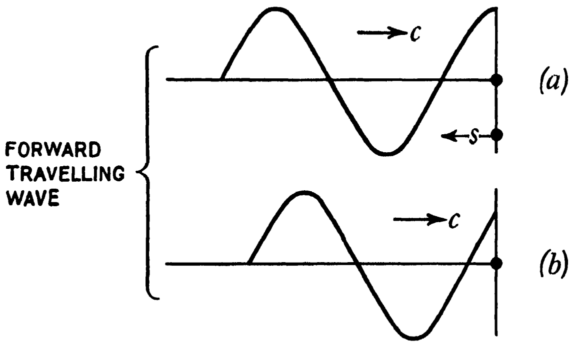


Fig. 22.

consumer's terminals, that is, for the moment $t = 0$. Then, if the speed is such that in the time $t = \frac{\pi}{2\omega}$ the wave has moved a distance $\frac{1}{4}\lambda$ km., that is, if

$$c = \frac{\frac{1}{4}\lambda}{\frac{\pi}{2\omega}} = \lambda . f \qquad . \quad . \quad . \quad . \quad (32)$$

the pressure at every point on the line will be in accordance with the expression $\frac{1}{2}V_{2m}e^{j(\omega t + \alpha s)}$. The wave shown in Fig. 22*b* is in the position the wave of Fig. 22*a* will occupy at the moment $t = \frac{\pi}{4\omega}$ second later (i.e. one-eighth of a period) than that to which Fig. 22*a* refers.

The speed of travel as defined in (32) is, of course, the same as that previously obtained for the velocity of propagation, viz.

$$c = 3 \times 10^5 \text{ km./sec.}$$

The wave shown in Figs. 22*a* and 22*b* is termed the *forward travelling wave.*

(*ii*) *The component* $\frac{1}{2}V_2e^{-j\alpha s} = \frac{1}{2}V_{2m}e^{j(\omega t - \alpha s)}$ *of the expression* (30).—Similar considerations to those of (*i*) show that this component can be represented by a sine wave of constant peak value $\frac{1}{2}V_{2m}$ travelling in the direction from the consumer's terminals to the generator at the speed c km./sec., that is a wave due to reflexion, at the consumer's terminals,

of the forward wave, and this is termed the *reflected wave.* In Fig. 23*a* and 23*b* are shown the positions of the reflected wave for the same two moments as those of Fig. 22*a* and 22*b*, and if the ordinates of the forward wave and of the reflected wave be added at every moment, it will be found that the standing wave of Fig. 21 is obtained.

The current conditions can be represented in exactly the same way as being due to the super-position of two travelling waves of the same magnitude and moving in opposite directions at the same speed c.

When the consumer's terminals are connected to a load, the reflected wave becomes smaller in magnitude than the forward wave, so that there will no longer be standing waves on the line. The r.m.s. values of the pressure and the current along the line will, in general, have different values at different points on the line.

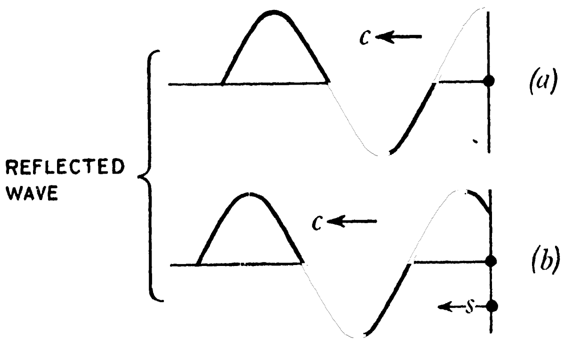


Fig. 23.

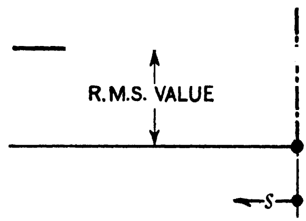


Fig. 24.

It is to be observed as a consequence of the foregoing considerations that the characteristic feature of the transmission of the natural load of the line is that the consumer's load is such that *the reflected wave disappears.* In this case the r.m.s. values of the current and pressure are respectively constant at every point of the line, as shown, for example, in Fig. 24.

Transmission Line Compensation : Reactive Power

When a line is transmitting its natural power it has been seen that :

(*i*) The pressure and current vectors are in phase at each point of the line and the magnitude of each of these vectors is constant throughout the line.

(*ii*) The pressure vector at the consumer's end of the line lags behind the pressure at the generator end of the line by the angle $\alpha . a = \frac{\omega}{c} a$ radian, where a is the distance of transmission, and similarly for the respective current vectors.

(*iii*) No " reactive power " (that is, no *wattless* current) is transmitted,

the whole transmission line being entirely occupied with "active power" (that is, *watt* current).

It has also been seen that when no power is being transmitted, that is, when the consumer's terminals are open-circuited, the current in the line will be a capacitance current and will cause a rise of pressure at the consumer's terminals (see Fig. 16). Such a pressure rise will also take place, although of less amount, when the power transmitted is less than the natural power, whereas if the power transmitted is greater than the natural power, there will be a pressure drop at the consumer's terminals.

Now, suppose that a load current I_x is taken by a consumer at the

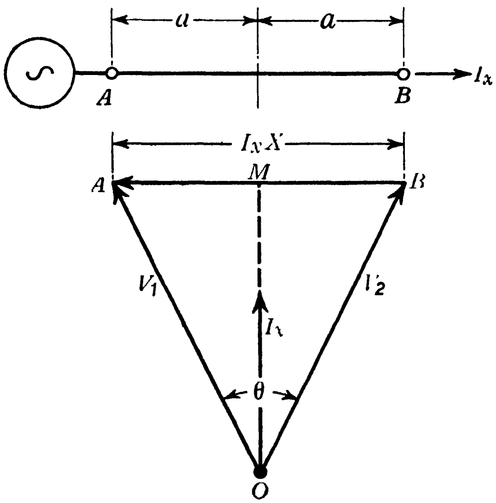


Fig. 25.

end B of the line (Fig. 25), and assume in the first place that the line capacitance current is negligibly small. The current I_x will flow through the total inductance $2a\omega L$ of the line, and if the pressure at each end of the line is the same, that is, if

$$V_1 = V_2 = V,$$

it will be seen that these pressure vectors $\mathfrak{V}_1$ and $\mathfrak{V}_2$ must move out of phase by the angle θ, so that the vector diagram will then be as shown in Fig. 25, where

$$AB = I_xX = I_x 2a\omega L$$

and

$$\sin\frac{\theta}{2} = \frac{AM}{OA} = \frac{I_x a\omega L}{V} \quad . \quad . \quad . \quad . \quad (33)$$

If the line capacitance current is now taken into account, then by suitably adjusting the magnitude of the consumer's current I_x, the

conditions which have already been considered on page 477, with reference to Fig. 20, can be established, i.e. that the current vector $\mathfrak{J}_1$ will come into phase with the pressure vector $\mathfrak{V}_1$ and the current vector $\mathfrak{J}_2$ will come into phase with the pressure vector $\mathfrak{V}_2$, as is shown in Fig. 26. When these conditions have been established, then

$$I_1 = I_2 = I : \quad V_1 = V_2 = V$$

and
$$\theta = \frac{V(2aC\omega)}{I} = \frac{I(2a\omega L)}{V} \quad . \quad . \quad . \quad (34)$$

as is seen from Fig. 26, so that

$$I^2(a\omega L) = V^2(a\omega C).$$

In practice, the value of $\frac{\theta}{2}$ will be, in general, not greater than about

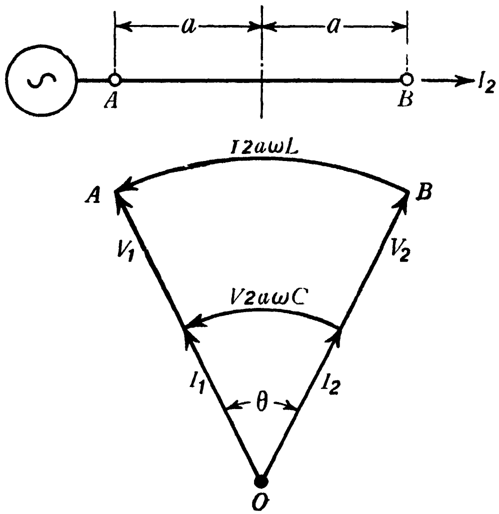


Fig. 26.

20° for reasons of stability, as will be explained on page 491. Then, from Fig. 26,

$$I = I_1 = I_2$$

and hence
$$\left.\begin{aligned} I_1{}^2(2a\omega L) &= V_1{}^2(2a\omega C) \\ I_2{}^2(2a\omega L) &= V_2{}^2(2a\omega C) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

that is to say, the reactive power due to the inductance of the line must be equal to the reactive power due to the capacitance of the line. When this condition holds, then

$$\left(\frac{V_1}{I_1}\right)^2 = \left(\frac{V_2}{I_2}\right)^2 = \frac{L}{C} = Z_0{}^2 \quad . \quad . \quad . \quad . \quad (36)$$

and the line is transmitting its "natural power" as has been explained already on page 478.

For this characteristic load, that is, the natural power of the line, the pressure and current vectors will remain in phase at every point of the line and the condition specified by equation (36) is automatically fulfilled without the necessity of feeding any reactive power into the line from outside sources. In general, however, the consumer's load will not be equal to this characteristic value of the power, that is to say, the line in general will not be transmitting its natural power and it then becomes necessary to feed into the line reactive power by some external appliance in order to maintain the pressure constant at the two ends of the line. The amount of reactive power which must be supplied at each end of the line is defined by the equation

$$W_r = V_2^2(a\omega C) - I_2^2(a\omega L) \quad . \quad . \quad . \quad . \quad (37)$$

If the load is less than the natural load, this reactive power will be capacitative and can be provided by suitably adjusting the field excitation of the machines at the ends A and B of the line, or, alternatively, instead of reactive power being drawn from the machine B, a symmetrical three-phase inductance of L_2 henry per phase may be connected to the line at B.

If the load is greater than the natural power of the line, then the reactive power W_r which must be supplied to each end of the line will be inductive and can be provided by suitably adjusting the field excitation of the machines at A and B, or, alternatively, instead of reactive power being drawn from the machine at B, a symmetrical three-phase capacitance may be connected to the line at B. When the natural power is being transmitted, then $W_r = 0$ and

$$I_2^2(2a\omega L) = V_2^2(2a\omega C),$$

as already found in expression (35).

This procedure of feeding reactive power into the line is sometimes termed "exciting" the line.

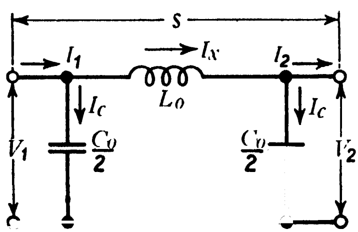


Fig. 27.

For many practical purposes the performance of a transmission line may conveniently be investigated by means of the equivalent circuit shown in Fig. 27, which represents one phase of a three-phase system. The total distributed inductance of the line is $L_0 = L.s$ henry, where L henry is the inductance per kilometre and s km. is the length of the line. The total distributed capacitance is C_0 farad, that is, $C.s$ farad, where C farad is the capacitance per unit length of the line, so that in the equivalent circuit $\frac{1}{2}C_0$ is assumed to be concentrated at each end of the line. For the conditions that the pressure at each end of the line is

of the same magnitude and that the line is transmitting its natural power, the vector diagram as obtained from the equivalent circuit is shown in Fig. 28. If the transmitted power is less than the natural power the

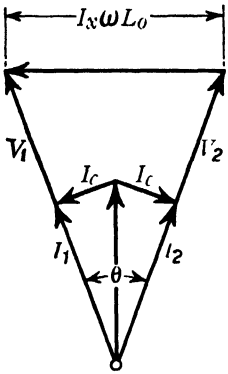


Fig. 28.

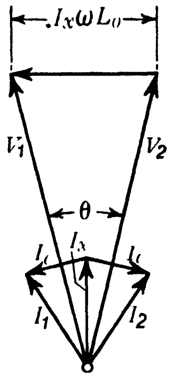


Fig. 29.

corresponding vector diagram will be as shown in Fig. 29. This representation of a transmission line by means of the equivalent circuit, as shown in Fig. 27, gives results which are practically exact for transmission distances up to about 200 km.

The Circle Diagram for a Transmission Line

In Fig. 30 is shown a transmission line of length $s = 2a$ and the end A is connected to a generator G. The active power W_a is transmitted to the consumer at B, the generator also supplying the reactive power W_r to the half-line AM. As before, it will be assumed that there are no losses in

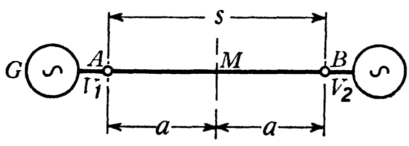


Fig. 30.

the line. At the consumer's end B a synchronous machine is shown, of which the function is to supply the reactive power W_r to the half-line BM, which is necessary to maintain the pressure at B constant and equal to the pressure at A.

If $W_n = \dfrac{V_1^2}{Z_0}$ is the natural power per phase of the line, then the ratio $\dfrac{W_a}{W_n} = w_a$ is termed the "specific active power" transmitted and

the ratio $\frac{W_r}{W_n} = w_r$ is the "specific reactive power" which is supplied to each end of the line. The total power supplied by the generator is

$$\mathfrak{W} = W_a + jW_r$$

and the specific power supplied by the generator is

$$\frac{\mathfrak{W}}{W_n} = \mathfrak{w} = w_a + jw_r \quad . \quad . \quad . \quad . \quad (38)$$

It can be shown that the relationship between the quantities w_a and w_r can be represented by a Circle Diagram as follows.

The general equations (25) for a "no loss" transmission line, as already obtained on page 472, are

$$\left.\begin{aligned} \mathfrak{V}_s &= \mathfrak{V}_2 \cos \alpha s + Z_0 \mathfrak{J}_2 j \sin \alpha s \\ \mathfrak{J}_s &= \mathfrak{J}_2 \cos \alpha s + \frac{1}{Z_0} \mathfrak{V}_2 j \sin \alpha s \end{aligned}\right\} \quad . \quad . \quad . \quad (39)$$

The pressure at both ends of the line is to be maintained constant and equal in magnitude, and the phase of the pressure at the consumer's end is assumed to be lagging by the angle Θ. That is,

$$V_1 = V_2 : \quad \mathfrak{V}_2 = V_1 e^{-j\Theta},$$

where V_1 refers to the generator end of the line and V_2 to the consumer end of the line. Eliminating $\mathfrak{J}_2$ between the equation (39) gives

$$\mathfrak{J}_1 = \frac{\mathfrak{V}_1 - V_2 \cos \alpha s}{Z_0 j \sin \alpha s} \cos \alpha s + \frac{1}{Z_0} \mathfrak{V}_2 j \sin \alpha s,$$

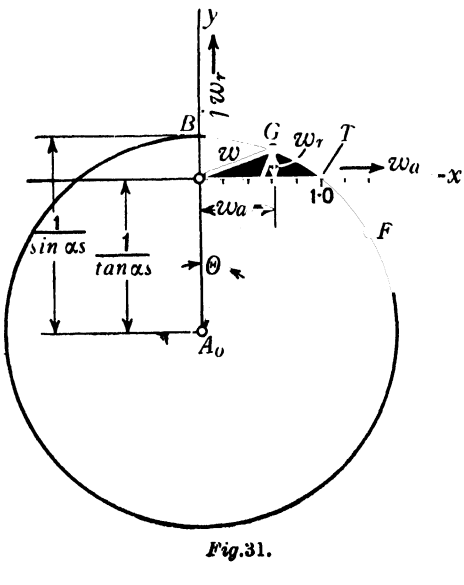


*Fig.*31.

which reduces to

$$\mathfrak{J}_1 = \frac{V_1}{Z_0}\left[- \frac{j}{\tan \alpha s} + \frac{j}{\sin \alpha s} e^{-j\Theta} \right].$$

Making use of the results already derived in Chapter XI, page 365, the power per phase which is supplied to the line from the generator is

$$\mathfrak{W} = \mathfrak{V}_1{}^*\mathfrak{J}_1,$$

where $\mathfrak{V}_1{}^*$ is the conjugate vector to $\mathfrak{V}_1$, and leading current is assumed to correspond to positive reactive current. Since the pressure vector is drawn in the direction of the ordinate axis, it follows that

$$\mathfrak{V}_1{}^* = \mathfrak{V}_1 = V_1,$$

so that

$$\frac{\mathfrak{W}}{W_n} = w_a + jw_r = \left[- \frac{j}{\tan \alpha s} + \frac{j}{\sin \alpha s} e^{-j\Theta} \right] \quad . \quad . \quad (40)$$

The expression on the right-hand side of (40) defines a circle as shown in Fig. 31 in which Ox and Oy are the co-ordinate axes for the specific active and reactive power, respectively. The Circle Diagram shown is drawn with the centre A_0 and passes through B, where

$$OA_0 = -j\frac{1}{\tan \alpha s} : \quad A_0B = j\frac{1}{\sin \alpha s}, \text{ and is the radius of the circle.}$$

The total length of the transmission line is $AB = s$ km. (Fig. 30) and

$$\alpha = \frac{\omega}{c} = \frac{2\pi f}{3 \times 10^5},$$

as has already been defined on page 473.

Then, for any radius such as A_0G, the angle Θ will be the angle between the pressure vectors at the ends of the line. The ordinate at G will be the specific reactive power w_r which the generator will be required to feed into the line and the abscissa of G will be the specific active power w_a which is transmitted to the consumer. For points on the Circle such as G, which are above the abscissa axis Ox, the reactive power will be capacitative, while for points such as F, which are below the abscissa axis, the reactive power will be inductive.

For the point T at which the circle cuts the abscissa axis, the reactive power is zero and the specific power is $w = w_a = 1$: that is to say, the line is transmitting its natural power under these conditions For the point B the active power is zero, that is, the consumer's terminals are open-circuited and OB is the specific reactive power which the generator must feed into the line.

An exactly similar Circle applies for the conditions at the consumer's end of the line, and in Fig. 32 are shown both Circles. It will be seen that, since the line losses have been neglected, then for any given load such as OD the reactive power fed into the transmitting end of the line

will be equal to the reactive power which must be fed in at the consumer's end of the line if the pressure at each end of the line is to be maintained at the same value for all conditions of load.

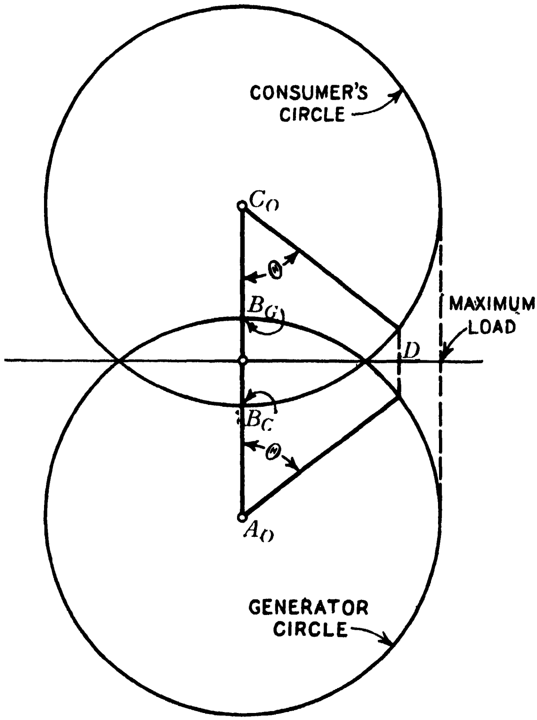


Fig. 32.

The Stability of a Three-phase Transmission System

Suppose two similar generators are working in parallel on a transmission line and let E_1 and E_2 be the respective induced e.m.f.s per phase. Further, let $x_G \, \Omega$ be the reactance of each generator and x_T the leakage reactance of each transformer when reduced to a transformation ratio of unity. The inductive reactance of the transmission line is denoted by $x_L \Omega$, and the capacitance currents of the line will be neglected.

The diagrammatical representation of one phase of this system will then be as shown in Fig. 33, and the corresponding vector diagram is also shown in Fig. 33. The power transmitted per phase will be

$$W_1 = E_1 . I \cos \frac{\Theta}{2} = E_1 I \sin \beta = \frac{E_1 I . X_T}{X_T} \sin \Theta$$

or $$W_1 = \frac{E_1 \times AB}{X_T} \sin \beta$$

where $$X_T = \{2(x_G + x_T) + x_L\}$$

that is $$W_1 = \frac{2\,(\text{Area of the triangle } OAB)}{X_T} = \frac{E_1 E_2}{X_T} \sin \Theta$$

or $$W_1 = \frac{E_1^2}{X_T} \sin \Theta : \text{ since } E_1 = E_2.$$

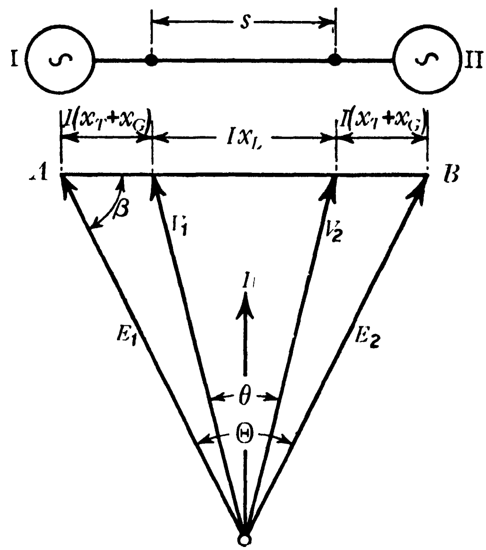


Fig. 33.

The total power for the three phases will then be

$$W_0 = 3\frac{E_1^2}{X_T} 10^3 \sin \Theta \text{ kW.},$$

when E_1 kV. is the phase e.m.f.

or $$W_0 = \frac{E_l^2}{X_T} 10^3 \sin \Theta \text{ kW.} \quad . \quad . \quad . \quad . \quad (41)$$

when E_l kV. is the e.m.f. between any two lines.

If the torque is τ_{kgm} in kilogram-meter units, and the speed of each generator is n_0 r.p.s., then

$$\tau_{kgm} \times 2\pi n_0 \times 9{\cdot}81 = \frac{E_l^2}{X_T} 10^6 \sin \Theta,$$

or $$\tau = \frac{E_l^2 \times 10^6}{2\pi n_0 9{\cdot}81 X_T} \sin \Theta = K \sin \Theta \text{ kg.m.} \quad (42)$$

for E_l in kilo-volts.

The torque is, therefore, a sine function of the angle Θ, that is, the angle of phase displacement between the pole system of the two machines. The maximum torque is, therefore, developed when $\Theta = 90°$, as shown in Fig. 34. In practice, however, it is not possible to operate the system at anything like the theoretical maximum torque, otherwise there would be no margin of stability, and it is therefore usual for normal operation to keep the value of Θ down to about 40°, as indicated by the point P in Fig. 34 (see, however, the following considerations).

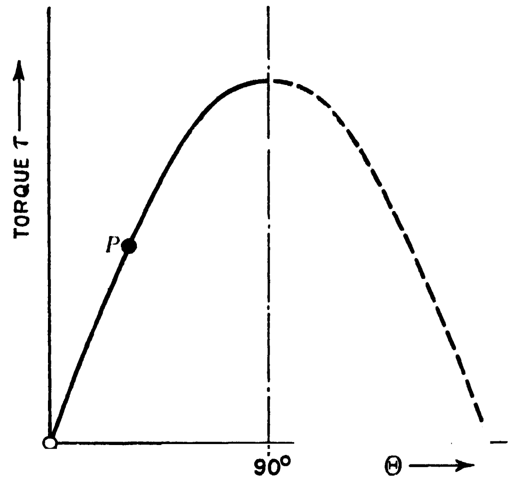


Fig. 34.

Let W_0 be the normal value of the transmitted power, that is,

$$W_0 = \sqrt{3} V_l I_a \text{ kW.},$$

where E_l is the line pressure in kV. and I_a is the watt component of the current in amperes (that is to say, the "active" current per phase). The percentage reactance per phase will then be

$$\varepsilon\% = \frac{I_a X_T}{E_p \times 10^3} 100 = \frac{I_a X_T \sqrt{3}}{E_l 10^3} 100,$$

where E_p kV. is the phase pressure and E_l kV. is the e.m.f. between any two lines.

From the expression (41) it is seen that

$$X_T = \frac{E_l^2 10^3}{W} \sin \Theta = \frac{E_l^2 10^3}{\sqrt{3} E_l I_a} \sin \Theta,$$

so that the percentage reactance per phase may be written

$$\varepsilon\% = (\sin \Theta) \times 100 \quad . \quad . \quad . \quad . \quad (43)$$

Taking a normal value of Θ to be about 40° as already stated, in the foregoing, then the total permissible percentage reactance will be about 64 per cent. Assuming the reactance x_G of each generator to be about 10 per cent. and the reactance of each transformer to be about 10 per cent., then

$$64\% = X_T = \{2(x_G + x_T) + x_L\} = \{2(10 + 10) + x_L\},$$

so that x_L must be not greater than about 24 per cent. That is to say, the angle θ between the line pressure vectors in Fig. 33 must not be greater than the value given by $\sin \theta = 0{\cdot}24$, so that θ must be not greater than about 14°.

For example, suppose a transmission line is operating at 220 kV. line pressure, and assuming the surge impedance to be $Z_0 = 350\ \Omega$ per phase, the natural power of the transmission line will then be (see page 478),

$$W_n = \frac{220^2}{350} 10^3 \text{ kW.} = 140{,}000 \text{ kW.}$$

If the reactance of the line is taken to be $x_L = 0{\cdot}4\ \Omega$ per phase per kilometre, which is a reasonable practical value for overhead lines, then the maximum distance of transmission which is permissible if the angle θ is to be not greater than about 14° when the transmitted power is 140,000 kW. will be s km., where (see Fig. 33),

$$\frac{\frac{1}{2}s(I_a 0{\cdot}4)}{\left(\frac{220}{\sqrt{3}}\right)10^3} = \sin 7° = 0{\cdot}12$$

and $$I_a = \frac{140{,}000}{220\sqrt{3}},$$

so that $$s = \frac{220^2 \times 10^3 \times 0{\cdot}24}{140{,}000 \times 0{\cdot}4} = 207 \text{ km.}$$

The limitation of the transmission distance due to the stability requirements is thus found to be about 200 km. If it is required to transmit power over greater distances, one method is to run several lines in parallel, thus reducing the percentage reactance of the line : this method is, however, very expensive. For very long distance transmission a more economical method is to sectionalise the line into lengths of about 200 km. and at each junction of neighbouring sections to connect a synchronous machine. This machine will not be required to supply any active power, but will apply to the line an e.m.f. equal to the line e.m.f. The angle of the pole displacement Θ between neighbouring machines will then be within the limit fixed by stability requirements, and by such means it becomes possible to transmit power economically over distances of practically unlimited length.

Chapter XVI

THE PROPAGATION OF ELECTROMAGNETIC WAVES THROUGH SPACE

Closed and Open Electromagnetic Circuits

WITH the exception of Chapter XV the electric circuits which have been dealt with in the previous parts of this book have been those which comprise an essentially concentrated inductance and an essentially concentrated capacitance, and the fundamental characteristics have been investigated for oscillatory circuits such as that which is shown, for example, in Fig. 1. In such circuits, when the condenser is charged the electric field which is produced is confined almost

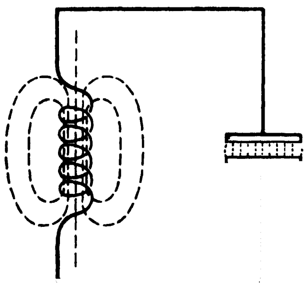


Fig. 1.

Fig. 2.

entirely to the dielectric enclosed between the charged plates. As this electric field disappears, the corresponding energy is transformed into the electromagnetic energy associated with the magnetic field linked with the coils of the inductance. The oscillating current due to this periodic interchange of energy between the condenser and the inductance gradually decays in amplitude mainly on account of the loss of heat energy in the resistance of the circuit. Even if no such circuit resistance loss occurred, however, there would still be a gradual dissipation of energy due to the fact that there will always be a certain amount of stray lines of electric force which do not complete the path across the dielectric between the condenser plates. These stray portions of the electric field and the consequent energy associated with them are not confined to the local circuit of inductance and capacitance, although, in practice, the amount of this stray energy is, in general, relatively very small and need not be taken into account here.

If the arrangement of the components of the closed oscillatory circuit is modified to form a single straight conductor to each end of which a capacitance plate is fixed, then an open oscillatory circuit is obtained, as is shown diagrammatically in Fig. 2, and this arrangement forms, in fact, a simple dipole antenna. In Fig. 3 the lines of electric force (dotted lines) are shown for such a dipole antenna and they are seen to extend over a wide range of the surrounding space : in Fig. 3 are also shown a few representative equipotential (chain-dotted) lines, P_0, P_1, P_2 : that

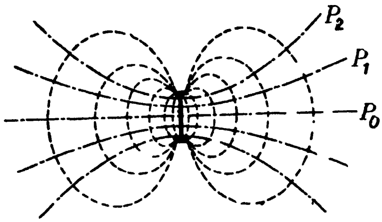


Fig. 3.

is to say, the lines of force are traces in the plane of the paper of corresponding surface of electric force and the equipotential lines are the traces in the plane of the paper of equipotential surfaces which everywhere cut the surfaces of electric force at right angles. As the condenser charges and discharges and so produces an oscillatory current then, owing to the finite velocity (3×10^5 km./sec.) of the movement of the lines of force in space, they cannot all return to the antenna in the time of one-quarter cycle of the oscillating current and consequently the corresponding energy associated with the lines (or tubes) of force is

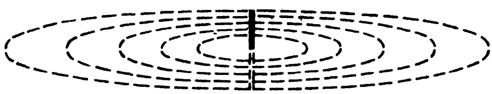


Fig. 4.

radiated out into space, and from what has been said it will be clear that the higher the frequency of the oscillating current the higher will be the proportion of radiated energy.

Associated with the lines of electric force, as shown in Fig. 3, are lines of magnetic force which are not shown in that diagram, but are shown in Fig. 4 ; the two sets of lines of force are inextricably intermingled in contrast to the more definite separation which is typical of the closed circuit of Fig. 1. Such lines of magnetic force are a series of concentric circles which are at right angles to the lines of force, these concentric circles again being traces in the plane of the paper of surfaces of magnetic force cutting at right angles the surfaces of electric force. For the

plane through the midpoint of a dipole antenna the magnetic lines of force are shown in perspective in Fig. 4.

As the antenna charge oscillates, the change of the magnetic field involves, and give rise to, a corresponding change of the electric field and *vice versa*, the change of the electric field involves and gives rise to a corrésponding change of the magnetic field. Such mutual changes are illustrated in Fig. 5*a* and 5*b*, and the electric field which surrounds the antenna can therefore be specified at any place either by the p.d. between two adjacent equipotential surfaces expressed in volts per metre or by the strength of the magnetic field expressed in oersted.

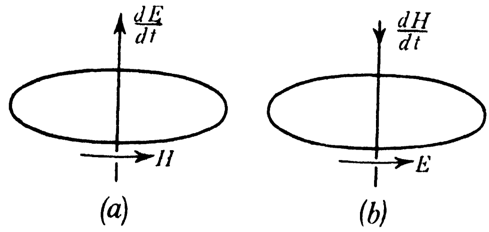


Fig. 5.

The Maxwell equations define the associated electric and magnetic phenomena of an electromagnetic field which is propagated in space, and the Maxwell-Hertz equations define the phenomena associated with the propagation of the electric and magnetic fields due to a simple dipole antenna as exemplified in Fig. 2.

In Chapter XV the electric and magnetic fields between transmission lines have been dealt with, this problem having involved a consideration of the distributed inductance of the lines and the distributed capacitance between the lines. Such transmission line problems do, in fact, provide a link between the completely closed oscillatory circuit of Fig. 1 and the essentially open oscillatory circuit of a radiating antenna.

The Maxwell Equations

In Fig. 6 is shown diagrammatically the flux of electric force of intensity $\mathfrak{E}$ electromagnetic units crossing at right angles, the surface of Q sq. cm. area. The density σ of the electric displacement charge over this area is given by the relationship [see also Chapter III, expression (89)]

$$\sigma = \frac{\varepsilon\mathfrak{E}}{c^2 4\pi} \text{ in electromagnetic units,}$$

where ε is the dielectric constant in electrostatic units and $\frac{\varepsilon}{c^2}$ is the dielectric constant in electromagnetic units (see also Chapter I, page 6): for open space $\varepsilon = 1$.

The "electric displacement current" $\mathfrak{J}$ across the area of Q square centimetres will then be given by the equation

$$\mathfrak{J} = \frac{d}{dt}(Q\sigma) = \frac{d}{dt}\left(\frac{\varepsilon Q\mathfrak{E}}{4\pi c^2}\right) \text{ in electromagnetic units,}$$

that is

$$4\pi\mathfrak{J} = \frac{\varepsilon}{c^2}\frac{d}{dt}(Q\mathfrak{E}). \qquad . \quad . \quad . \quad . \quad . \quad (1)$$

But $4\pi\mathfrak{J}$ is the magnetomotive force round the boundary of the area Q, that is

$$4\pi\mathfrak{J} = \oint \mathfrak{H}\, dl. \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\mathfrak{H}$ is the intensity of magnetic force along the boundary of the area Q. Hence, combining (1) and (2) gives

$$\oint \mathfrak{H}\, dl = \frac{\varepsilon}{c^2}\frac{d}{dt}(Q\mathfrak{E}) \qquad . \quad . \quad . \quad . \quad . \quad (3)$$

since for open space $\varepsilon = 1$, as already stated in the foregoing.

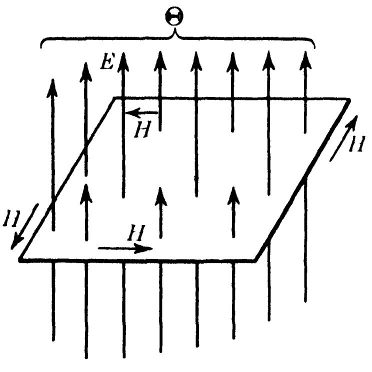


Fig. 6.

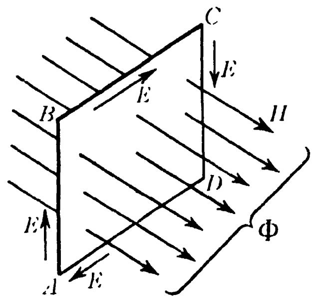


Fig. 7.

In Fig. 7 is shown, diagrammatically, the magnetic flux ϕ of intensity $\mathfrak{B}$ in electromagnetic units, that is gauss, crossing at right angles the surface of S sq. cm. area, so that the "magnetic displacement current" $\mathfrak{M}$ in electromagnetic units is given by the equation

$$4\pi\mathfrak{M} = \frac{d\phi}{dt} = \frac{d}{dt}(S\mathfrak{B}) = \frac{d}{dt}(S\mu\mathfrak{H}) \qquad . \quad . \quad . \quad . \quad (4)$$

the "magnetic displacement current" $\mathfrak{M}$ of (4) being the counterpart of the "electric displacement current" $\mathfrak{J}$ of (1). The magnetic intensity is measured in electromagnetic units, that is oersted, the magnetic permeability for open space being $\mu = 1$, also in electromagnetic units.

But $$4\pi\mathfrak{M} = \oint \mathfrak{E}\, dl \qquad (5)$$

where the integration is taken round the boundary of the surface S. Combining (4) and (5) gives

$$\oint \mathfrak{E}\, dl = \frac{d}{dt}(S\mu\mathfrak{H}) \quad . \quad . \quad . \quad . \quad (6)$$

Now, suppose that an electromagnetic wave-front of uniform intensity and of infinitely large surface is at the moment t_0 in the position shown by the square surface *abcd* in Fig. 8, the direction of travel of the wave being at right angles to its front, that is, in the direction of the x axis as is shown in Fig. 8. In the element of time δt the wave front will move through the distance δx cm., so that at the time $(t_0 + \delta t)$ it will

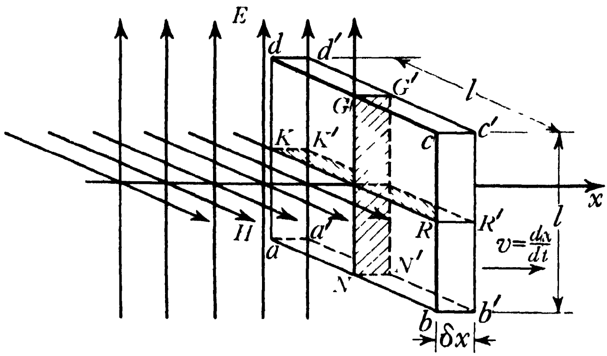


Fig. 8.

have reached the position defined by the square surface $a'b'c'd'$. In this element of time δt, the magnetic flux which enters the shaded vertical area ($GG'NN' = S = l.\delta x$ sq. cm.) will be

$$d\phi = \mathfrak{B}.S = \mathfrak{B}.l.\delta x = \mathfrak{H}.\mu.l.\delta x,$$

so that, by equation (6), the induced e.m.f. round the shaded vertical area $GG'NN'$ will be given by

$$\oint \mathfrak{E}\, dl = \mathfrak{H}.\mu.l\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad (7)$$

Similarly, the flux of electric force which enters the horizontal shaded area ($RR'KK' = Q = l.\delta x$ sq. cm.) in Fig. 8 will be

$$\Theta = \frac{\varepsilon}{c^2}Q.\mathfrak{E} \text{ in electromagnetic units,}$$

that is [as in equation (1)],

$$4\pi\mathfrak{J} = \frac{d\Theta}{dt} = \frac{\varepsilon}{c^2}\frac{d}{dt}(Q.\mathfrak{E}) = \frac{\varepsilon}{c^2}\mathfrak{E}.l\frac{dx}{dt} \quad . \quad . \quad . \quad (8)$$

so that, from equations (3) and (8),

$$\oint \mathfrak{H} \cdot dl = \frac{\varepsilon}{c^2} \mathfrak{E} \cdot l \frac{dx}{dt} \qquad (9)$$

Now, the expression $\oint \mathfrak{E}\, dl$ on the left-hand side of (7) is the line integral of the electric force $\mathfrak{E}$ taken round the boundary of the area $GG'NN'$ and due to the magnetic flux $d\phi$ which crosses this area. The initial value of this line integral is $\mathfrak{E}.l$ and the final value is zero, so that the line integral in (7) may be written $\oint \mathfrak{E}.l = \mathfrak{E}.l$. Similarly, the line integral in (9) may be written $\mathfrak{H}.l$, and consequently the equations (7) and (9) become, respectively,

$$\left.\begin{aligned} \mathfrak{E}.l &= \mathfrak{H}.\mu.l\frac{dx}{dt} \\ \mathfrak{H}.l &= \mathfrak{E}\frac{\varepsilon}{c^2}l\frac{dx}{dt} \end{aligned}\right\} \qquad (10)$$

If these two equations are now multiplied together it is found that the speed of travel of the wave-front is

$$\boxed{v = \frac{dx}{dt} = \frac{c}{\sqrt{\mu\varepsilon}} \text{ cm. per second}} \qquad (11)$$

and since, for open space, $\mu = 1$ and $\varepsilon = 1$, the speed is then

$$\boxed{v = c = 3 \times 10^{10} \text{ cm. per second}} \qquad (12)$$

The Maxwell-Hertz Equations for the Propagation of the Electromagnetic Field due to a Dipole Antenna

The Maxwell-Hertz equations are based on the assumption that the length of the dipole is relatively so small that the current may be assumed to be uniformly distributed throughout its length. Fig. 9*a* and 9*b* show an element of a spherical shell forming the envelope of the electromagnetic field due to the dipole, it being observed that at distances which are great in comparison with the length of the dipole, the electromagnetic field can be assumed to be of spherical shape as illustrated in Fig. 9*a* and 9*b*. The equations which define the propagation of the field will be derived from the relationships which exist between the electric and magnetic forces associated with the element of the spherical shell as shown in Fig. 9*a* and 9*b*.

This element of the spherical shell is bounded by radial planes which are relatively displaced by the longitudinal angle $d\Theta$ and by the two

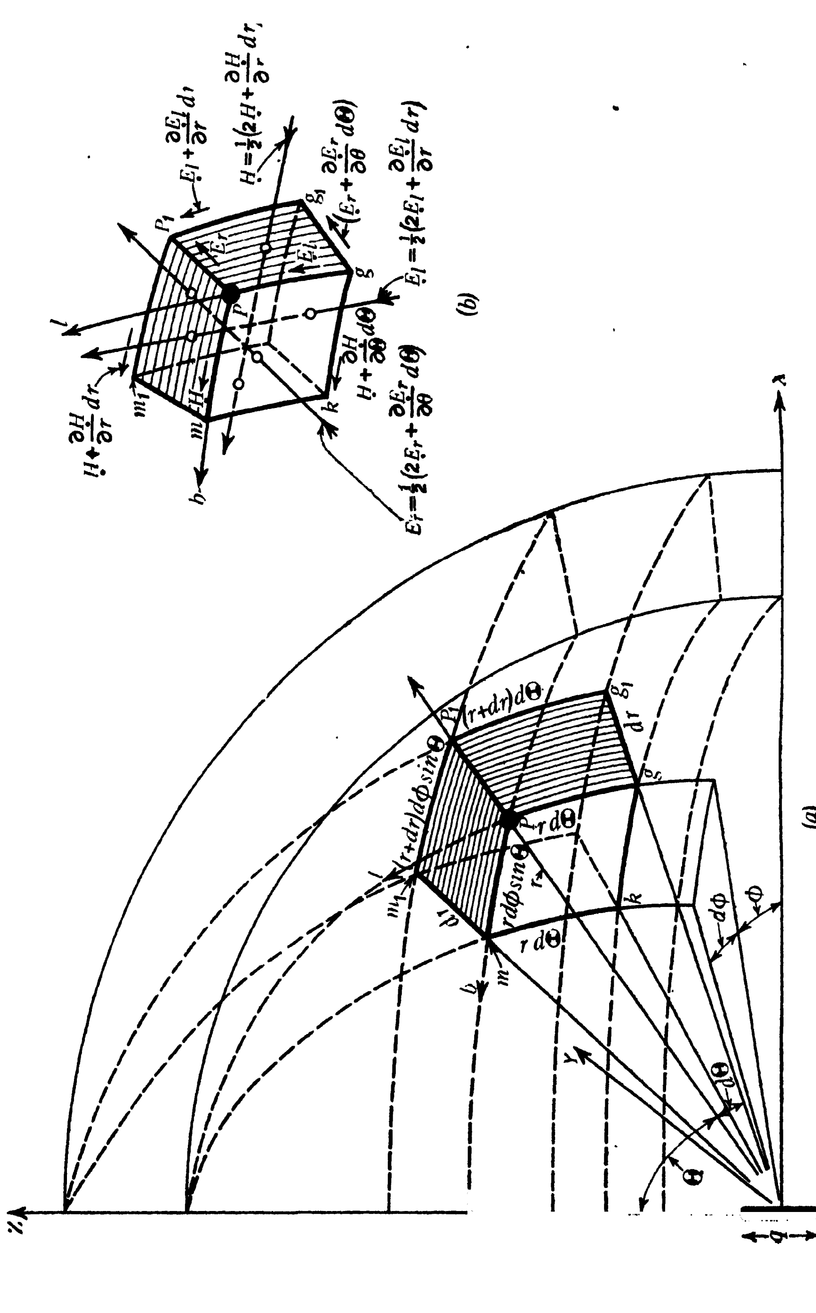


Fig. 9.

vertical planes enclosing the angled $d\phi$. The lines of magnetic force will only be developed in horizontal planes, whilst the lines of electric force will have a component $\mathfrak{E}_r$ in the radial direction of the sphere and a component $\mathfrak{E}_l$ in the longitudinal direction as shown in Fig. 9*b*. The Maxwell equations for the line integral of each of the components of the electric force and the line integral of the magnetic force may then be written down as follows:

(*i*) FOR THE SURFACE PP_1, gg_1, FIG. 9*a* and 9*b*.

$$\oint \mathfrak{E} dl = \left(\mathfrak{E}_l + \frac{\partial \mathfrak{E}_l}{\partial r} dr\right)(r + dr)d\Theta + \left(\mathfrak{E}_r + \frac{\partial \mathfrak{E}_r}{\partial \Theta} d\Theta\right)dr - \mathfrak{E}_r dr - \mathfrak{E}_l r d\Theta \quad . \quad (13)$$

and since from equation (6) on page

$$\oint \mathfrak{E}\, dl = \frac{d}{dt}(S\mathfrak{H}), \text{ where } S = r.dr.d\Theta.$$

Then, after expanding the terms on the right-hand side of (13), this equation reduces to

$$\frac{\partial \mathfrak{E}_l}{\partial r} + \frac{\mathfrak{E}_l}{r} + \frac{1}{r}\frac{\partial \mathfrak{E}_r}{\partial \Theta} = \frac{d\mathfrak{H}}{dt} \quad . \quad . \quad . \quad . \quad (14)$$

(*ii*) FOR THE SURFACE *Pgkm* OF FIG. 9*a* and 9*b*.

$$\oint \mathfrak{H}\, dl = -\mathfrak{H}.r.d\phi \sin \Theta + \left(\mathfrak{H} + \frac{\partial \mathfrak{H}}{\partial \Theta} d\Theta\right) r.d\phi \sin(\Theta + d\Theta) \quad . \quad (15)$$

and since from equation (3) on page 495, viz.,

$$\oint \mathfrak{H}\, dl = \frac{1}{c^2}\frac{d}{dt}(Q.\mathfrak{E}_r) : \text{ where } Q = r.d\Theta(r.d\phi \sin \Theta)$$

it follows that, after expanding the right-hand side of (15), this equation reduces to

$$\frac{1}{r}\mathfrak{H} \cot \Theta + \frac{1}{r}\frac{\partial \mathfrak{H}}{\partial \Theta} = \frac{1}{c^2}\frac{\partial \mathfrak{E}_1}{\partial t} \quad . \quad . \quad . \quad (16)$$

(*iii*) FOR THE SURFACE $PP_1\ mm_1$ OF FIG. 9*a* and 9*b*,

$$\oint \mathfrak{H}\, dl = -\mathfrak{H} r d\phi \sin \Theta + \left(\mathfrak{H} + \frac{\partial \mathfrak{H}}{\partial r} dr\right)(r + dr)d\phi \sin \Theta \quad . \quad (17)$$

and again, using the relationship of equation (3), viz.,

$$\oint \mathfrak{H}\, dl = \frac{1}{c^2}\frac{d}{dt}(Q\mathfrak{E}_r) : \text{ where } Q = dr(r.d\phi \sin \Theta),$$

equation (17) becomes

$$\frac{1}{r}\mathfrak{H} + \frac{\partial \mathfrak{H}}{\partial r} = \frac{1}{c^2}\frac{\partial \mathfrak{E}_1}{\partial t} \quad . \quad . \quad . \quad . \quad (18)$$

The expressions (14), (16) and (18) are the Maxwell-Hertz equations for the propagation through space of the electromagnetic field of a dipole

antenna. For convenience these three fundamental equations are assembled below :

$$\boxed{\begin{aligned} \frac{1}{r}\frac{\partial \mathfrak{E}_r}{\partial \Theta} + \frac{\partial \mathfrak{E}_l}{\partial r} + \frac{\mathfrak{E}_l}{r} &= \frac{d\mathfrak{H}}{dt} \\ \frac{1}{r}\frac{\partial \mathfrak{H}}{\partial \Theta} + \frac{1}{r}\mathfrak{H}\cot\Theta &= \frac{1}{c^2}\frac{\partial \mathfrak{E}_r}{\partial t} \\ \frac{\partial \mathfrak{H}}{\partial r} + \frac{1}{r}\mathfrak{H} \qquad &= \frac{1}{c^2}\frac{\partial \mathfrak{E}_l}{\partial t} \end{aligned}} \quad . \quad . \quad . \quad (19)$$

The three simultaneous equations (19) may be solved by means of the "Hertz Vector $\mathfrak{Z}_H$", which is dependent only upon r and t. In accordance with a proposition enunciated by Hertz, the vectors $\mathfrak{E}_r$, $\mathfrak{E}_l$ and $\mathfrak{H}$, may be respectively written

$$\left.\begin{aligned} \mathfrak{E}_r &= -\frac{2c^2}{r}\cos\Theta\frac{\partial \mathfrak{Z}_H}{\partial r} \\ \mathfrak{E}_l &= -c^2\sin\Theta\left(\frac{1}{r}\frac{\partial \mathfrak{Z}_H}{\partial r} + \frac{\partial^2 \mathfrak{Z}_H}{\partial r^2}\right) \\ \mathfrak{H} &= -\sin\Theta\frac{\partial^2 \mathfrak{Z}_H}{\partial r . \partial t} \end{aligned}\right\} \quad . \quad . \quad (20)$$

If, in addition, $\mathfrak{Z}_H$ also satisfies the equation,

$$\frac{1}{c^2}\frac{\partial^2 \mathfrak{Z}_H}{\partial t^2} = \frac{2}{r}\frac{\partial \mathfrak{Z}_H}{\partial r} + \frac{\partial^2 \mathfrak{Z}_H}{\partial r^2} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which will be the case if

$$\mathfrak{Z}_H = \frac{Z_H}{r}\sin\omega\left(t - \frac{r}{c}\right) = \frac{Z_H}{r}\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)$$

that is, if

$$\mathfrak{Z}_H = \frac{Z_H}{r}\sin\alpha \quad . \quad . \quad . \quad . \quad . \quad (22)$$

then the solution of the simultaneous equations (19) for the propagation of the electromagnetic field through space will be

$$\boxed{\begin{aligned} \mathfrak{E}_r &= \frac{2c^2 Z_H}{r^3}\cos\Theta\left[\sin\alpha + \frac{2\pi r}{\lambda}\cos\alpha\right] \\ \mathfrak{E}_l &= -\frac{c^2 Z_H}{r^3}\sin\Theta\left[\sin\alpha + \frac{2\pi r}{\lambda}\cos\alpha - \left(\frac{2\pi r}{\lambda}\right)^2\sin\alpha\right] \\ \mathfrak{H} &= \frac{c Z_H}{r^3}\sin\Theta\frac{2\pi r}{\lambda}\left[\cos\alpha - \frac{2\pi r}{\lambda}\sin\alpha\right] \end{aligned}} \quad . \quad (23)$$

In applying these equations it is to be observed that two distinct cases have to be considered,

(i) The distance r is great in comparison with the length q of the dipole, but small as compared with the quantity $\frac{\lambda}{2\pi}$.

(ii) The distance r is great in comparison with the quantity $\frac{\lambda}{2\pi}$.

CASE I.—*The Distance r is Small in Comparison with $\frac{\lambda}{2\pi}$.*—In equations (23) the quantity $r\frac{2\pi}{\lambda}$ can be neglected and writing for α its equivalent $\omega\left(t - \frac{r}{c}\right)$, the equations (23) reduce to

$$\begin{aligned} \mathfrak{E}_r &= \frac{2c^2Z_H}{r^3}\cos\Theta\sin\omega\left(t-\frac{r}{c}\right) \quad . \quad . \quad . \quad (a) \\ \mathfrak{E}_l &= -\frac{c^2Z_H}{r^3}\sin\Theta\sin\omega\left(t-\frac{r}{c}\right) \quad . \quad . \quad . \quad (b) \\ \mathfrak{H} &= \frac{2\pi cZ_H}{r^2\lambda}\sin\Theta\cos\omega\left(t-\frac{r}{c}\right) \quad . \quad . \quad . \quad (c) \end{aligned} \qquad (24)$$

It is to be noticed here that the equation (24c) for the vector H is in complete accordance with the Biot-Savart-Laplace Law (see Chapter VIII, page 211), viz.,

$$\mathfrak{H} = \frac{iq\sin\Theta}{r^2} = \frac{\sqrt{2}Iq\sin\Theta}{r^2}\sin(\omega t \quad \phi)$$

where I is the r.m.s. value of the current measured in electromagnetic units. If this expression is equated to (24c) it is seen that

$$\frac{2\pi cZ_H}{r^2\lambda}\sin\Theta\cos\omega\left(t-\frac{r}{c}\right) = \frac{\sqrt{2}Iq}{r^2}\sin\Theta\sin(\omega t - \phi)$$

from which it follows that

$$Z_H = \frac{\sqrt{2}Iq\lambda}{2\pi c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

since for small values of $\frac{r}{c}$, $\cos\left(\omega t - \frac{r}{c}\right) \doteq \cos\omega t$, and since from equations (24) the vectors $\mathfrak{H}$ and $\mathfrak{E}$ will be 90° out of phase in time, that is, $\phi = \frac{\pi}{2}$, which also means that the current is 90° out of phase with the electric force.

If the value of Z_H given by the expression (25) is substituted in the equations (24) it is found that

$$\left.\begin{aligned}
\mathfrak{E}_r &= \frac{c\sqrt{2}Iq\lambda}{\pi r^3}\cos\Theta\sin\omega\left(t-\frac{r}{c}\right)\\
\mathfrak{E}_1 &= -\frac{c\sqrt{2}Iq\lambda}{2\pi r^3}\sin\Theta\sin\omega\left(t-\frac{r}{c}\right)\\
\mathfrak{H} &= \frac{\sqrt{2}Iq}{r^2}\sin\Theta\cos\omega\left(t-\frac{r}{c}\right)
\end{aligned}\right\} \quad . \quad . \quad . \quad (26)$$

These equations show the important and remarkable result that the electric and magnetic field vectors are displaced in time by one-quarter of a period relatively to each other. This relationship signifies therefore, that during one half-period energy is radiated outwards from the dipole, whilst during the next half-period energy is flowing back into the dipole. It follows, therefore, that within the distance r from the dipole where r is small compared with $\frac{\lambda}{2\pi}$, the greater portion of the energy stream is wattless.

CASE II.—*When the Distance r is Large in Comparison with* $\frac{\lambda}{2\pi}$.—In this case the simultaneous equations (23) become simplified by neglecting those terms in which $\frac{1}{r^2}$ or $\frac{1}{r^3}$ appears as a multiplying factor so that after substituting for Z_H the value given in expression (25) the equations become :

$$\left.\begin{aligned}
\mathfrak{E}_r &= 0\\
\mathfrak{E}_1 &= \frac{2\pi c\sqrt{2}Iq}{r\lambda}\sin\Theta\sin\omega\left(t-\frac{r}{c}\right)\\
\mathfrak{H} &= -\frac{2\pi\sqrt{2}Iq}{r\lambda}\sin\Theta\sin\omega\left(t-\frac{r}{c}\right)
\end{aligned}\right\} \quad . \quad . \quad (27)$$

The extremely important and remarkable result is now obtained that the electric and magnetic field vectors are in phase with one another in time so that the energy will now be streaming continuously outwards.

Radiation in the Direction Perpendicular to the Axis of the Dipole

This direction of radiation is of the greatest practical importance and is defined by (see Fig. 9*a*),

$$\Theta = \frac{\pi}{2} : \quad \cos\Theta = 0 : \quad \sin\Theta = 1.$$

If these substitutions are made in the general simultaneous equations (23),

and if the value of Z_H given by the expression (25) is also substituted, the following relationships will be obtained,

$$\left.\begin{aligned}\mathfrak{E}_r &= 0\\ \mathfrak{E}_1 &= \left(\frac{c\sqrt{2}Iq\lambda}{2\pi r^3} - \frac{2\pi c\sqrt{2}Iq}{r\lambda}\right) + j\frac{c\sqrt{2}Iq}{r^2}\\ \mathfrak{H} &= -\frac{2\pi\sqrt{2}Iq}{r\lambda} + j\frac{\sqrt{2}Iq}{r^2}\end{aligned}\right\} \quad . \quad . \quad (28)$$

In this form of the expressions for the vectors $\mathfrak{E}_1$ and $\mathfrak{H}$ the terms in equation (23) which contain $\sin\alpha$ are taken as real quantities and the terms which contain $\cos\alpha$ as imaginary quantities. The expressions for $\mathfrak{E}_1$ and $\mathfrak{H}$ in equations (28) give the respective *peak* values.

The linear quantities $r : \lambda : q :$ in equations (28) are expressed in centimetres, the electric force $\mathfrak{E}_1$ in electromagnetic units per centimetre, the magnetic force $\mathfrak{H}$ in electromagnetic units, and the current I in electromagnetic units. For practical purposes, however, it is more convenient to express the *electric force in volts per metre*; the *magnetic force in oersted*; the *current in r.m.s. amperes*; and the linear quantities $r : \lambda : q :$ *in metres.*

The length q of the dipole may be expressed in terms of the "equivalent height" of the antenna, $q = 2h$ (see also page 507). The equations (28) may then be written in the form :

$$\left.\begin{aligned}\mathfrak{E} = \mathfrak{E}_1 &= \left\{30\frac{I\lambda h}{\pi r^3} - 377\frac{Ih}{r\lambda}\right\} + j60\frac{Ih}{r^2} \text{ r.m.s. volts per metre}\\ \mathfrak{H} &= -\frac{\pi Ih}{250\lambda r} + j\frac{Ih}{500r^2} \text{ r.m.s. oersted}\end{aligned}\right\} \quad . \quad (29)$$

Reference should also be made to the vector relationships shown in Fig. 9*b* for $\mathfrak{E}$ and $\mathfrak{H}$.

The distance beyond which the energy may be assumed to stream continuously outwards is $r > 4\lambda$ and the equations (29) then reduce to the very simple form

$$\boxed{\begin{aligned}\mathfrak{E} &= 377\frac{I.h}{r.\lambda} \text{ r.m.s. volts per metre}\\ \mathfrak{H} &= \frac{\pi.I.h}{250\lambda.r} \text{ r.m.s. oersted}\end{aligned}} \quad . \quad . \quad (30)$$

The Magnitude of the Radiated Electromagnetic Energy

It has been pointed out already on page 501 that the Maxwell-Hertz equations assume that the length q of the dipole is so small relatively to the radiated wave-length that the current may be assumed to be uniformly distributed throughout its length, and although in the case of an actual antenna, the current is not uniformly distributed throughout

its length, the Maxwell-Hertz equations may still be applied if the "equivalent height" of the antenna h be used in the formulae instead of the geometrical height H_A (see also Fig. 14, page 507), that is

$$q = 2h = 2 . \frac{2}{\pi} H_A \quad . \quad . \quad . \quad . \quad (31)$$

At distances from the antenna, which are great in comparison with

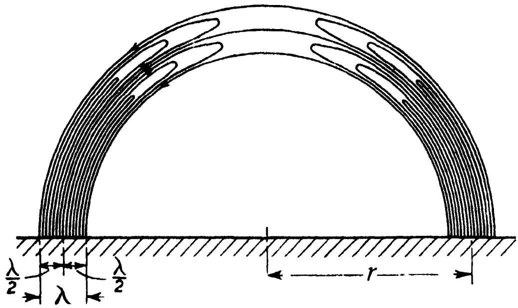


Fig. 10.

the wave-length λ, that is to say, within the "radiating range" of the antenna [see page 502, Case II], the electromagnetic field due to the antenna of equivalent height h may be taken to be radiated in the form of hemispherical waves as shown in Fig. 10, in which a diagrammatical representation of a section in the plane of the paper of one such hemispherical wave is shown. In Fig. 11 is shown the radiated electric field

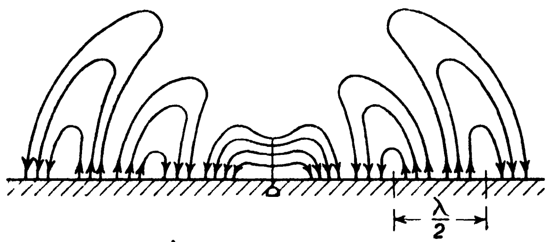


Fig. 11.

of the antenna, whilst the distribution of the electric field in the neighbourhood of the antenna is shown in Fig. 12 and the distribution of the magnetic field in Fig. 13.

Now it is to be observed that the field due to a dipole as already considered on page 493, and shown diagrammatically in Fig. 12 is symmetrical about a plane through the midpoint of the dipole and at right angles to its length. If the earth's surface is assumed to be a perfect conductor and consequently does not absorb energy from the travelling

waves, then the dipole of length q may be considered as being equivalent to a vertical antenna of height h in combination with its image in the earth's surface as shown in Fig. 12, so that the expression for the electric

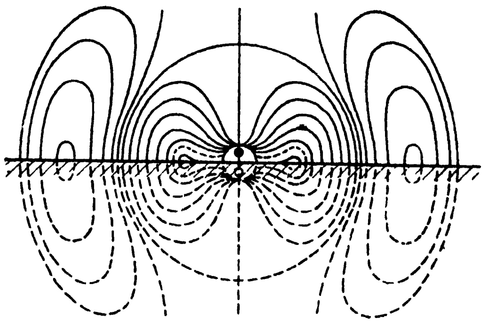


Fig. 12.

and magnetic fields of a dipole may be directly applied to the case of a vertical antenna. Reference to Figs. 10 to 13 will make this clear.

The volume of the hemispherical shell shown in Fig. 10, for which r is much greater than λ, that is, for the radiating range of the antenna, is very approximately

$$V = 2\pi r^2 \lambda \text{ c.cm.} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

when r and λ are expressed in centimetres.

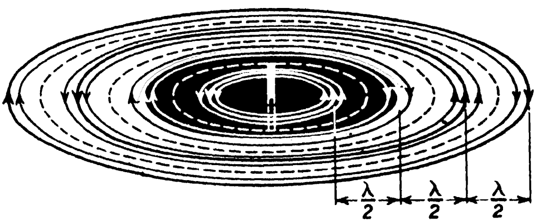


Fig 13.

The energy stored in such a hemispherical shell and due to the electric force of intensity $\mathfrak{E}$ is (see page 118, Chapter IV)

$$U_E = \frac{\epsilon}{8\pi}\int \mathfrak{E}^2 \, dv \text{ ergs} \quad . \quad . \quad . \quad . \quad (33)$$

where $\mathfrak{E}$ is the intensity of the electric force in electrostatic units, dv is the element of volume in cubic centimetres and $\epsilon = 1$ for open space. Now, it has been shown on page 502, expression (27), that

$$\mathfrak{E} = \frac{2\pi . c\sqrt{2}I . q}{r . \lambda} \sin \Theta \sin \omega\left(t - \frac{r}{c}\right) \text{ in electromagnetic units,}$$

that is

$$\mathfrak{E} = \frac{2\pi\sqrt{2}I.q}{r.\lambda} \sin\Theta \sin\omega\left(t - \frac{r}{c}\right) \text{ in electrostatic units.} \qquad (34)$$

where I is the r.m.s. value of the current in electromagnetic units: $\lambda : r$ and q: are each expressed in centimetres and $c = 3 \times 10^{10}$ cm. per second, that is, the velocity of light in open space.

The peak value of $\mathfrak{E}$ will be obtained at the surface of the earth, that is, when $\Theta = \frac{\pi}{2}$, so that

$$E_{max.} = \frac{2\pi\sqrt{2}Iq}{r\lambda} \text{ in electrostatic units} \qquad (35)$$

But the magnitude of E will vary sinusoidally in the radial direction from r to $r \pm \frac{\lambda}{2}$ (see Fig. 10) and will also vary sinusoidally in the circumferential direction from $\Theta = \frac{\pi}{2}$ to $\Theta = 0$. The mean value of E^2 for the whole volume of the hemispherical shell can then be shown to be $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$ times the maximum value $(E_{max.})^2$.

The energy stored in the hemispherical shell due to the electric force may then be written [see equations (32), (33) and (35)]:

$$U_E = \frac{1}{8\pi}\frac{1}{3}\left\{\frac{2\pi\sqrt{2}Iq}{r\lambda}\right\}^2 2\pi r^2\lambda \text{ ergs,}$$

that is

$$U_E = \frac{2}{3}\frac{\pi^2\lambda}{10^7}\left(\frac{Iq}{\lambda}\right)^2 \text{ joules.}$$

An equal amount of energy will also be stored in the hemispherical shell due to the magnetic intensity $\mathfrak{H}$, which is inextricably associated with the electric force $\mathfrak{E}$, so that the total amount of stored energy in the shell will be (see also Chapter XV, page 463)

$$U_T = 2U_E = \frac{4}{3}\frac{\pi^2\lambda}{10^7}\left(\frac{Iq}{\lambda}\right)^2 \text{ joules,}$$

and substituting $q = 2h$, then the total energy stored in the hemispherical shell will be

$$U_T = \left(\frac{16}{3}\right)\frac{\pi^2\lambda}{10^7}\left(\frac{Ih}{\lambda}\right)^2 \text{ joules} \qquad (36)$$

If the transmitter frequency is f hertz, then $f = \frac{c}{\lambda}$ and the periodic time is $\tau = \frac{1}{f} = \frac{\lambda}{c}$, so that the mean rate of flow of energy through the hemispherical shell, that is, the *radiated power*, will be

$$W = \frac{U_T}{\tau} = U_T\frac{c}{\lambda} = \frac{16\pi^2 c}{3 \times 10^7}\left(\frac{Ih}{\lambda}\right)^2 \text{ watts}$$

where $c = 3 \times 10^{10}$ cm./sec. and I is the r.m.s. value of the current in electromagnetic units.

If the current is expressed in *r.m.s. amperes* and h and λ in *metres*, the radiated power will be

$$\boxed{\begin{aligned} & W = \left\{160\pi^2\left(\frac{h}{\lambda}\right)^2\right\}I^2 \\ \text{that is}\quad & W = R_R I^2 \end{aligned}}\quad \text{watts} \qquad . \quad . \quad (37)$$

where R_R ohms is the " radiation resistance " of the antenna and represents the load on the antenna due to the radiated power. The current I is assumed to be measured at the base of the antenna, that is, at the position of maximum current. For example, in the case of a quarter-wave antenna

$$H_A = \frac{\pi}{2}h = \frac{\lambda}{4},$$

so that

$$h = \frac{\lambda}{2\pi}$$

and the radiation resistance will be

$$R_R = 40\ \Omega.$$

Making the appropriate correction to take into account the fact that the current is not distributed quite sinusoidally along the antenna, the radiation resistance then becomes

$$\boxed{R_R = 36{\cdot}6\ \Omega} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In Fig. 14*a* and 14*b* is shown a vertical quarter-wave anetnna of height H and having a sinusoidal distribution of current, the magnitude of which is I at the base of the antenna. Then it is easily shown (see

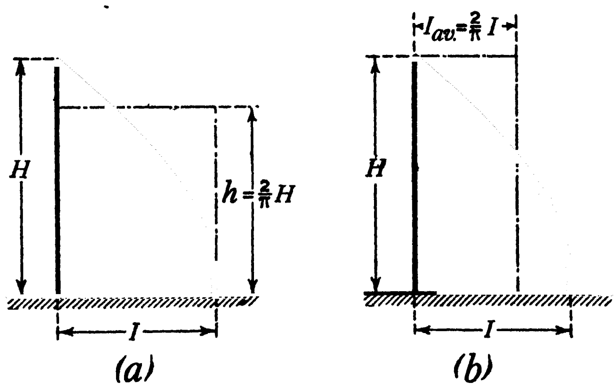


Fig. 14.

Fig. 14*a*) that if the current were to be uniformly distributed along the antenna and of magnitude I the height h for which the current volume would be the same as in the actual current distribution, will be given by the equation

$$h = \frac{2}{\pi}H$$

and, consequently, h is termed the " equivalent height " of the antenna.

Alternatively, it is easily shown that the uniformly distributed current in the actual antenna of height H which would give the same current volume as that of the sinusoidally distributed current will be,

$$I_{av} = \frac{2}{\pi}I.$$

as is shown in Fig. 14*b*.

Antenna Characteristics

It has been seen in Chapter XV, page 479 (see also Fig. 16, page 473), that when, for example, a single-phase transmission line is open-circuited at the receiver's end, standing waves of pressure and current are produced. It has also been seen that when the length of the transmission line is equal to one-quarter of the length of the standing wave, then, neglecting the resistance of the line, the pressure at the generator end of the line necessary to maintain the standing wave will be negligibly small. The relationship between the generator frequency f hertz, the wavelength λ and the speed of propagation c is

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} \text{ cm.}$$

when $c = 3 \times 10^{10}$ cm. per second.

Now, suppose that a straight conductor be supported vertically, the lower end being connected through a generator to earth so that the earth forms one pole of the circuit and the vertical conductor forms the other, as shown in Fig. 15. If it be assumed that the electrical capacitance per unit length of the conductor is C farad and is uniformly distributed throughout the length, and that the inductance per unit length is L henry and is also uniformly distributed, then the surge impedance will be (see Chapter XV, page 460)

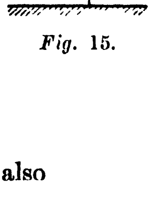


Fig. 15.

$$Z_0 = \sqrt{\frac{L}{C}} \text{ ohms,}$$

also
$$c = \frac{1}{\sqrt{LC}} = 3 \times 10^{10} \text{ cm. per second,}$$

when L and C are the respective value of inductance and capacitance per centimetre length.

If the generator supply frequency is f hertz and the height of the conductor H is such that

$$4H = \lambda = \frac{c}{f} = \frac{1}{f\sqrt{L.C}},$$

then a condition of resonance will exist and the current and pressure will be defined by the standing waves as shown in Fig. 16.

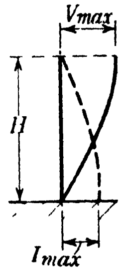


Fig. 16.

In the case of such a quarter-wave antenna the current will always be zero at the upper end and will increase sinusoidally towards the lower end, where the current will have its maximum value I_{max}. It will be seen, therefore, that the e.m.f. due to the inductance per unit length near the foot of the antenna will be greater than that due to the inductance per unit length near the top. Similarly, the e.m.f. due to the capacitance per unit length near the foot will be greater than that near the top. If, however, it is assmued that the total current which is flowing in the antenna at any moment is uniformly distributed throughout its length, then it is easy to see that the magnitude of this equivalent uniformly distributed current will be $\frac{2}{\pi}I_{max}$. The total e.m.f. induced in the antenna due to the sinusoidally distributed current will be $\frac{2}{\pi}\omega L I_{max}$ and this will, of course, be the same as that due to the uniformly distributed current of magnitude $\frac{2}{\pi}I_{max}$. The same result will clearly be obtained if the sinusoidal current distribution is maintained and the effective inductance per unit length of the antenna be taken to be $\frac{2}{\pi}L$ henry. In other words, instead of a circuit with distributed inductance of total value $L \times H$ it may be assumed that the inductance is concentrated or "lumped" into a total value $L_0 = \frac{2}{\pi}L \times H$, and for similar reasons a "lumped" capacitance $C_0 = \frac{2}{\pi}C \times H$ can be assumed to replace the distributed capacitance and a "lumped" resistance $R_0 = \frac{2}{\pi}RH$ to replace the distributed resistance.

The antenna circuit can then be considered as being equivalent to the circuit shown in Fig. 17. For this circuit the natural frequency is

$$f = \frac{1}{\tau} = \frac{1}{2\pi\sqrt{L_0C_0}} = \frac{1}{2\pi\sqrt{\left(\frac{2}{\pi}LH\right)\left(\frac{2}{\pi}CH\right)}} = \frac{1}{4H\sqrt{LC}},$$

but since $\lambda = \frac{c}{f}$ where $c = \frac{1}{\sqrt{LC}}$

it follows that $\lambda = 4H,$

which is in agreement with the wavelength of the actual antenna.

The logarithmic decrement of the oscillations in the circuit of Fig. 17 (see also Chapter X, page 321) will be (see Fig. 18)

$$\Delta = \log_e \frac{I_2}{I_1}.$$

that is

$$\Delta = \log_e e^{\frac{R_0}{2L_0}\tau} = \frac{R_0}{2L_0}\tau$$

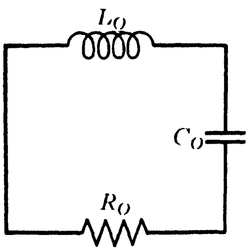


Fig. 17.

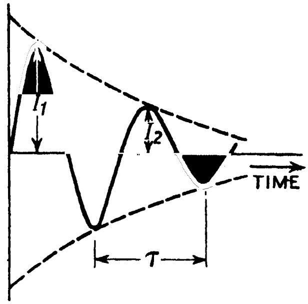


Fig. 18.

where $\tau = 2\pi\sqrt{L_0C_0}$ and is the time of one complete oscillation. Hence

$$\Delta = \frac{R_0}{2L_0}2\pi\sqrt{L_0C_0} = \pi R_0\sqrt{\frac{C_0}{L_0}} = \pi\frac{R_0}{Z_0},$$

where $Z_0 = \sqrt{\frac{L_0}{C_0}}$ and is the surge impedance of the antenna.

This result may also be written

$$\Delta = \pi\left[\frac{\frac{2}{\pi}RH}{Z_0}\right] = \left(\frac{\pi R}{Z_0}\right)h \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where $h = \frac{2}{\pi}H$ and is termed the " equivalent height " of the antenna.

If, instead of the sinusoidal distribution of the standing wave of current, it is assumed that the current is uniformly distributed and has a magnitude I_{max}, then for h measured in metres

$$I_{max}.h \text{ metre-amperes} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

will be of the same magnitude as the corresponding quantity for the sinusoidally distributed current in the antenna of height H metres.

The quantity $I_{max}h$ meter-amperes is used as a measure of the power of a transmitting station.

The expression (39) for the logarithmic decrement Δ may be derived in a somewhat different way, as follows. In the diagram for the standing waves shown in Fig. 16 it is assumed that the waves are at their maxima so that the current I_{max} is the maximum in time as well as in space. The electromagnetic energy which is then stored in the inductance L_0, viz. $\frac{1}{2}L_0(I_{max})^2$ will be a maximum. After one half-cycle the whole of this energy will have become transformed into electrostatic energy stored in the dielectric of the capacitance of C_0, the amount being

$\frac{1}{2}C_0(V_{max})^2$, which is therefore equal to $\frac{1}{2}L_0(I_{max})^2$ joules.

That is to say, twice during each cycle the stored electromagnetic energy of amount $\frac{1}{2}L_0(I_{max})^2$ joules will become transformed into electrostatic energy stored in the capacitance. The number of such transformations per second will then be $2f$ and the corresponding rate of energy transformation will then be

$$2f(\tfrac{1}{2}L_0 I_{max}{}^2) \text{ watts.}$$

The rate of dissipation of energy in the resistance of the circuit during this process will be

$$R_0(I_{rms})^2 = \tfrac{1}{2}R_0(I_{max})^2 \text{ watts.}$$

The damping will then be given by

$$\Delta = \frac{\frac{1}{2}R_0(I_{max})^2}{2f[\frac{1}{2}L_0(I_{max})^2]} = \frac{R_0}{2f.L_0} = \frac{R\frac{2}{\pi}H}{2f\frac{2}{\pi}LH} = \frac{R}{2f.L}$$

$$\left.\begin{aligned}
&\text{that is} && \Delta = \frac{R}{2}\frac{\lambda}{c.L} = \frac{R}{2}\lambda\sqrt{\frac{C}{L}} = \lambda\frac{R}{2Z_0},\\
&\text{or} && \Delta = \lambda.\beta \quad . \quad . \quad . \quad . \\
&\text{where} && \beta = \frac{R}{2Z_0} \text{ neper per km.}\\
&\text{Otherwise} && \Delta = \frac{R}{2}\frac{\lambda}{cL} = \frac{R}{2}\frac{4H\sqrt{LC}}{L} = 2R.\frac{\pi}{2}.h\sqrt{\frac{C}{L}} = \frac{\pi Rh}{Z_0}
\end{aligned}\right\} \quad . \quad (41)$$

since $\lambda = 4H$ for a quarter-wave antenna. This expression for Δ is the same as has been obtained already in expression (39) (see also Chapter X, expression (29), page 321).

Since the antenna will be required to radiate not only the carrier-wave but also the side bands, the damping must be such that no serious distortion of the amplitudes of the high-frequency waves will take place. In the case of an overhead transmission line the damping factor β is given by the relationship (see Test Paper, Chapter XV, Example 14)

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{R}{2Z_0} \text{ neper per km.}$$

where R ohms per km. is the resistance of the line and $Z_0 = \sqrt{\frac{L}{C}}$ and is the surge impedance of the line.

For a vertical antenna the damping is given by the same general expression as will be seen by reference to expression (41), page 511, and the damping resistance R can be assessed by the following method. First assume that the antenna has no damping and also that the current

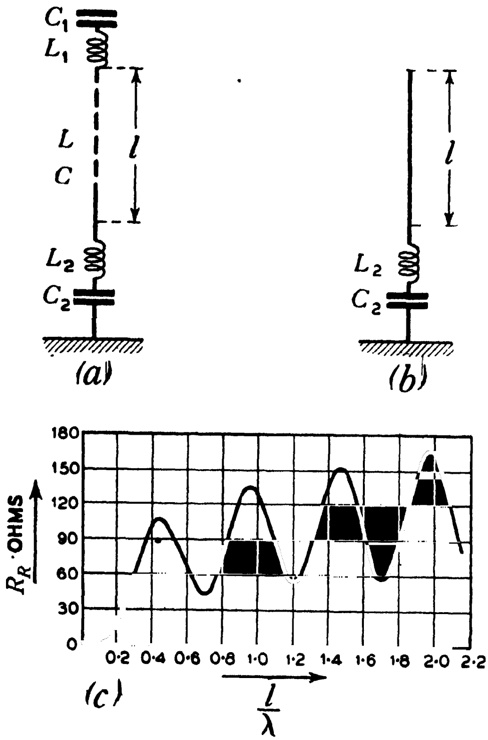


Fig. 19.

is sinusoidally distributed along the antenna. Then, provided the antenna is not too short in comparison with the wave-length, the damping resistance R in the above expression may be assumed to comprise the resistance loss in the antenna conductor itself, and the radiation resistance R_R and, as a first degree of approximation, the resistance $R.h$ (where h metres is the height of the antenna) may be assumed to comprise exclusively, the radiation resistance R_R (see expression (38), page 507). It is now to be assumed that this radiation resistance is uniformly

distributed throughout the whole length of the antenna, the radiated power being then expressed as follows :

$$W_R = I^2 R_R = \int_0^h i^2 R\, ds$$

where $$i = I \sin \frac{2\pi s}{\lambda}$$

and s metres is the distance measured from the top of the antenna. Then

$$W_R = \int_0^h I^2 R \sin^2 \left(\frac{2\pi s}{\lambda}\right) ds = I^2 R_R$$

so that

$$R.h = \frac{2R_R}{\left[1 - \dfrac{\sin \frac{4\pi}{\lambda}h}{\frac{4\pi}{\lambda}h}\right]} \quad . \quad . \quad . \quad . \quad (41a)$$

and this gives the value of $R.h$ which is to be inserted in the expression (41) for the logarithm decrement of the whole antenna, that is

$$\Lambda = 2\pi \frac{R.h}{2} \sqrt{\frac{C}{L}} \text{ nepers}$$

or $$\Lambda = 2\pi\beta.h$$

where $$\beta = \frac{R}{2Z_0}.$$

It is to be observed that the radiation resistance of an antenna depends not only on its height and the wave-length used, but also on the way in which the current is distributed along the antenna conductor. This is particularly the case when the antenna is " loaded " with auxiliary capacitance or inductance, the most general type of such a loaded antenna being shown in Fig. 19a, where l is the length of the antenna conductor. A very important example of a loaded antenna is shown in Fig. 19b, and Fig. 19c gives the radiation resistance R_R of this system as a function of the ratio $\frac{l}{\lambda}$.

The total effective capacitance of an antenna of height H will be the total capacitance of an antenna of effective height $h = \frac{2}{\pi}H$, that is (see Chapter IV, page 104, formula (14), Fig. 10)

$$C_0 = \frac{h}{\left(2 \log_e \frac{2h}{r}\right) 9 \times 10^{11}} \text{ farad,}$$

and the total effective inductance will be

$$L_0 = \left(2h \log_e \frac{2h}{r}\right) 10^{-9} \text{ henry.}$$

The surge impedance will then be

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\frac{\pi}{2}\frac{L_0}{h}}{\frac{\pi}{2}\frac{C_0}{h}}} = \sqrt{\frac{L_0}{C_0}}$$

or

$$Z_0 = 60 \log_e \frac{2h}{r} \text{ ohms} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

For example, if the antenna wire has a radius r of $\frac{1}{8}$-inch and is 150 metres high, then

$$h = \frac{2}{\pi} 150 \times 10^2 \text{ cm., and } r = 0{\cdot}32 \text{ cm.,}$$

so that $$Z_0 = 60 \log_e \frac{2 \times 3 \times 10^4}{0{\cdot}32\pi} = 660\ \Omega.$$

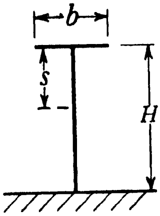


Fig. 20.

If a vertical antenna of height H is provided with a horizontal cross-wire at the upper end, as shown in Fig. 20, this horizontal member provides, in effect, a capacitance of K farad connected to the upper end. If the resistance of the antenna system is negligible the equation for the pressure at any distance s from the upper end is then [see Chapter XV, page 472, equation (24)]

$$\mathfrak{V}_s = \tfrac{1}{2}[\mathfrak{V}_2 + \mathfrak{I}_2 Z_0]e^{j\alpha s} + \tfrac{1}{2}[\mathfrak{V}_2 - \mathfrak{I}_2 Z_0]e^{-j\alpha s}$$

where $$\mathfrak{I}_2 = \frac{\mathfrak{V}_2}{-j\frac{1}{\omega K}} = j\omega K \mathfrak{V}_2,$$

so that $$\mathfrak{V}_s = \mathfrak{V}_2[\cos \alpha s - \omega K Z_0 \sin \alpha s]$$

$$= \tfrac{1}{2}\mathfrak{V}_2[\cos \alpha s \cos \theta - \sin \alpha s \sin \theta]\sqrt{1 + (\omega K Z_0)^2}$$

where $\tan \theta = \omega K Z_0$.

Hence $$\mathfrak{V}_s = \mathfrak{V}_2\sqrt{1 + (\omega K Z_0)^2} \cos (\alpha s + \theta).$$

For a *pressure node* at the foot of the antenna, that is, for $s = H$, then

$$\mathfrak{V}_H = 0 = \tfrac{1}{2}\mathfrak{V}_2\sqrt{1 + (\omega K Z_0)^2} \cos (\alpha H + \theta),$$

that is $$\alpha H + \theta = \frac{\pi}{2} : \frac{3\pi}{2} : \frac{5\pi}{2} : \ldots$$

or $\qquad \alpha H = \frac{\pi}{2} - \theta : \ \frac{3\pi}{2} - \theta : \ldots$

and since $\qquad \alpha = \frac{2\pi}{\lambda}$

$$H = \frac{\lambda_1}{2}\left(\frac{1}{2} - \frac{\theta}{\pi}\right) : \ \frac{\lambda_3}{2}\left(\frac{3}{2} - \frac{\theta}{\pi}\right) : \ldots$$

so that, if $\quad K = 0$, then $\theta = 0$: and

$$H = \frac{\lambda_1}{4} : \ \frac{3\lambda_3}{4} : \ldots$$

and the first two of these wavelengths are shown in Fig. 21.

For $K = \infty$: $\theta = \frac{\pi}{2}$: and $\lambda = \infty$ for all values of θ.

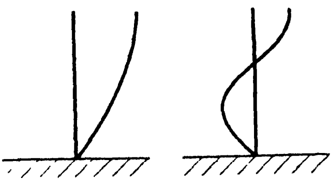


Fig. 21.

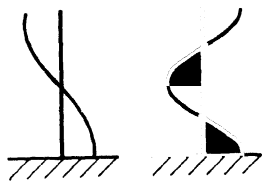


Fig. 22.

The *current* at any point in the antenna distant s, from the upper end is

$$\mathfrak{J}_s = \frac{1}{2}\frac{1}{Z_0}[\mathfrak{V}_2 + \mathfrak{J}_2 Z_0]e^{j\alpha s} - \frac{1}{2}\frac{1}{Z_0}[\mathfrak{V}_2 - \mathfrak{J}_2 Z_0]e^{-j\alpha s}$$

$$= \frac{1}{2}\frac{\mathfrak{V}_2}{Z_0}[e^{j\alpha s} - e^{-j\alpha s}] + \tfrac{1}{2}\mathfrak{J}_2[e^{j\alpha s} + e^{-j\alpha s}]$$

$$= \frac{\mathfrak{V}_2}{Z_0}j \sin \alpha s + j\omega K \mathfrak{V}_2 \cos \alpha s$$

$$= j\frac{\mathfrak{V}_2}{Z_0}[\sin \alpha s + \omega K Z_0 \cos \alpha s].$$

At the foot of the antenna:

$$\mathfrak{J}_H = j\frac{\mathfrak{V}_2}{Z_0}\sqrt{1 + \omega K Z_0}\sin(\alpha H + \theta) : \text{ where } \tan\theta = \omega K Z_0 :$$

so that for I_H a maximum: $\alpha H + \theta = \frac{\pi}{2} : \ \frac{3\pi}{2} : \ldots$

that is $\qquad H = \frac{\lambda_1}{2\pi}\left(\frac{\pi}{2} - \theta\right) : \ \frac{\lambda_3}{2\pi}\left(\frac{3\pi}{2} - \theta\right) : \ldots$

If $K = 0$, then $\quad \theta = 0$, and

$$H = \frac{\lambda_1}{4} : \ \frac{3}{4}\lambda_3 : \ldots$$

If a *pressure anti-node* is to appear at the foot of the attenna, then V_H will be a maximum, so that

$$(\alpha H + \theta) = 0 : \ \pi : \ 2\pi : \ . \ . \ .$$

The equation $(\alpha H + \theta) = 0$ gives a negative value for H, so that this has no practical significance and the successive values for the height of the antenna are :

$$H = \frac{1}{\alpha_2}(\pi - \theta) = \frac{\lambda_2}{2\pi}(\pi - \theta) = \frac{\lambda_2}{2}\left(1 - \frac{\theta}{\pi}\right)$$

$$H = \frac{1}{\alpha_4}(2\pi - \theta) = \frac{\lambda_4}{2\pi}(2\pi - \theta) = \frac{\lambda_4}{2}\left(2 - \frac{\theta}{\pi}\right),$$

and so on, for successive values of the wavelengths.

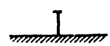
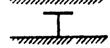
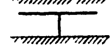
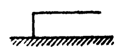
For $\qquad K = 0 :$ then $\theta = 0$, and $H = \frac{\lambda_2}{2} : \ \lambda_4$

the first two of these wavelengths being shown in Fig. 22.

A general relationship connecting the wavelength λ of an antenna and the height H (see Fig. 20) is as follows :

$$\lambda = k\left(H + \frac{b}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

for which the constant k can be obtained from the following table :

Type of Antenna	k	
Single vertical wire	4·1*	
Single inclined wire	4·2	
Single straight wire at height of 1 metre above the earth	5	
T antenna with short cross-wire . . .	4·5 to 5	
T antenna with longer cross-wire . . .	5 to 7	
T antenna with long cross-wire $b \simeq (2 \text{ to } 3)$ H	9 to 10	
Umbrella antenna	6 to 8	
Umbrella antenna with many wires . .	8 to 10	
L antenna	5·5	

* Due to the capacitance not being uniformly distributed so that the standing waves are not strictly sinusoidal.

Tuning an Antenna

In Fig. 23 is shown a vertical antenna in the base of which a reactance of X_A ohms has been connected in series, and for the investigation of the current and pressure relationships for this system the general transmission line equations (24) on page 472, Chapter XV, may be used.

$$\mathfrak{V}_s = \tfrac{1}{2}(\mathfrak{V}_2 + \mathfrak{I}_2 Z_0)e^{j\alpha s} + \tfrac{1}{2}(\mathfrak{V}_2 - \mathfrak{I}_2 Z_0)e^{-j\alpha s}$$

$$\mathfrak{I}_s = \frac{1}{2}\frac{1}{Z_0}(\mathfrak{V}_2 + \mathfrak{I}_2 Z_0)e^{j\alpha s} - \frac{1}{2}\frac{1}{Z_0}(\mathfrak{V}_2 - \mathfrak{I}_2 Z_0)e^{-j\alpha s}.$$

In these equations as applied to an antenna system, the current $\mathfrak{I}_2 = 0$.

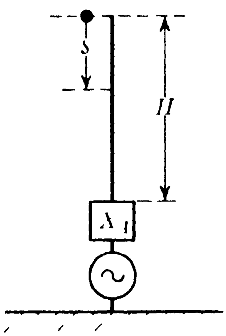


Fig. 23.

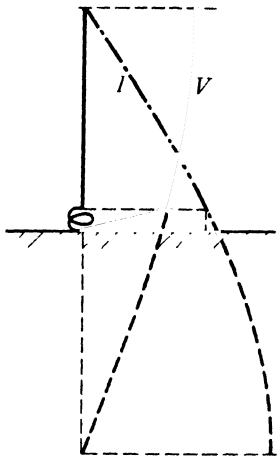


Fig. 24.

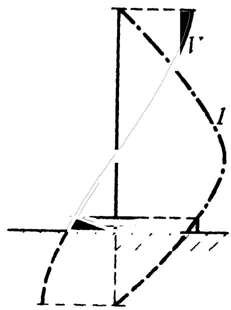


Fig 25.

At the base of the antenna $s = H$, so that

$$\mathfrak{V}_H = \mathfrak{V}_2 \cos \alpha H : \mathfrak{I}_H = \mathfrak{V}_2 \frac{1}{Z_0} j \sin \alpha H,$$

Also, the generator e.m.f. is $\mathfrak{E}_A$ where

$$\mathfrak{E}_A = \mathfrak{V}_H + \mathfrak{I}_H \mathfrak{X}_A = \mathfrak{V}_2 \cos(\alpha H) + \mathfrak{V}_2 \frac{\mathfrak{X}_A}{Z_0} j \sin(\alpha H).$$

For resonance $\mathfrak{E}_A = 0 = \cos(\alpha H) + \frac{\mathfrak{X}_A}{Z_0} j \sin(\alpha H)$.

Case I.—If $\mathfrak{X}_A = j\omega L_A$, that is, the series reactance is inductive, then

$$0 = \cos(\alpha H) - \frac{\omega L_A}{Z_0} \sin(\alpha H),$$

so that $\cos(\alpha H + \theta) = 0$: where $\tan \theta = \frac{\omega L_A}{Z_0}$.

Hence for the fundamental wave-length

$$(\alpha H + \theta) = \frac{\pi}{2}: \quad H = \frac{\lambda_1}{2\pi}\left(\frac{\pi}{2} - \theta\right) = \frac{\lambda_1}{2}\left(\frac{1}{2} - \frac{\theta}{\pi}\right)$$

and if, for example, $\theta = \frac{\pi}{4}$; then $H = \frac{\lambda_1}{8}$,

so that the fundamental wavelength is $\lambda_1 = 8H$ and this condition is shown in Fig. 24. It is seen, therefore, that the effect of connecting an inductance in series with the antenna is to bring the antenna into resonance for a *longer wavelength*, that is, for a *lower frequency* than that corresponding to the geometrical length of the antenna.

Case II.—If $\mathfrak{X}_A = -j\frac{1}{\omega K}$, where K is the capacitance in farad of the condenser shown in Fig. 25, then

$$\begin{aligned} \mathfrak{E}_A &= \mathfrak{V}_2\left\{\cos(\alpha H) + \frac{1}{\omega K Z_0}\sin(\alpha H)\right\} \\ &= \mathfrak{V}_2 \cos(\alpha H - \theta), \end{aligned}$$

where $\tan\theta = \frac{1}{\omega K Z_0}$.

For resonance, $\cos(\alpha H - \theta) = 0$, so that for the fundamental wavelength λ_1

$$(\alpha H - \theta) = \frac{\pi}{2} \text{ or } H = \frac{\lambda_1}{2}\left(\frac{1}{2} + \frac{\theta}{\pi}\right),$$

if, for example, $\theta = \frac{\pi}{4}$, then $H = \frac{3}{8}\lambda_1$,

that is, $\lambda_1 = \frac{8}{3}H$,

and this condition is shown in Fig. 25. That is to say, the effect of connecting a capacitance in series with the antenna is to bring the antenna into resonance for a *shorter wavelength*, that is, for a *higher frequency* than that which corresponds to the geometrical length of the antenna.

Matching Antenna Circuits

It has been seen in Chapter XV that when a transmission line is delivering its " natural load " to a consumer, the impedance Z_2 of the consumer's load will be equal to the surge impedance Z_0 of the line. It has also been seen that when this is the case, no reflexion effects at the consumer's terminals will develop and, consequently, no standing waves will be produced. Such a condition of operation is ideally satisfactory and, consequently, the natural load is taken to be the datum of reference in the investigation of transmission line problems.

When the impedance of the consumer's load is $Z_2 = Z_0$, that is to say, when the line is transmitting its natural power, the vector of pressure and the vector of current in the line will be defined by the relationships

$$\frac{\mathfrak{V}_1}{\mathfrak{V}_2} = \frac{\mathfrak{V}_s}{\mathfrak{J}_s} = \frac{\mathfrak{V}_2}{\mathfrak{J}_2} = \mathfrak{Z}_2 = Z_0.$$

If the resistance of the line is relatively small, then

$$\mathfrak{Z}_2 = Z_0 = \sqrt{\frac{L}{C}},$$

and the surge impedance will be a " real " quantity, in consequence of which, the vectors of pressure and current will be in phase at every point throughout the line.

When the consumer's load is not equal to the natural load of the line, it has been seen in Chapter XV, page 491, that the ideal condition

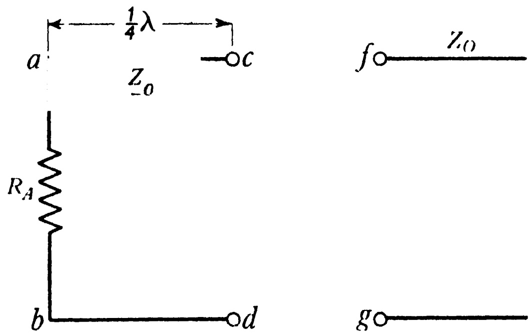


Fig. 26.

of operation may be very closely realized by injecting at each end of the line as well as at selected intermediate points, a suitable amount of reactive power. This process may be described as " matching " the transmission line to the load circuit.

The problem of " matching " is of equal importance when dealing with the transmission of high-frequency currents and the nature of this aspect of the problem may be seen by means of the following example. Suppose a loaded antenna is represented by the resistance R_A in Fig. 26 and is connected to a feeder line of length $\frac{\lambda}{4}$ and of surge impedance Z_0. It can then be shown that, if R^1 is the total impedance in series with the feeder lines *ac* and *bd*, that is, the impedance of the circuit *cabd* (Fig. 27), is

$$Z_F = \frac{{Z_0}^2}{R_A} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Now, if the antenna circuit of total impedance Z_F were to be connected directly to the transmission line fg in Fig. 26, there would be, in general, reflexion effects produced at the junctions and corresponding standing waves would develop. It is, however, of vital importance that such standing waves shall not appear, otherwise the whole of the energy supplied by the generating station will not be transferred to the antenna. A further objection to the development of standing waves on such feeder lines is that the radiation which they would produce would cause serious interference with the radiation characteristics of the antenna. In order to ensure that such reflexion effects and consequent standing waves shall not appear at the junction fg, it is necessary to connect the antenna circuit *cabd* to the transmission line by means of a " matching line " of impedance Z_m as shown in Fig. 27. The magnitude of the requisite impedance Z_m for this purpose can be found as follows.

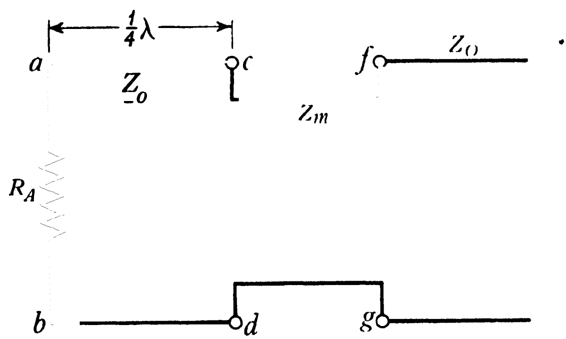


Fig. 27.

Since the impedance of the antenna circuit *cabd* is

$$Z_F = \frac{Z_0^2}{R_A},$$

the impedance of the system when the matching line is connected to the antenna circuit, that is, the impedance of the circuit *fcabdg* is

$$(Z_F)_m = \frac{Z_m^2}{Z_F} = \frac{Z_m^2}{\frac{Z_0^2}{R_A}} \quad . \quad . \quad . \quad . \quad (45)$$

But if no reflexion effects are to be produced at the junction the impedance of the aerial circuit must be equal to the impedance of the transmission line, that is,

$$(Z_F)_m \text{ must be equal to } Z_0,$$

so that

$$Z_m = Z_0\sqrt{\frac{Z_0}{R_A}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Appendix I

SOME MATHEMATICAL RELATIONSHIPS

The Binomial Theorem

$$(a \pm b)^n = a^n \pm na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3 + \ldots$$

The series on the right-hand side is finite if n is a positive integer and is convergent for $a > b$: it is infinite when n is not a whole number and also when n is negative.

If $a = 1$, and $b = \frac{1}{n}$ then

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1 - \frac{1}{n}}{\underline{|2}} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{\underline{|3}} + \ldots$$

If $n = \infty$, then

$$\left(1 + \frac{1}{\infty}\right)^\infty = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots$$

and this is the definition of the base e of natural logarithms, so that

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots = 2{\cdot}718282.$$

Some Applications of the Binomial Theorem for all Values of n and for $-1 < x < +1$

$$\frac{1}{1 \pm x} = (1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \ldots$$

$$\sqrt{1 \pm x} = (1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{3}{48}x^3 - \ldots$$

$$\frac{1}{\sqrt{1 \pm x}} = (1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{15}{48}x^3 + \ldots$$

Some Square Roots

$\sqrt{a^2 \pm b} \simeq a \pm \frac{b}{2a}$: when b is very small in comparison with a.

$\sqrt{a^2 + b^2} \simeq 0{\cdot}960a + 0{\cdot}368b$: for $a > b$, the error being $<$ 4 per cent. of the true value.

$\sqrt{a^2 + b^2 + c^2} \simeq 0{\cdot}939a + 0{\cdot}389b + 0{\cdot}297c$ for $a > b > c$, the error being $<$ 6 per cent. of the true value

Formulae Relating to Plane Triangles

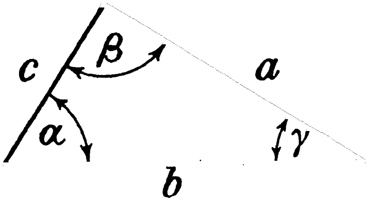


Fig. 1.

$$s = \tfrac{1}{2}(a + b + c)$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$a = b \cos \gamma + c \cos \beta :$$
$$b = c \cos \alpha + a \cos \gamma :$$
$$c = a \cos \beta + b \cos \alpha$$
$$a^2 = b^2 + c^2 - 2bc \cos \alpha :$$

Area of triangle $= \tfrac{1}{2}a \times b \sin \gamma$
$= \tfrac{1}{2} b \times c \sin \alpha = \tfrac{1}{2} c \times a \sin \beta$

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{b \times c}} : \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{b \times c}}$$

$$\tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\rho}{s-a},$$

where ρ is the radius of the inscribed circle.

$$\rho = 4r \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$= \frac{a \times b \times c}{4r \times s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

where r is the radius of the circumscribed circle.

Some Mathematical Series

The base e of natural logarithms is

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots = 2{\cdot}718282$$

also
$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots$$

and, writing $j = \sqrt{-1}$

$$e^{jx} = 1 + jx - \frac{x^2}{\lfloor 2} - j\frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \ldots$$

$$e^{-jx} = 1 - jx - \frac{x^2}{\lfloor 2} + j\frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$$

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \dots$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$$

$$\frac{x}{\sin x} = 1 + \frac{x^2}{\underline{|3}} + \frac{14x^4}{\underline{|6}} + \dots$$

$$\sin^{-1} x = x + \frac{1x^3}{2 \times 3} + \frac{1 \times 3x^5}{2 \times 4 \times 5} + \frac{1 \times 3 \times 5x^7}{2 \times 4 \times 6 \times 7} + \dots \text{ for } -1 \leq x \leq +1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \text{ for } -1 \leq x \leq +1$$

$$\cos^2 x = 1 - x^2 + \frac{2^3x^4}{\underline{|4}} - \frac{2^5x^6}{\underline{|6}} + \dots$$

$$\tan\left(\frac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \dots$$

$$\log_e \tan\left(\frac{\pi}{4} + x\right) = 2x + \frac{4}{3}x^2 + \frac{4}{3}x^5 + \dots$$

$$\sec x = 1 + \frac{x^2}{\underline{|2}} + \frac{5x^4}{\underline{|4}} + \frac{61x^6}{\underline{|6}} + \dots$$

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{1}{6}\frac{x^2}{\underline{|2}} - \frac{1}{30}\frac{x^4}{\underline{|4}} + \dots$$

$$\log_e (1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \dots \text{ for } -1 < x < +1$$

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right)$$

$$\cos x = \tfrac{1}{2}(e^{jx} + e^{-jx})$$

$$j \sin x = \tfrac{1}{2}(e^{jx} - e^{-jx})$$

$$\cos x + j \sin x = e^{jx}$$

$$\cos x - j \sin x = e^{-jx}.$$

De Moivre's Theorem for all Real Values of n

$$(\cos x \pm j \sin x)^n = \cos nx \pm j \sin nx$$

Some Trigonometrical Relationships

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}.$$

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$
$$\sin \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{2}} : \cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}} : \tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}.$$

Hyperbolic Functions

Certain combinations of the exponential functions, known as hyperbolic functions, have properties closely analogous in form to the circular functions, and in some respects they bear the same relationships to the rectangular hyperbola that the circular functions bear to the circle. Thus in Fig. 2 is shown a pair of conjugate rectangular hyperbolae of which the half-axis is $OA = 1$. If from any point B on the curve a line BB' is drawn at right angles to the axis OA, and if OB and OB' are joined and from A a line AD is drawn at right angles to OA and cutting OB at D, then

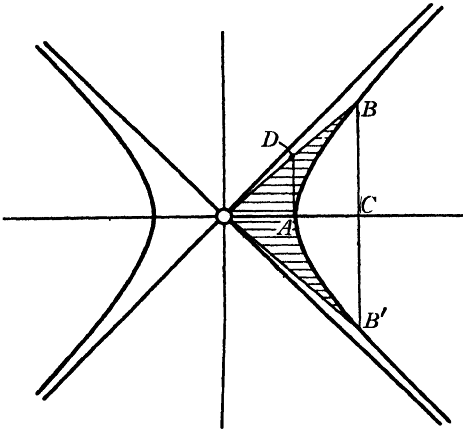


Fig. 2.

$$BC = \text{Sinh}\ \phi : \quad OC = \text{Cosh}\ \phi : \quad DA = \text{Tanh}\ \phi :$$

where ϕ is the area of the shaded portion $OBAB'O$. It can then be shown that the following relationships will hold, viz.

$$\text{Sinh}\ \phi = \tfrac{1}{2}(e^{\phi} - e^{-\phi}) = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots$$

$$\text{Cosh}\ \phi = \tfrac{1}{2}(e^{\phi} + e^{-\phi}) = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots$$

$$\text{Tanh}\ \phi = \frac{\text{Sinh}\ \phi}{\text{Cosh}\ \phi} = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

$$\text{Sinh}\ (-\phi) = -\text{Sinh}\ \phi : \quad \text{Cosh}\ (-\phi) = +\text{Cosh}\ \phi$$

$$\text{Cosh}\ \phi + \text{Sinh}\ \phi = e^{\phi} : \quad \text{Cosh}\ \phi - \text{Sinh}\ \phi = e^{-\phi}$$

$$\text{Cosh}^2\ \phi - \text{Sinh}^2\ \phi = 1$$

$$\text{Cosh}\ j\phi + \text{Sinh}\ j\phi = e^{j\phi}$$

$$\text{Sinh}\ (x \pm y) = \text{Sinh}\ x\ \text{Cosh}\ y \pm \text{Cosh}\ x\ \text{Sinh}\ y$$
$$\text{Cosh}\ (x \pm y) = \text{Cosh}\ x\ \text{Cosh}\ y \pm \text{Sinh}\ x\ \text{Sinh}\ y$$

$$\text{Sinh}\ 2x = 2\ \text{Sinh}\ x\ \text{Cosh}\ x$$
$$\text{Cosh}\ 2x = \text{Cosh}^2 x + \text{Sinh}^2 x$$

$$\operatorname{Tanh}(x-y) = \frac{\operatorname{Tanh} x - \operatorname{Tanh} y}{1 - \operatorname{Tanh} x \operatorname{Tanh} y}$$

$$\operatorname{Tanh}(x+y) = \frac{\operatorname{Tanh} x + \operatorname{Tanh} y}{1 + \operatorname{Tanh} x \operatorname{Tanh} y}$$

$$\operatorname{Sech}^2 x = 1 - \operatorname{Tanh}^2 x$$

$$\operatorname{Cosech}^2 x = \operatorname{Coth}^2 x - 1$$

$$\operatorname{Sinh}^{-1} x = \log_e \{x + \sqrt{x^2+1}\} :$$

$$\operatorname{Cosh}^{-1} x = \log_e \{x \pm \sqrt{x^2-1}\}$$

In the expression for the inverse Cosh x either sign is admissible (see also Fig. 3).

In Fig. 3 are shown the graphs of Sinh x and Cosh x respectively, and Fig. 4 shows the graph of Tanh x.

The accompanying tables give the hyperbolic functions Sinh ϕ and Cosh ϕ respectively, for values of ϕ from 0 to 4.

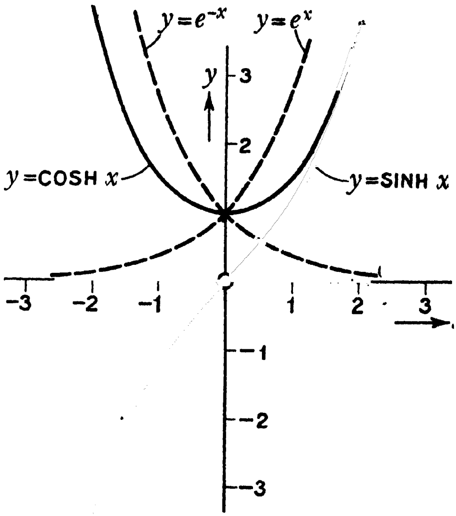


Fig. 3.

Relationships between the Circular and the Hyperbolic Functions

$$\sin x = -j \operatorname{Sinh} jx = \frac{e^{jx} - e^{-jx}}{2j} : \qquad \cos x = \operatorname{Cosh} jx = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tan x = -j \operatorname{Tanh} jx = -j\frac{e^{jx} - e^{-jx}}{e^{jx} + e^{-jx}} : \qquad \cot x = j \operatorname{Coth} jx = j\frac{e^{jx} + e^{-jx}}{e^{jx} - e^{-jx}}$$

$$\sin jx = j \operatorname{Sinh} x = j\frac{e^{x} - e^{-x}}{2} : \qquad \cos jx = \operatorname{Cosh} x = \frac{e^{x} + e^{-x}}{2}$$

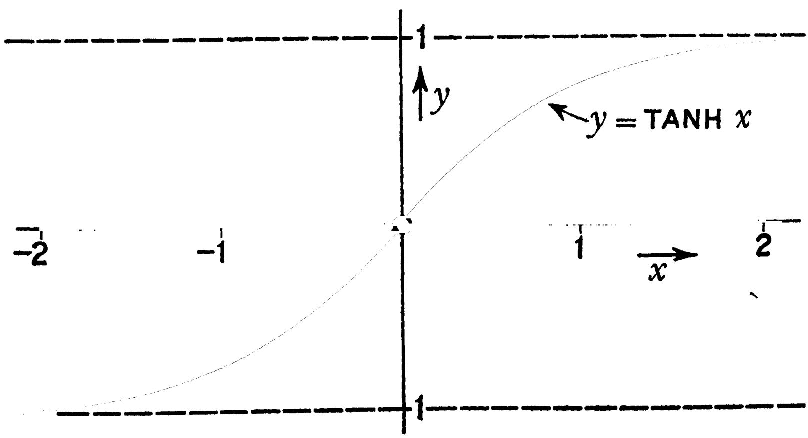


Fig. 4.

HYPERBOLIC FUNCTION

$$Sinh\ \phi = \tfrac{1}{2}(e^{\phi} - e^{-\phi})$$

for values of ϕ from 0 to 4

ϕ	0	1	2	3	4	5	6	7	8	9	D
0·0	0 0000	0 0100	0·0200	0 0300	0 0400	0 0500	0·0600	0 0701	0 0801	0 0901	101
0·1	0·1002	0 1102	0 1203	0 1304	0 1405	0 1506	0 1607	0 1708	0 1810	0 1911	102
0·2	0·2013	0 2115	0 2218	0 2320	0 2423	0 2526	0 2629	0 2733	0 2837	0 2941	104
0·3	0 3045	0 3150	0 3255	0 3360	0 3466	0 3572	0 3678	0 3785	0 3892	0 4000	108
0·4	0 4108	0 4216	0 4325	0 4434	0 4543	0 4653	0 4764	0 4875	0 4986	0 5098	113
0·5	0 5211	0 5324	0 5438	0 5552	0 5666	0 5782	0 5897	0 6014	0 6131	0 6248	119
0·6	0·6367	0·6485	0 6605	0 6725	0 6846	0 6967	0 7090	0 7213	0 7336	0 7461	125
0 7	0 7586	0 7712	0 7838	0 7966	0 8094	0 8223	0 8353	0 8484	0 8615	0 8748	133
0·8	0 8881	0 9015	0 9150	0 9286	0 9423	0 9561	0 9700	0 9840	0 9981	1 0122	143
0·9	1 0265	1·0409	1 0554	1 0700	1 0847	1 0995	1 1144	1 1294	1 1440	1 1598	154
1·0	1 1752	1·1907	1 2063	1 2220	1 2379	1 2539	1 2700	1 2862	1 3025	1 3190	166
1·1	1 3356	1 3524	1 3693	1 3863	1 4035	1 4208	1 4382	1 4558	1 4735	1 4914	181
1·2	1·5095	1 5276	1 5460	1 5645	1 5831	1 6019	1·6209	1 6400	1 6593	1 6788	196
1·3	1·6984	1 7182	1 7381	1 7583	1 7786	1 7991	1 8198	1 8406	1 8617	1 8829	214
1·4	1 9043	1 9259	1 9477	1 9697	1 9919	2 0143	2 0369	2 0597	2 0827	2 1059	234
1·5	2 1293	2 1529	2 1768	2 2008	2 2251	2 2496	2 2743	2 2993	2 3245	2 3499	257
1·6	2 3756	2 4015	2 4276	2 4540	2 4806	2 5075	2 5346	2 5620	2 5896	2 6175	281
1 7	2·6456	2 6740	2 7027	2 7317	2 7609	2 7904	2 8202	2 8503	2 8806	2 9112	310
1 8	2 9422	2 9734	3 0049	3 0367	3 0689	3 1013	3 1340	3 1671	3 2005	3 2341	341
1·9	3 2682	3 3025	3 3372	3 3722	3 4075	3 4432	3 4792	3 5156	3 5523	3 5894	375
2·0	3 6269	3 6647	3 7028	3 7414	3 7803	3 8196	3 8593	3 8993	3 9398	3 9806	413
2·1	4·0219	4 0635	4 1056	4 1480	4 1909	4 2342	4 2779	4 3221	4 3666	4 4117	454
2 2	4 4571	4 5030	4 5494	4 5962	4 6434	4 6912	4 7394	4 7880	4 8372	4 8868	502
2 3	4 9370	4 9876	5 0387	5 0903	5 1425	5 1951	5 2483	5 3020	5 3562	5 4109	553
2·4	5 4662	5 5221	5 5785	5 6354	5 6929	5 7510	5 8097	5 8689	5 9288	5 9892	610
2·5	6 0502	6 1118	6 1741	6 2369	6 3004	6 3645	6 4293	6 4946	6 5607	6 6274	673
2·6	6 6947	6 7628	6 8315	6 9009	6 9709	7 0417	7 1132	7 1854	7 2583	7 3319	744
2·7	7·4063	7 4814	7 5572	7 6338	7 7112	7 7894	7 8683	7 9480	8 0285	8 1098	821
2·8	8 1919	8 2749	8 3586	8 4432	8 5287	8 6150	8 7021	8 7902	8 8791	8 9689	907
2·9	9 0596	9·1512	9 2437	9 3371	9 4315	9 5268	9 6231	9 7203	9 8185	9 9177	1,002
3·0	10 0179	10 1191	10 2212	10 3245	10 4287	10 5340	10 6403	10 7477	10 8562	10 9658	1,107
3·1	11·0765	11 1882	11·3011	11 4151	11 5303	11 6466	11 7641	11 8827	12 0026	12 1236	1,223
3·2	12 2459	12 3694	12 4941	12 6201	12 7473	12 8758	13 0056	13 1367	13 2691	13 4028	1,351
3·3	13 5379	13 6743	13 8121	13 9513	14 0919	14 2338	14 3772	14 5221	14 6684	14 8161	1,493
3·4	14 965	15·116	15 268	15 422	15 577	15 731	15 893	16 053	16 214	16 378	165
3·5	16 543	16 709	16 877	17 047	17 219	17 392	17 567	17 744	17 923	18 103	182
3·6	18·285	18·470	18 655	18 843	19 033	19 224	19 418	19 613	19 811	20 010	201
3·7	20 211	20 415	20 620	20 828	21 037	21 249	21 463	21 679	21 897	22 117	220
3·8	22 339	22·564	22 791	23 020	23 252	23 486	23 722	23 961	24 202	24·445	246
3·9	24·691	24 939	25 190	25 444	25 700	25 958	26 219	26 483	26 749	27 018	272
4·0	27 290	27 564	27 842	28 122	28 404	28 690	28 979	29 270	29 564	29·862	300

HYPERBOLIC FUNCTION

$$Cosh\ \phi = \tfrac{1}{2}(e^{\phi} + e^{-\phi})$$

for values of ϕ from 0 to 4

ϕ	0	1	2	3	4	5	6	7	8	9	D
0·0	1·0000	1·0001	1·0002	1·0005	1·0008	1·0013	1·0018	1·0025	1·0032	1·0041	9
0·1	1·0050	1·0061	1·0072	1·0085	1·0098	1·0113	1·0128	1·0145	1·0162	1·0181	20
0·2	1·0201	1·0221	1·0243	1·0266	1·0289	1·0314	1·0340	1·0367	1·0395	1·0423	30
0·3	1·0453	1·0484	1·0516	1·0549	1·0584	1·0619	1·0655	1·0692	1·0731	1·0770	41
0·4	1·0811	1·0852	1·0895	1·0939	1·0984	1·1030	1·1077	1·1125	1·1174	1·1225	51
0·5	1·1276	1·1329	1·1383	1·1438	1·1494	1·1551	1·1609	1·1669	1·1730	1·1792	63
0·6	1·1855	1·1919	1·1984	1·2051	1·2119	1·2188	1·2258	1·2330	1·2402	1·2476	76
0·7	1·2552	1·2628	1·2706	1·2785	1·2865	1·2947	1·3030	1·3114	1·3199	1·3286	88
0·8	1·3374	1·3464	1·3555	1·3647	1·3740	1·3835	1·3932	1·4029	1·4128	1·4229	102
0·9	1·4331	1·4434	1·4539	1·4645	1·4753	1·4862	1·4973	1·5085	1·5199	1·5314	117
1·0	1·5431	1·5549	1·5669	1·5790	1·5913	1·6038	1·6164	1·6292	1·6421	1·6552	133
1·1	1·6685	1·6820	1·6956	1·7093	1·7233	1·7374	1·7517	1·7662	1·7808	1·7956	151
1·2	1·8107	1·8258	1·8412	1·8568	1·8725	1·8884	1·9045	1·9208	1·9373	1·9540	169
1·3	1·9709	1·9880	2·0053	2·0228	2·0404	2·0583	2·0764	2·0947	2·1132	2·1320	189
1·4	2·1509	2·1700	2·1894	2·2090	2·2288	2·2488	2·2691	2·2896	2·3103	2·3312	212
1·5	2·3524	2·3738	2·3955	2·4174	2·4395	2·4619	2·4845	2·5073	2·5305	2·5538	237
1·6	2·5775	2·6013	2·6255	2·6499	2·6746	2·6995	2·7247	2·7502	2·7760	2·8020	263
1·7	2·8283	2·8549	2·8818	2·9090	2·9364	2·9642	2·9922	3·0206	3·0492	3·0782	293
1·8	3·1075	3·1371	3·1669	3·1972	3·2277	3·2585	3·2897	3·3212	3·3530	3·3852	325
1·9	3·4177	3·4506	3·4838	3·5173	3·5512	3·5855	3·6201	3·6551	3·6904	3·7261	361
2·0	3·7622	3·7987	3·8355	3·8727	3·9103	3·9483	3·9867	4·0255	4·0647	4·1043	400
2·1	4·1443	4·1847	4·2256	4·2668	4·3085	4·3507	4·3932	4·4362	4·4797	4·5236	443
2·2	4·5679	4·6127	4·6580	4·7037	4·7499	4·7966	4·8437	4·8914	4·9395	4·9881	491
2·3	5·0372	5·0868	5·1370	5·1876	5·2388	5·2905	5·3427	5·3954	5·4487	5·5026	543
2·4	5·5569	5·6119	5·6674	5·7235	5·7801	5·8373	5·8951	5·9535	6·0125	6·0721	602
2·5	6·1323	6·1931	6·2545	6·3166	6·3793	6·4426	6·5066	6·5712	6·6365	6·7024	666
2·6	6·7690	6·8363	6·9043	6·9729	7·0423	7·1123	7·1831	7·2546	7·3268	7·3998	737
2·7	7·4735	7·5479	7·6231	7·6990	7·7758	7·8533	7·9136	8·0106	8·0905	8·1712	815
2·8	8·2527	8·3351	8·4182	8·5022	8·5871	8·6728	8·7594	8·8469	8·9352	9·0244	902
2·9	9·1146	9·2056	9·2976	9·3905	9·4844	9·5791	9·6749	9·7716	9·8693	9·9680	998
3·0	10·0678	10·1683	10·2700	10·3728	10·4765	10·5813	10·6872	10·7942	10·9022	11·0113	1,102
3·1	11·1215	11·2328	11·3453	11·4588	11·5736	11·6895	11·8065	11·9247	12·0442	12·1648	1,218
3·2	12·2866	12·4097	12·5340	12·6596	12·7864	12·9146	13·0440	13·1747	13·3067	13·4401	1,347
3·3	13·5748	13·7108	13·8482	13·9871	14·1273	14·2689	14·4120	14·5565	14·7024	14·8498	1,489
3·4	14·999	15·149	15·301	15·455	15·610	15·766	15·924	16·084	16·245	16·408	165
3·5	16·573	16·739	16·907	17·077	17·248	17·421	17·596	17·772	17·951	18·131	182
3·6	18·313	18·497	18·682	18·870	19·059	19·250	19·444	19·639	19·836	20·035	201
3·7	20·236	20·439	20·644	20·852	21·061	21·272	21·486	21·702	21·919	22·139	222
3·8	22·362	22·586	22·813	23·042	23·273	23·507	23·743	23·982	24·222	24·466	245
3·9	24·711	24·959	25·210	25·463	25·719	25·977	26·238	26·502	26·768	27·037	271
4·0	27·308	27·582	27·860	28·139	28·422	28·707	28·996	29·287	29·581	29·878	300

$$\tan jx = j\,\mathrm{Tanh}\,x \quad = j\frac{e^x - e^{-x}}{e^x + e^{-x}}: \qquad \cot jx = -j\,\mathrm{Coth}\,x = -j\frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\sin^{-1} x = -j\,\mathrm{Sinh}^{-1} jx \quad = -j \log_e \{jx + \sqrt{1 - x^2}\}$$

$$\cos^{-1} x = -j\,\mathrm{Cosh}^{-1} x \quad = -j \log_e \{x + j\sqrt{1 - x^2}\}$$

$$\tan^{-1} x = -j\,\mathrm{Tanh}^{-1} jx = \frac{1}{2j} \log_e \left\{\frac{1 + jx}{1 - jx}\right\}.$$

Values of e^{-x} intermediate to and beyond those tabulated on page 529 may be obtained by interpolation, thus:

$$e^{-1\cdot03} = \tfrac{1}{2}(e^{-1\cdot02} + e^{-1\cdot04}) = \tfrac{1}{2}(0\cdot361 + 0\cdot353) = 0\cdot357.$$

The following examples show how to obtain values of e^{-x} when x has

TABLE OF CIRCULAR FUNCTIONS

$\theta°$	sin θ	tan θ	cot θ	cos θ	
0	0	0	$+\infty$	1	90
5	0·087	0·087	11·430	0·996	85
10	0·174	0·176	5·671	0·985	80
15	0·259	0·268	3·732	0·966	75
20	0·342	0·364	2·747	0·940	70
25	0·423	0·466	2·144	0·906	65
30	0·500	0·577	1·732	0·866	60
35	0·574	0·700	1·428	0 819	55
40	0·643	0·839	1·192	0·766	50
45	0·707	1·000	1·000	0·707	45
	cos θ	cot θ	tan θ	sin θ	$\theta°$

TABLE OF LOGARITHMS TO BASE e

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
1	0	0·095	0·182	0·262	0·336	0·405	0·470	0·531	0·588	0·642
2	0·693	0·742	0·788	0·833	0·875	0·916	0·956	0·993	1·030	1·065
3	1·099	1·131	1·163	1·194	1·224	1·253	1·281	1·308	1·335	1·361
4	1·386	1·411	1·435	1·459	1·482	1·504	1·526	1·548	1·569	1·589
5	1·609	1·629	1·649	1·668	1·686	1·705	1·723	1·740	1·758	1·775
6	1·792	1·808	1·825	1·841	1·856	1·872	1·887	1·902	1·917	1·932
7	1·946	1·960	1·974	1·988	2·001	2·015	2·028	2·041	2·054	2·067
8	2·079	2·092	2·104	2·116	2·128	2·140	2·152	2·163	2·175	2·186
9	2·197	2·208	2·219	2·230	2·241	2·251	2·262	2·272	2·282	2·293

$\log_e 10 = 2\cdot303$: $\log_e 10^2 = 4\cdot605$: $\log_e 10^3 = 6\cdot908$: $\log_e x = 2\cdot3 \log_{10} x$:

a value greater than the highest value given in the table—i.e. for x greater than 2·08.

$$e^{-3} = e^{-2} \times e^{-1} = 0{\cdot}135 \times 0{\cdot}368 = 0{\cdot}050$$
$$e^{-2\cdot66} = e^{-2} \times e^{-0\cdot66} = 0{\cdot}135 \times 0{\cdot}517 = 0{\cdot}070$$
$$e^{-6} = (e^{-2})^3 = (0{\cdot}135)^3 = 0{\cdot}002.$$

It is also useful to note that

$$e^{-2\cdot303} = 0{\cdot}1.$$

TABLE OF VALUES OF e^{-x}

x	0	2	4	6	8
0·0	1·000	0·980	0·961	0·942	0·923
0·1	0·905	0·887	0·869	0·852	0·835
0·2	0·819	0·803	0·787	0·771	0·756
0·3	0·741	0·726	0·712	0·698	0·684
0·4	0·670	0·657	0·644	0·631	0·619
0·5	0·607	0·595	0·583	0·571	0·560
0·6	0·549	0·538	0·527	0·517	0·507
0·7	0·497	0·487	0·477	0·468	0·458
0·8	0·449	0·440	0·432	0·423	0·415
0·9	0·407	0·398	0·391	0·383	0·375
1·0	0·368	0·361	0·353	0·346	0·340
1·1	0·333	0·326	0·320	0·313	0·307
1·2	0·301	0·295	0·289	0·284	0·278
1·3	0·273	0·267	0·262	0·257	0·252
1·4	0·247	0·242	0·237	0·232	0·228
1·5	0·223	0·219	0·214	0·210	0·206
1·6	0·202	0·198	0·194	0·190	0·186
1·7	0·183	0·179	0·176	0·172	0·169
1·8	0·165	0·162	0·159	0·156	0·153
1·9	0·150	0·147	0·144	0·141	0·138
2·0	0·135	0·133	0·130	0·127	0·125

Appendix II

SKIN EFFECT

(See also Chapter XIV)

Calculation of the Skin Effect in a Straight Wire, showing the Results given by the Exact Method and by the Approximate Formulae.

This comparison is given in Table I (on the next page) for a copper wire, 1 mm. diameter.

In Table II are shown the comparative data for some of the practical formulae applied to an iron wire 5 mm. diameter.

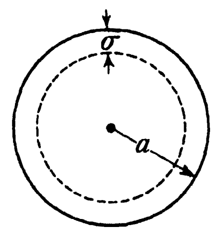


Fig. 5.

M M

TABLE I

Copper Wire: 1 mm. diameter: section = 0·00796 cm.^{2}; $\rho = 1{\cdot}77 \times 10^{-6}\Omega$/cm./cm.^{2}

$$Y = \sqrt{\frac{4\pi\mu\omega}{\rho \times 10^9}} = 0{\cdot}211\sqrt{f}: \text{ radius } a = 0{\cdot}05 \text{ cm.}: \ \sigma = 5{,}030\sqrt{\frac{\rho}{\mu f}} = 6{\cdot}7\frac{1}{\sqrt{f}} \text{ cm.}:$$

(See Fig. 5.)

$R_{d.c.}$ for Copper Wire 1 mm. diameter = $222 \times 10^{-6}\Omega$/cm.

hertz	$Y \cdot a$	Y		$\frac{R_{a.c.}}{R_{d.c.}}$	$J_0\sqrt{-1}\lvert Y \cdot a$	$J_1\sqrt{-1}\lvert Y \cdot a$	*Exact Values* *Res. to a.c. / Res. to d.c.*	*Exact Values* *Reactance / Res. to d.c.*	*Resistance to a.c. / Resistance to d.c.* (*see Chapter XIV, Example* 7), as derived from the penetration depth $\sigma = \frac{\sqrt{2}}{Y}$: $\sigma = \frac{\sqrt{2}}{Y}$ *cm.*	*Res. to a.c. / Res. to d.c.* $\frac{a}{2\sigma}$
9,000	1	20	$1 + \frac{(Ya)^4}{190} - \frac{(Ya)^8}{46{,}100}$	1·0052	$1{\cdot}015e^{j14\ 2^\circ}$	$0{\cdot}50e^{-j37\ 8^\circ}$	1·015	0 123	—	—
	1·25	25			$1{\cdot}037e^{j22^\circ}$	$0{\cdot}63e^{\ j33\ 8^\circ}$				
	1·50	30			$1{\cdot}077e^{j31\ 2^\circ}$	$0{\cdot}76e^{-j30^\circ}$				
	1·75	35			$1{\cdot}14e^{j41\ 4^\circ}$	$0{\cdot}80e^{-j23\ 2^\circ}$				
36,000	2·0	40		1·052	$1{\cdot}23e^{j52\ 3^\circ}$	$1{\cdot}04e^{-j16\ 7^\circ}$	1·085	0·46	—	—
	2·25	45			$1{\cdot}35e^{j63\ 5^\circ}$	$1{\cdot}20e^{-j9\ 5^\circ}$				
	2·5	50			$1{\cdot}51e^{j74\ 6^\circ}$	$1{\cdot}37e^{-j1\ 6^\circ}$				
	2·82	56·4		1·26	$1\ 76e^{j88^\circ}$	$1{\cdot}62e^{\ j9^\circ}$	1·28	0 835		
81,000	3·0		$0{\cdot}352Ya + 0{\cdot}25 + \frac{0{\cdot}134}{Ya}$	1·351	$1{\cdot}95e^{j96\ 5^\circ}$	$1{\cdot}80e^{j15\ 7^\circ}$	1·31	0·946	—	—
	3·25				$2{\cdot}23e^{j107^\circ}$	$2{\cdot}06e^{j25^\circ}$				
	3·50				$2{\cdot}58e^{j117\ 6^\circ}$	$2{\cdot}38e^{j34\ 4^\circ}$				
	3·75				$2{\cdot}95e^{j127\ 5^\circ}$	$2{\cdot}74e^{j44^\circ}$				
144,000	4·0	80		1·691	$3{\cdot}44e^{j138\ 2^\circ}$	$3{\cdot}17e^{j53\ 9^\circ}$	1·67	1·35	—	—
	4·25				$4{\cdot}0e_j{}^{148^\circ}$	$3{\cdot}7e^{j63\ 7^\circ}$				
	4·50				$4\ 62e^{j158\ 6^\circ}$	$4{\cdot}28e^{j73\ 7^\circ}$				
	4·75				$5{\cdot}36e^{j168\ 7^\circ}$	$5{\cdot}0e^{j83\ 5^\circ}$				
225,000	5·0	100		2·037	$6\ 23e^{j180^\circ}$	$5{\cdot}8e^{j93\ 5^\circ}$	2·005	1·68	0·0014	1·8
324,000	6·0	120		2·38	$11{\cdot}5e^{j219\ 6^\circ}$	$10{\cdot}85e^{j133\ 4^\circ}$	2·40	2·09	0·00117	2·15
441,000	7·0	140		2·73	$21{\cdot}55e^{j260\ 3^\circ}$	$20{\cdot}5e^{j173{\cdot}5^\circ}$	2·76	2·43	0·00101	2·48
576,000	8·0	160		3·08	$40{\cdot}82e^{j300\ 9^\circ}$	$39{\cdot}07e^{j213\ 7^\circ}$	3·14	2·8	0·00088	2·84
729,000	9·0	180		3·43	$77{\cdot}96e^{j341\ 5^\circ}$	$74{\cdot}97c^{j253\ 9^\circ}$	3·45	3·18	0·00079	3·15
900,000	10·0	200		3·783	$149{\cdot}83e^{j382\ 1^\circ}$	$144{\cdot}6e^{j294\ 3^\circ}$	3·78	3·525	0·000707	3·52
$3{\cdot}6\times10^6$	20	400		7·0	For large values of τa the real component of the ratio $\frac{J_0}{J_1}$ becomes $\cos\varphi$, i.e. $\frac{1}{\sqrt{2}}$, and the ratio $\frac{\text{Res. to a.c.}}{\text{Res. to d.c.}} = \frac{Ya}{3\sqrt{2}} = 0{\cdot}353\tau a$ as in the adjacent column to ←the left.		7·0	7·0	0·000353	7·0
$14{\cdot}4\times10^6$	40	800		14·1			14·1	14·1	0·000177	14·2
$32{\cdot}4\times10^6$	60	1,200		21·2			21·2	21·2	0·000117	21·2
$57{\cdot}6\times10^6$	80	1,600		28·2			28·2	28·2	0·000088	28·2

TABLE II

Iron Wire: 0·5 cm, diameter: $\rho = 15 \times 10^{-6}/\Omega/\text{cm./cm.}^2$: $\mu = 2{,}500$

$$a = 0{\cdot}25 \text{ cm.}: \; Y = \sqrt{\frac{8\pi^2\mu f}{\rho \times 10^9}} = 3{\cdot}62\sqrt{f}: \; f = \frac{Y^2}{13{\cdot}2}: \; \sigma = \frac{\sqrt{2}}{Y}$$

f hertz	Ya	Y		$\frac{\text{Res. to a.c.}}{\text{Res. to d.c.}}$	$\sigma = \frac{\sqrt{2}}{Y}$ cm.	$\frac{\text{Res. to a.c.}}{\text{Res. to d.c.}}$ $\frac{a}{2\sigma}$
1·21	1	4	$1+\frac{(Ya)^4}{190}-\frac{(Ya)^8}{46{,}100}$	1·005	—	—
4·86	2	8		1·085	—	—
10·9	3	12	$0{\cdot}352(Ya)+0{\cdot}25+\frac{0{\cdot}134}{(Ya)}$	1·35	—	—
19·5	4	16		1·69	0·088	1·42
30·5	5	20		2·04	0·071	1·80
43·0	6	24		2·38	0·059	2·10
59·0	7	28		2·73	0·0505	2·47
77·0	8	32		2·98	0·044	2·85
122	10	40		3·78	0·035	3·5
274	15	60		5·3	0·023	5·3
485	20	80		7·0	0·0177	7·1

Appendix III

GERMAN SCRIPT AND ENGLISH EQUIVALENTS

$\mathfrak{A}$	=	A	$\mathfrak{W}$	=	W
$\mathfrak{B}$	=	B	$\mathfrak{X}$	=	X
$\mathfrak{C}$	=	C	$\mathfrak{Y}$	=	Y
$\mathfrak{E}$	=	E	$\mathfrak{Z}$	=	Z
$\mathfrak{H}$	=	H	$\mathscr{D}$	=	Manuscript D
$\mathfrak{I}$	=	I or J	$\mathfrak{e}$	=	e
$\mathfrak{K}$	=	K	$\mathfrak{i}$	=	i
$\mathfrak{M}$	=	M	$\mathfrak{q}$	=	q
$\mathfrak{S}$	=	S	$\mathfrak{u}$	=	u
$\mathfrak{U}$	=	U	$\mathfrak{w}$	=	w
$\mathfrak{V}$	=	V	$\mathfrak{z}$	=	z

These Script letters are used to denote vector quantities.

LIST OF GREEK SYMBOLS

α	alpha	μ	mu
β	beta	ν	nu
γ	gamma	ξ	xi
Δ, δ	delta	π	pi
		ρ	rho
ε	epsilon	Σ, σ	sigma
ζ	zeta		
η	eta	τ	tau
Θ, θ	theta	Φ, ϕ	phi
ι	iota	χ	chi
κ	kappa	ψ	psi
Λ, λ	lambda	Ω, ω	omega

Appendix IV

SOME USEFUL PHYSICAL CONSTANTS

10^{-1} cm. = 1 mm.
10^{-4} cm. = 10^{-6} m. = 1 micron (μ).
10^{-7} cm. = 1 milli-micron ($m\mu$).
10^{-8} cm. = 1 Ångström (Å).
10^{-10} cm. = 1 micro-micron ($\mu\mu$).

Gravitation constant for the attraction between two masses :

$$F_g = H\frac{m_1 \times m_2}{r^2} \text{ dynes}$$

where m_1 and m_2 are in gms., r in cm.,

$$H = 6{\cdot}65 \times 10^{-8}.$$

EXAMPLE.—r = 5 cm., m_1 = 100 gm., m_2 = 10^8 gm.

$$F_g = \frac{6{\cdot}65 \times 10^{-8} \times 10^8 \times 10^2}{25} = 26{\cdot}6 \text{ dynes.}$$

---

1 *Mol weight M* of a substance is an amount such that the number of grams weight is the same as the number which defines the molecular weight, e.g.,

1 mol of hydrogen = 2 gm.

1 mol weight of every gas at 0° C. and 760 mm. pressure occupies 22·4 litres.

Avogadro's Number (also called Loschmidt's Number) at 0° C. and 760 mm. pressure, is :

$$\frac{\text{Mol weight}}{\text{Weight of one molecule}} = 6{\cdot}06 \times 10^{23}.$$

$$1\ \textit{Physical Atmosphere} = 1{\cdot}033\ \text{kg./cm.}^2$$
$$= 1{,}013 \times 10^3\ \text{dynes/cm.}^2.$$

This quantity is the weight of a column of mercury 760 mm. long and 1 sq. cm. cross-section, the density of the mercury being 13·6 gm./cm.^{3}.

$$1\ \textit{Technical Atmosphere} = 1{\cdot}000\ \text{kg./cm.}^2.$$

Electromagnetic Waves

Type of Radiation	Wavelength in microns, μ
Short gamma-rays	$0{\cdot}466 \times 10^{-6}$
X-rays	$0{\cdot}158 \times 10^{-4}$ . . . 660×10^{-4}
Ultra-violet rays	$1{\cdot}3 \times 10^{-2}$ . . . 36×10^{-2}
Visible rays	0·36 . . . 0·78
Ultra-red rays } Heat wave	0·78 . . . 340
Electric waves	340 . . . ∞

$$1\ \text{micron}\ (\mu) = 0{\cdot}001\ \text{mm.}$$
$$1\ \text{micron}\ (\mu) = 10^4\ \text{Å (Ångström).}$$
$$1\ \text{Å} = 10^{-8}\ \text{cm.}$$

Fig. 6 shows the energy distribution of electromagnetic waves as a function of the wavelength for a series of temperature measured in ° K., i.e. Kelvin degrees : (° K. = ° C. + 273° C.).

This diagram shows how the maximum energy radiated becomes displaced on the abscissa scale for the different temperature values.

The shaded area refers to visible light.

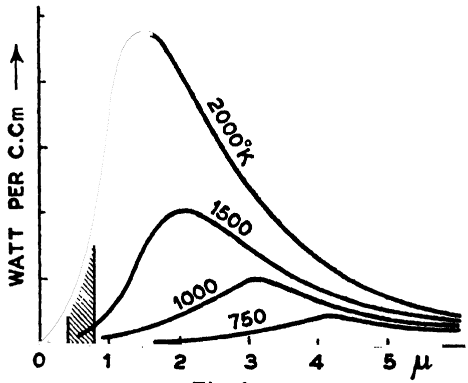


Fig. 6.

Heat Energy required to melt 1 Metric-ton of Steel

$$1 \text{ metric-ton} = 1{,}000 \text{ kg.} = 2{,}200 \text{ lbs.}$$
$$\text{Sensible heat at } 1{,}500^\circ \text{ C.} = 240 \times 1{,}000 \text{ kg.-calories.}$$
$$\text{Latent heat} = 49 \times 1{,}000 \quad \text{,,}$$
$$\text{Total heat} = 289 \times 10^3 \text{ kg.-calories.}$$

or
$$\frac{289 \times 10^3}{860} = \underline{336 \text{ kWh.}}$$

Specific resistance of high conductivity electrolytic copper:

$$\text{Annealed copper: } \rho \ngtr \frac{1}{57} \text{ ohm/m./mm.}^2.$$

$$\text{Cold-drawn copper wire: } \rho \ngtr \frac{1}{56} \text{ to } \frac{1}{55} \text{ ohm/m./mm.}^2.$$

General Expression for the Coupling Coefficient of Two Coupled Circuits

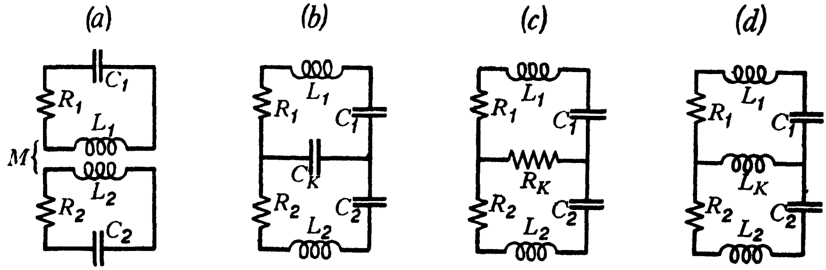
In Chapter X, page 337, expression (68), the general formula for the coupling coefficient of two coupled circuits is given, viz.,

$$k = \frac{X_K}{\sqrt{X_1 . X_2}},$$

and on page 338 are shown four representative diagrams of such coupled systems. The respective values of the four components of this expression,

$$k, \; X_K, \; X_1, \; X_2,$$

for each system is given in the following Table:



	(a)	(b)	(c)	(d)
X_K	ωM	$\dfrac{1}{\omega C_K}$	R_K	ωL_K
X_1	ωL_1	$\dfrac{1}{\omega}\left(\dfrac{1}{C_1} + \dfrac{1}{C_K}\right)$	$R_1 + R_K$	$\omega(L_1 + L_K)$
X_2	ωL_2	$\dfrac{1}{\omega}\left(\dfrac{1}{C_2} + \dfrac{1}{C_K}\right)$	$R_2 + R_K$	$\omega(L_2 + L_K)$
k	$\dfrac{M}{\sqrt{L_1 L_2}}$	$\sqrt{\dfrac{C_1 C_2}{(C_1 + C_K)(C_2 + C_K)}}$	$\dfrac{R_K}{\sqrt{(R_1 + R_K)(R_2 + R_K)}}$	$\dfrac{L_K}{\sqrt{(L_1 + L_K)(L_2 + L_K)}}$

TEST PAPERS

The Test Examples below are all chosen and arranged with a view to emphasising the different aspects of the principles which are dealt with in the individual chapters of this volume

Solutions to the questions will be found in the companion volume, " Test Papers and Solutions on Electrical Engineering " They will also serve to elucidate points which might otherwise be obscure to the reader

TEST PAPER ON CHAPTER I

1. Express the kinetic energy (i) of translation, (ii) of rotation in joules.

2. A train of mass G metric tons is travelling at a speed of V km./hour. Find the kinetic energy, assuming that the rotational energy is 25 per cent. of that of translation.

3. What driving force is necessary to produce an acceleration of A km./hour/sec. of a train on the level, neglecting the frictional resistance ?

4. An electrically operated vehicle of mass M kg. is running on the level at a speed of V metres/sec. If the vehicle is braked by the application of a constant torque, show by means of a diagram the relationship between speed and time, and between kinetic energy and time, during the braking period.

5. A train running at a speed of 60 miles/hour on the level is brought to rest in 1,000 yards. Find what uniform resistance must be applied to the train.

6. At what speed will an engine of 500 horse-power take a train of 300 tons weight up an incline of 1 : 80, if the frictional resistance is 15 lb. weight per ton.

7. A railway waggon weighing 30 tons strikes the buffers at a speed of 5 miles/hour on the level, and comes to rest in 1 foot. Find the mean resistance exerted by the buffer springs.

8. The specific heat of copper is 0·094 $\frac{\text{kcal.}}{\text{kg. }^\circ\text{C.}}$ *and the density is 8·9. Express the specific heat in terms of* $\frac{\text{joules}}{\text{c.cm. }^\circ\text{C.}}$.

9. Express the c.g.s. unit of mass in terms of joules $\frac{\text{sec.}^2}{\text{cm.}^2}$.

10. Express the c.g.s. unit of force in terms of joules/metre.

11. If the cost of coal is 12s. per ton and the heat value is 3,000 kcal. per lb., and if 4,500 kcal. are required for the generation of 1 kWh., find (i) the thermal efficiency of generation, and (ii) the cost of generation per kWh.

12. In a fuel-fired central heating plant 1 lb. of coal of calorific value 3,500 kcal. per lb. can produce about 0·9 kWh. of useful heat energy. Find the thermal efficiency of the plant. A fuel-fired boiler in continuous service can produce about 2·25 kWh. of useful heat energy per lb. of coal. Find the thermal efficiency.

TEST PAPER ON CHAPTER II

1. Describe the constructional features of a dry (e.g. torch) battery.

2. With the aid of a diagram of connections, show how the internal resistance of a cell may be measured.

3. If the resistance of wire used for a telephone line is 55 ohms per mile and its conductivity is 56 siemens/metre/mm.^{2}, find the cross-sectional area.

4. What is meant by " electrolytic corrosion " ? How may its destructive effects be kept under control ?

5. Experiment shows that when 1 gm. of hydrogen combines with oxygen to form water, 34 kcal. of heat energy are released. If, in the electrolysis of water, a current of I amperes flows for t seconds, find : (i) The necessary expenditure of energy. (ii) The minimum value of the applied p.d. for effective electrolysis. (iii) The energy required to release 1 cubic metre of hydrogen at 0° C. and 760 mm. pressure.

6. What quantity of electricity will be necessary to produce one ton of caustic soda by the electrolysis of brine ?

7. If the electrochemical equivalents of iron, lead, and copper, are respectively 0·000289, 0·0010714, 0·00066 gm. per coulomb, find what amount of each of these metals will become dissolved in electrolytic corrosion by a current of 1 ampere in 1 year, the number of active hours per annum being 4,200.

8. Show how a logarithmic heating curve may be derived graphically from a knowledge of the heating time constant, one point on the curve and the final steady temperature.

9. The field coil of a generator has a heating time constant of 20 minutes and heat is generated at a uniform rate such that the final steady temperature will be 80° C. If the room temperature is 20° C. construct the temperature/time curve for the first 20 minutes after switching on the coil when it is at room temperature. If the current is then switched off, construct the cooling curve for the next 10 minutes.

10. Explain why a mercury-arc rectifier is not very suitable for operating at low pressures.

11. Describe the principal features of a single-phase mercury-arc rectifier.

12. Explain the operation of the two-electrode thermionic valve (the diode).

13. Show how a three-electrode thermionic valve (a triode) may be used as an amplifier.

14. How may the cathode-glow discharge be applied as a surge arrester ?

15. Give an account of Townsend's " avalanche " theory of the breakdown of a spark-gap.

16. Give an account of the principal phenomena associated with an electric arc.

17. What is the International Electrotechnical Commission (I.E.C.) definition for the specific resistance of standard soft copper ? From this definition derive the temperature coefficient at 0° C.

18. Derive an expression which shows how the temperature rise of a coil which carries a current of I amperes depends upon the way in which the temperature coefficient of the material of the wire varies with the temperature. It is to be noted that the cooling is assumed to be forced so that the radiated heat is negligibly small as compared with the heat which is dissipated by convection.

19. Describe some modern form of the carbon microphone.

20. A source of d.c. supply of which the e.m.f. is E volts and the resistance is R_0 ohms is connected to a consumer's load of resistance R_x ohms. Find (i) the conditions for which the maximum power is delivered to the consumer, and (ii) the efficiency of the supply system under these conditions.

TEST PAPER ON CHAPTER III

1. *When the insulation between an overhead transmission line and the supporting mast fails, a current will flow to earth and, for many purposes, a sufficiently good approximation can be obtained by assuming that the current passes into the earth through a hemispheric surface bedded in the earth and representing the foundation of the mast. Obtain expressions for the resultant current density and the pressure drop per centimetre at points in the neighbourhood of the base of the mast and relate these quantities to corresponding quantities for the electric field due to a charged sphere.*

2. *Assuming the current radiates from a hemispherical surface bedded in the earth, find the potential and the current density at a point distant x cm. from the centre of the hemisphere.*

3. *An animal is standing in the neighbourhood of a live mast. Find the magnitude of the p.d. which will be short-circuited by the body of the animal.*

4. *A man is walking in the neighbourhood of a live transmission mast. Find the magnitude of the current which will pass through his body.*

5. *A sphere of 1 cm. diameter is charged with 5 electrostatic units of positive electricity and is suspended by an insulating support at a height of 1 metre above a large flat metal plate placed on the ground. Find the force exerted on the metal plate and the potential of the sphere. It is to be assumed that the height of the sphere above the plate is large in comparison with the radius of the sphere.*

6. *Two concentrated quantities each of 2 electrostatic units of positive electricity are placed at 5 cm. apart. Plot the traces of the equipotential surfaces and the lines of force.*

7. *Give an account of the main types of insulation materials for heavy and light current work respectively. Describe the principle of one method of testing the insulation strength by means of high tension rectified alternating current pressure.*

8. *Give an account of the electron theory of the action of a crystal detector.*

TEST PAPER ON CHAPTER IV

1. *A condenser discharges through a resistance and spark-gap in series. Obtain the current and pressure relationship showing how the time taken for the condenser to discharge may be found.*

2. *A condenser is charged from a source of direct current by means of an arc across the switch gap. Give an account of the current and pressure relationships and derive an expression for the time taken to charge the condenser.*

3. *Derive an expression for the pressure distribution along a string of high tension insulators. (NOTE.—To answer this question knowledge of the contents of Chapter XV is desirable.)*

4. *Express the relationship*

$$Q \text{ in coulombs} = (V \text{ in volts}) \times (C \text{ in farads})$$

in electromagnetic units.

5. *Find the potential at any point due to a long straight wire which is free in space and is charged with q coulombs per centimetre length. Apply the result to the determination of the capacitance per kilometre of a symmetrically arranged three-phase transmission line.*

6. *Two metal spheres each of radius r cm. and distant a cm. apart in air, are equally and oppositely charged with Q electrostatic units. Find :*

(*i*) *The potential difference between the spheres.*
(*ii*) *The capacitance of the condenser formed by the two spheres.*
(*iii*) *The energy stored in this charged condenser.*
(*iv*) *The force acting on each sphere.*

7. Three long straight horizontal and parallel wires are supported at different heights above the earth's surface and each is charged with q coulombs per centimetre length. Find the potential at any point and apply the result to the calculation of the capacitance per phase of a symmetrically arranged three-phase transmission line.

8. A transformer bushing is built up as follows :

Layer No.				Dielectric constant
0	Copper rod	outside radius	2·0 cm.	
1	Layer of varnished paper . .	outside radius	2·3 cm.	4
2	Layer of "compound" . .	outside radius	5·0 cm.	2·6
3	Porcelain shell	outside radius	7·6 cm.	5

Maximum pressure gradients :
- *For the paper insulation . . . . 30,000 volts/cm.*
- *For the "compound" . . . . . 40,000 ,,*
- *For the porcelain . . . . . 9,500 ,,*

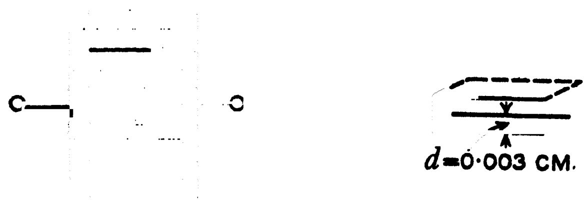
Find the potential of the copper rod and the pressure gradient in the air at the outside surface of the porcelain shell.

9. What significance has corona phenomena for the design of overhead transmission lines ?

10. What significance has corona phenomena for the design of underground cables ?

11. A capacitance is formed from two flat parallel plates as electrodes. A slab of solid insulation material of thickness t_1 mm. and of dielectric constant 4 is close against one electrode and the air-gap between the other electrode and the insulation is t_2 mm. wide. If a pressure V is applied across the electrodes, find the pressure drop across the air-gap.

12. A condenser is built up of 11 plates as shown in the accompanying diagram, the surface area of each plate is 50 sq. cm. and the distance between adjacent plates is



(a) (b)

Question 12.

0·003 cm. If the condenser is charged to a pressure of 250 volts and the dielectric constant is ε = 9, find—
- *(i) The capacitance of the condenser.*
- *(ii) The energy stored.*
- *(iii) The force of attraction between two adjacent plates.*

13. Describe a method for determining the dielectric resistance by measuring the rate at which a charged and insulated cable loses its potential.

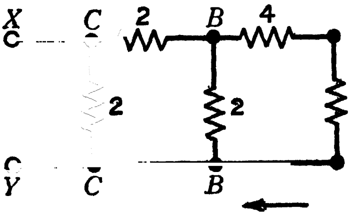
14. Find the capacitance due to the two long cylindrical conductors of an overhead single-phase transmission line the diameter of each of which is comparable with the distance between the axes of the cylinders.

TEST PAPER ON CHAPTER V

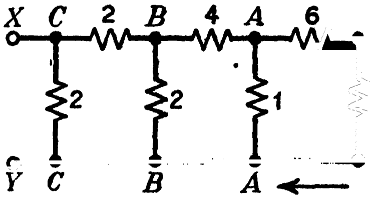
1. *A chain of two resistance cells is shown in the accompanying diagram. Find the resistance as measured between the terminals XY.*

2. *A chain of three resistance cells is shown in the accompanying diagram. Find the resistance measured between the terminals XY.*

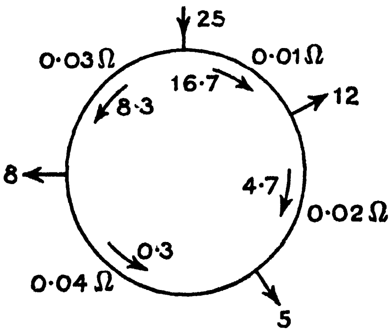
3. *A ring-main is fed at one point and there are three consumer's tapping points, as shown in the accompanying diagram. Find the current distribution in the ring.*



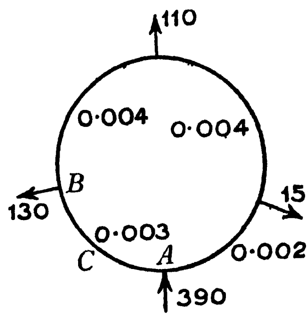
Question 1.



Question 2.



Question 3.

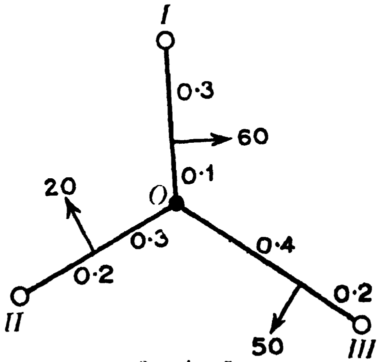


Question 4.

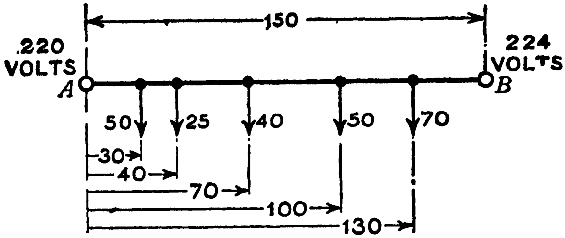
4. *A closed ring-main, shown in the accompanying diagram, is fed at one point A and supplies three separate consumers. Find the current distribution by assuming the main is cut at any point, such as B.*

5. *A star connected system of resistance is loaded, as shown in the accompanying diagram, and the supply points I, II and III are all at the same potential. Find (i) the current distribution in the system, and (ii) the potential difference between the star point and the supply points.*

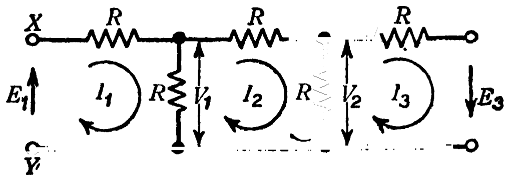
6. *A distributor of copper wire 150 metres long is fed from each of the ends A and B, the end A being supplied at 220 volts and the end B at 224 volts, as shown in the*



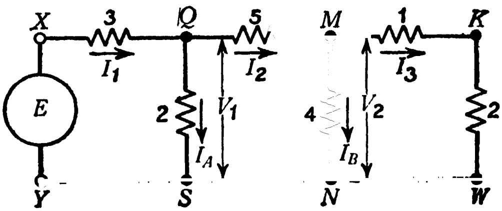
Question 5.



Question 6.



Question 7.



Question 8.

accompanying diagram. Loads are connected to tapping points as follows (distances measured from the end A):

50 amperes at 30 metres, 25 amperes at 40 metres, 40 amperes at 70 metres, 50 amperes at 100 metres; 70 amperes at 130 metres.

Find what current is fed in at each end of the line and the maximum percentage drop. The cross section of the distributor conductor is 20 sq. mm.

7. Find by the method of determinants the current distribution in the network shown in the accompanying diagram.

8. Apply Thévenin's Theorem to determine the current distribution in the network of the accompanying diagram.

9. Give an account of (i) the Varley test and (ii) the Murray test for determining the position of faults in cables.

10. Describe the " universal shunt " and explain its purpose.

TEST PAPER ON CHAPTER VI

1. When an electric current passes across a junction of two different metals the " Joule effect " and the " Peltier effect " are superposed. How may these two effects be separated?

2. Show how the Peltier cooling action may be calculated from the e.m.f./temperature relationship of a thermo-junction.

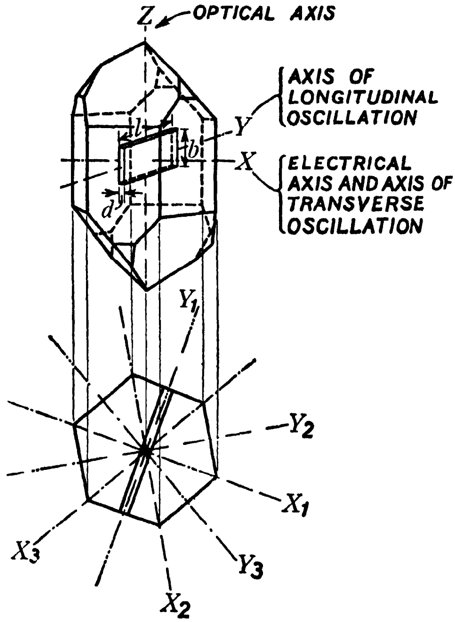
3. Quartz plates for piezo-electric oscillating systems are usually cut from the crystal, as shown in the accompanying diagram. If such a quartz plate is held between two metal plates so that the electric axis of the crystal lies at right angles to the metal plates, then, according to the frequency of the p.d. which is applied to the plates, longitudinal or transverse (i.e. thickness) oscillations of the quartz will be produced. What is the structure of the electric circuit which is equivalent to this system of quartz and metal plates?

4. Give a short account of the problem of measuring high temperatures such as are required for many industrial purposes.

5. Describe some form of resistance thermometer which is suitable for measuring temperatures in the neighbourhood of 1,000° C.

6. Give a diagram of connections for a Bridge method of measuring temperatures by means of a thermocouple.

7. What is the technical significance of a "black body" and what is its practical significance?



Question 3, Chapter VI.

TEST PAPER ON CHAPTER VII

1. Describe a method for measuring the horizontal component H of the earth's magnetic field.

2. Find the potential at a point P in the field of a small bar magnet.

3. What is a "magnetic shell of uniform strength," and what is its practical significance?

4. Find the potential of a magnetic shell at a point P.

5. What change of direction takes place when lines of magnetic induction pass from one medium into another?

6. Explain the principle in accordance with which a space may be screened from a magnetic field.

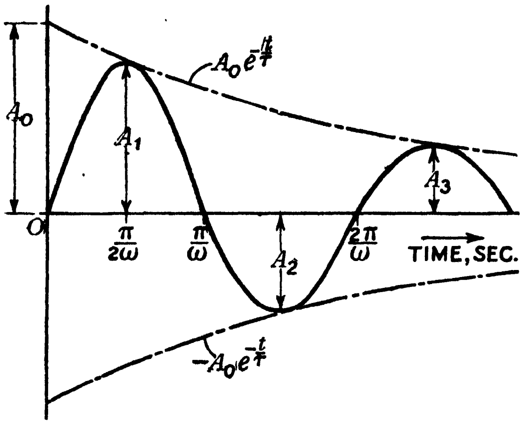
7. Discuss the magnetic force conditions which will exist when two magnetised surfaces are placed parallel to each other in a gaseous or liquid material of permeability μ_1.

8. What mechanical force is necessary to drag a block of iron from the end of a magnetised bar of circular section and 2 cm. diameter when the induction density at the end of the bar is $B = 3{,}200$ *gauss?*

9. Successive deflections of a ballistic galvanometer with small damping are as shown in the accompanying diagram, viz.: $A_1 : A_2 : A_3 \ldots$, *where* $A_1 = 10$ *cm.:* $A_3 = 9{\cdot}5$ *cm., and so on. Find the undamped deflection* A_0 *in terms of the following data.*

(*i*) *The logarithmic decrement is* λ *nepers per* $\frac{1}{2}$ *period of swing.*

(*ii*) *The logarithmic decrement is* Δ *nepers per period of swing.*



Question 9.

10. Explain the principle of action of the thermo-magnetic motor.

11. Show how the principle of magneto striction may be applied to the measurement of Young's modulus for a rod or wire of iron, nickel, or cobalt.

12. Explain the principle of operation of the Magnetic-Potential Meter.

13. On page 204, Fig. 30, of "Principles", a method is explained for calibrating a magnetic-potential meter having a flexible coil. Describe two methods for calibrating the non-flexible semi-circular coil system shown on page 204, Fig. 31.

14. Describe the procedure for measuring static magnetic fields by means of a "bismuth spiral".

15. What is the "Hall Effect" and how may it be applied to the measurement of the strength of a magnetic field?

TEST PAPER ON CHAPTER VIII

1. The pole shoe of a direct current dynamo is shaped as shown in the accompanying diagram. Explain how the total flux per pole crossing the air-gap per pole may be determined.

2. A magnetic separator, capable of dealing with large blocks of iron, comprises a "magnetic pulley" driven by a belt on which the material to be separated travels. The pulley is magnetised by means of an exciting coil as shown diagrammatically in the accompanying diagram (a). Find what magnetic pull is exerted on the block when it is symmetrically placed over the axis of the pulley as shown in the diagram (b). The diameter of the pulley is 36 inches and the belt is $\frac{5}{8}$-inch thick. The surface of the block is $6\frac{1}{4} \times 24$ inches, the effective length in the direction of the pulley axis being 12 inches. The exciting ampere-turns of the magnetising coil is 12,000.

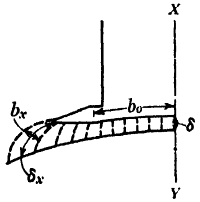
3. A current of 8 amperes is flowing in a long straight wire. What is the force on a unit magnetic pole placed at a distance of 7 cm. from the axis of the wire? What will be the radius of a circular coil of one turn which will produce the same force at the centre of the coil for the same current of 8 amperes in the coil?

4. A lifting electromagnet and armature of soft iron are shown in the accompanying diagram, the air-gap δ being 2 mm. and the total length of the magnetic circuit in the iron being 130 cm. The section of the iron is 9×9 cm. and the permeability is 2,500. What flux density in the air-gap will be required to hold a load of 1 metric ton and what number of ampere-turns must be provided to produce this flux density?

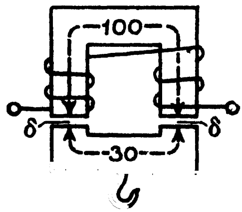
5. In order to generate the required direct current e.m.f., the flux in the armature of the two-pole dynamo of the accompanying diagram must be $5{\cdot}75 \times 10^6$ c.g.s. lines, the magnetic permeability being $\mu = 3{,}000$. The magnetic leakage factor is $1{\cdot}2$: the mean length of path in the poles and yoke is 100 cm. and the cross-section is 650 sq. cm.: the length of the single air-gap is 1 cm. and the cross-section is 1,300 sq. cm.: the length of path in the armature is 15 cm. and the cross-section is 500 sq. cm. Find what exciting ampere-turns will be required.

6. A short-circuited circular coil of 150 turns has a mean diameter of 25·5 cm. and is arranged so that it can be rotated about a vertical axis. The coil is set with its plane perpendicular to the horizontal component of the earth's magnetic field, the intensity of which is 0·2 oersted. If the resistance of the coil is 20 ohms, find what quantity of electricity will be set in motion when the coil is turned through 180°.

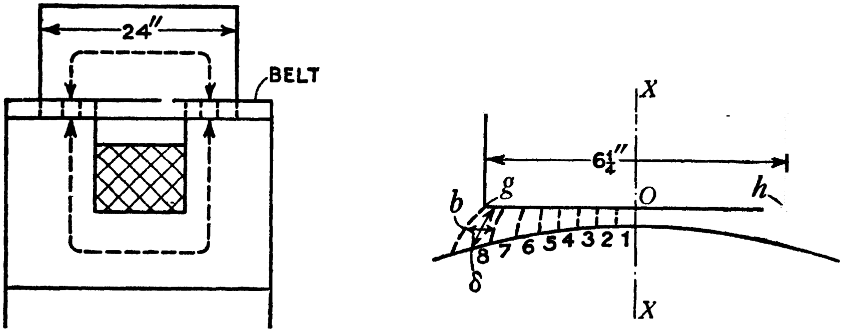
7. Show how the mutual inductance of two coils can be measured by means of a ballistic galvanometer test.



Question 1.

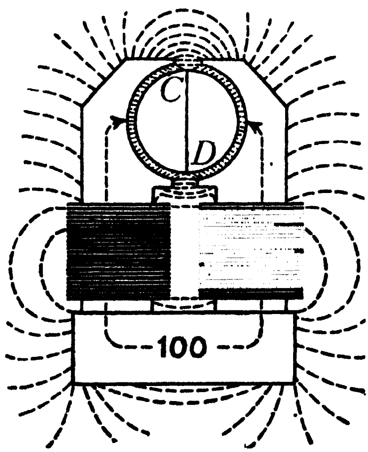


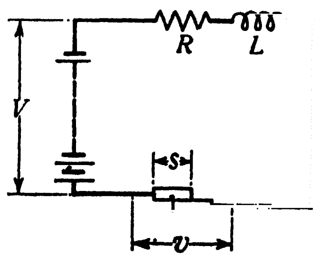
Question 4.



(*a*) (*b*)

Question 2.





Above—Question 11.
Left—Question 5.

8. Two coils, 1 and 2, are in inductive relationship to one another, the resistance of each coil being negligible. If a d.c. pressure of 10 volts is applied to coil 1, obtain a graph showing the current in each coil and the stored electromagnetic energy of the system as a function of the time, when $L_1 = 1$ henry : $L_2 = 2$ henry : $M = 1{\cdot}2$ henry.

9. A two-pole short-circuit occurs at a power station which is operating in parallel with a number of other stations on 10 kV. bus bars. The short-circuit current reaches a peak value of 93,500 amperes. If the length of conductor between two supports is 160 cm. and the mean distance between two neighbouring conductors is 40 cm., find what mechanical force will be developed between two conductors.

10. Obtain the relationship between torque and speed for a direct current series wound traction motor.

11. The switch shown in the accompanying diagram comprises two surfaces in contact, the process of opening the switch consisting in sliding these two surfaces so that the area of contact gradually becomes zero. Obtain the relationships between contact current and time and contact p.d. and time.

12. An electron is moving in a steady homogeneous electric field and a steady homogeneous magnetic field, these two fields acting in directions which are mutually at right angles. Obtain the equations which define the motion of the electron.

13. Derive the equations which define the path of the motion of Example 12, for the following initial conditions, viz., $t = 0$: $v_x = 0$: $v_y = 0$.

(This principle is applied in the "Magnetron" valve for the generation of ultra-high-frequency oscillations.)

TEST PAPER ON CHAPTER IX

1. A condenser of $3\mu F$ capacitance is placed in parallel with a resistance of 150 ohms and the parallel system is connected to an a.c. supply of 600 r.m.s. volts at 500 frequency. What is the power factor and the magnitude of the current which is supplied by the mains? If the resistance and capacitance are now placed in series across the same supply terminals, what will be the current and power factor?

2. In order to protect a group of relatively small power cables, from a short-circuit, which might produce a destructive effect, a choking coil can be arranged between the generator and the cables as shown in the accompanying diagram.

A water-driven three-phase power station of 150 MVA. capacity at $6{\cdot}6/\sqrt{3}$ kV. per phase supplies a nitrogen plant by means of cables. The reactance of the station is 15 per cent. and the reactance of the protective choke coil is 8 per cent. Find the value of the short-circuit current.

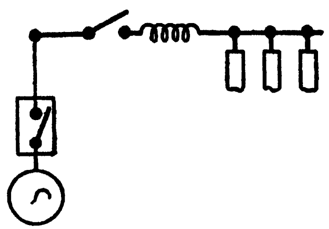
What will be the short-circuit current if there is no protective choking coil? What is the ratio of the consequent mechanical stress in the two cases?

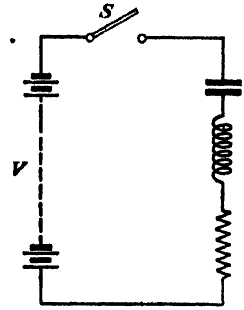
3. A laminated cylindrical iron ring of 45 sq. cm. cross-section, is uniformly wound with a coil of 100 turns and an alternating p.d. $v = V_m \cos \omega t$ is applied to the terminals of the coil. If the resistance of the winding is negligibly small, obtain an expression for the magnetic induction B as a function of the time and discuss the implications of this expression.

4. Show how the magnetising current wave of a transformer may be derived from the hysteresis loop.

5. In the accompanying diagram is shown an oscillatory circuit to the terminals of which is applied a d.c. pressure of V volts. Find the logarithmic decrement and the amplitude of the undamped current.

6. What is meant by the "Q" value of an alternating current circuit?



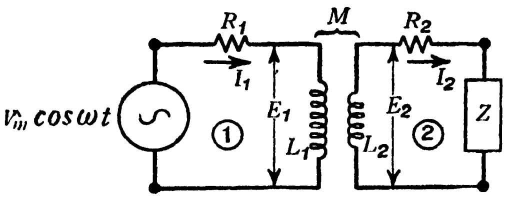


Above—Question 2.
Right—Question 5.

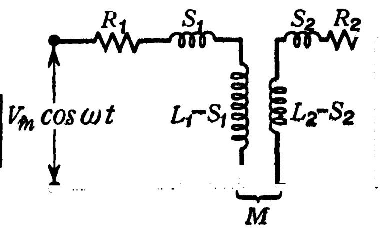
7. A condenser is connected in parallel with a series arrangement of resistance and inductance. This parallel system is then connected to a source of a.c. supply. Discuss the resonance conditions of the system.

8. The two circuits 1 and 2 shown in the accompanying diagram are electromagnetically coupled, the coefficient of mutual inductation being M henry, and the coefficients of self-induction respectively L_1 and L_2 henry, and the resistances R_1 and R_2. If it be assumed that there is no magnetic leakage, obtain an expression for the transformation ratio.

9. Two coupled circuits shown in the accompanying diagram, the primary circuit of which is connected to an a.c. supply pressure $V_m \cos \omega t$. Obtain (i) the two simultaneous equations which define the current in each of the coupled circuits, and (ii) the simple series circuit which is electromagnetically equivalent to the coupled system of circuits.



Question 8.



Question 9.

10. A trip coil for a circuit-breaker has an inductance of 0·1 mH and a very small capacitance, e.g. the leading-in conductor from the circuit-breaker the capacitance of which might be 10^{-4} μF. Find the natural frequency of this coil.

11. Find the natural frequency of oscillation of a protective choke coil of inductance $L = 5$ mH which is connected in front of a cable of capacitance $C = 2\mu F$.

12. Experiments show that when an arc is formed in oil, oil-gas is generated, the volume of which is about 50 c.cm. per kW. sec. at normal temperature and pressure. The relationship between the pressure, volume, and absolute temperature of a gas is $\frac{PV}{\Theta}$ = constant. If on opening a switch to interrupt $0{\cdot}5 \times 10^6$ kVA., the generated energy of the arc is 400 kW. sec. and the temperature of the oil-gas generated is 5,000° C., what will be the volume of gas generated if the gas pressure is 25 atmospheres?

13. Explain the action of the "shaded pole" motor—also known as the "Ferraris Disc".

14. Show how the short-circuit impedance of a transformer may be measured by means of a bridge test.

15. Compare the characteristic performance of a "polarised" and a "non-polarised" relay.

16. Compare the action of a magnetic relay operated by alternating current with that of a relay operated by direct current.

TEST PAPER ON CHAPTER X

1. Two circuits each comprising a series arrangement of capacitance, inductance, and resistance are coupled by mutual induction. Find the simple series circuit which is electromagnetically equivalent to this coupled circuit.

2. Find the magnitudes of the primary current and of the secondary current of the mutually coupled circuits of Example 1.

3. The two coupled circuits of Example 1 are separately tuned to the same resonance frequency, the individual constants of the circuits being as follows :

$$\left.\begin{array}{l} L_1 = 10^{-4}\ H.,\ C_1 = 5 \times 10^{-9}\ F.:\ R_1 = 10\ \Omega \\ L_2 = 1{\cdot}5 \times 10^{-4}\ H.,\ C_2 = 3{\cdot}3 \times 10^{-9}\ F.:\ R_2 = 12\ \Omega \end{array}\right\}\omega_0 = 1{\cdot}4 \times 10^6.$$

Find the current in the secondary circuit and obtain the value of the coupling factor for which this current reaches a maximum value.

4. Two coupled circuits, each tuned separately to the same resonance frequency, have the same numerical value as those given in Example 3 :

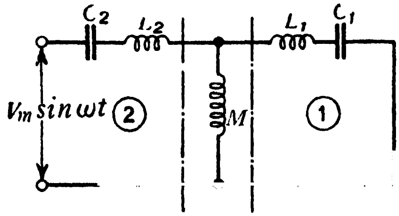
(i) Find the coupling factor which will give the maximum current in the secondary winding and plot the current-frequency relationships for each of the two circuits.

(ii) Taking a coupling factor equal to one-fifth of the critical value, plot the current-frequency relationships for each of the two circuits.

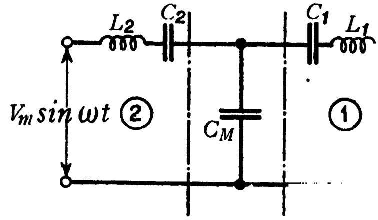
(iii) Taking a coupling factor equal to five times the critical value, plot the current frequency relationships for each of the two circuits.

5. Find the resonance frequency of two circuits which are coupled by mutual induction, each of the circuits being separately tuned to the same resonance frequency and the resistance of each circuit being relatively small.

6. Find the input impedance and the resonance frequencies of the compound circuit shown in the accompanying diagram when $L_1 = L_2$ *and* $C_1 = C_2$.



Question 6.



Question 7.

7. Find the input impedance and the resonance frequencies of the circuit system shown in the accompanying diagram when $C_1 = C_2$ *and* $L_1 = L_2$.

8. Explain the action of a simple valve oscillator.

9. A two-pole short-circuit occurs at a power station which is operating in parallel with a number of other power stations on 10 kV. bus bars. The short-circuit current reaches a peak value of $\sqrt{2} \times 66{,}100 = 93{,}500$ *amperes. If the length of conductor between two consecutive supports is* $\lambda = 160$ *cm. and the mean distance between the*

conductors is $D = 40$ *cm., find the mechanical force which will be developed between them due to the short circuit.*

10. The natural frequency of oscillation of a stretched conductor supported at each end is

$$n = 112\sqrt{\frac{EI}{gl^4}} \text{ oscillations per second,}$$

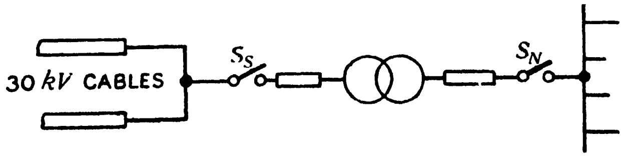
where g *kg. per centimetre length is the weight of the conductor,*

E *is Young's modulus and for copper is* $1{\cdot}15 \times 10^6$ *kg. per centimetre,*

J *is the moment of inertia of the cross-section of the conductor in* cm.^{4} *units.*

l *cm. is the free length of the stretched conductor.*

If the copper bus bars of Example 9 have a cross-section of 10×1 cm.^{2} *and the distance between two supports is 160 cm., find (i) the natural frequency of oscillation, and (ii) the stress in the conductors due to the force of* $P = 720$ *kg. as calculated for the short-circuit conditions of Example 9.*



Question 11.

11. A transformer connects a 30-kV. supply cable with a 6-kV. network as shown in the accompanying diagram, a circuit-breaker being installed on each side of the transformer. The three-phase 30/6-kV. transformer is rated at 15 MVA., the leakage reactance is 7·7 per cent., no-load current (i.e. magnetising current) is 7·1 per cent. of the rated full-load current. The low-tension switch opens by means of an arc and it is required to find the oscillation frequencies to which it will give rise.

12. Three simple series circuits are respectively defined by the following data, viz. :

(i) $C = 200\mu F$: $L = 0{\cdot}1$ henry : $R = 5$ ohms

(ii) $C = 200\mu F$: $L = 0{\cdot}1$ henry : $R = 0$,,

(iii) $C = 40\mu F$: $L = 0{\cdot}5$ henry : $R = 0$,,

Draw the graphs showing the relationship between current and supply frequency for the following arrangements of these circuits :

I. The simple series circuit 1 is alone connected across the supply terminals.

II. The circuits 1 and 2 are connected in series across the supply terminals.

III. The circuits 1 and 3 are connected in series across the supply terminals.

In each case the peak value of the supply pressure is 1,000 volts.

13. Show how the alternating current of constant frequency may be generated by means of a mechanical oscillating system.

14. Give a short account of the vibrating reed rectifier.

15. Show how a quartz crystal may be used to stabilize the frequency of a valve generator.

TEST PAPER ON CHAPTER XI

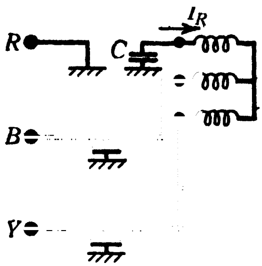
1. Explain the principle of action of some form of relay by means of which the power factor of the current which is being transmitted from one station to another may be automatically controlled.

2. A symmetrical three-phase supply of which the line pressure is 380 volts feeds a mesh connected load as follows :

Between lines	*1 and 2 :*	*19 kVA.*	*at* $\cos\phi = 0{\cdot}5$	*lagging.*
,, ,,	2 ,, 3 :	*30 kVA.*	*at* $\cos\phi = 0{\cdot}8$	*lagging.*
,, ,,	3 ,, 1 :	*10 kVA.*	*at* $\cos\phi = 0{\cdot}9$	*leading.*

Draw the vector diagram for the mesh loads and derive the vector diagram for the line currents.

3. Convert the mesh load of Example 2 into the equivalent star load. Find the pressure of the load star-point as referred to the star-point of the supply system and derive the vector diagram for the line currents.



Question 6.

4. An unsymmetrical three-phase pressure system is connected to a star arrangement of reactances, all three reactances being of equal magnitude. Show that the star-point pressure of the reactance will lie at the centre of gravity of the supply line pressure triangle.

5. It is required to obtain a current of variable magnitude and in phase with one of the phase pressures. Show how this may be done and derive the magnitude so obtained (i) by Thévenin's Theorem, (ii) by means of vector diagram analysis.

6. One conductor of a three-phase overhead transmission system breaks and one of the free ends falls to earth ; the receiver end of this line is connected to a three-phase transformer, as shown in the accompanying diagram. Draw the vector diagrams for the three-phase current and pressure at the consumer's terminals.

7. Show how the total reactive volt-amperes denoted by the symbol var taken by a balanced three-phase load, may be measured by means of two wattmeters.

TEST PAPER ON CHAPTER XII

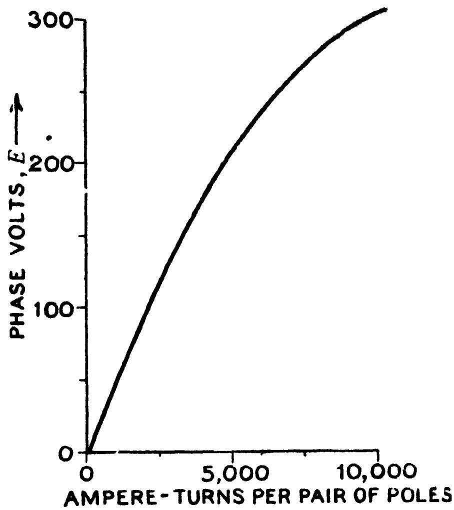
1. A transformer has a 10 per cent. impedance and a short-circuit power factor of 0·4. Find the percentage pressure drop when the transformer is operating at its full rated output and (i) at unity power factor, (ii) at a power factor of 0·8 lagging.

2. Explain the general principle of operation of the synchronous motor.

3. Draw the vector diagram for a three-phase synchronous machine which is running in parallel with a large supply system and show that the locus of the extremity of the current vector is a circle.

4. Show how the torque and power of a synchronous motor may be derived from the circle diagram.

5. A synchronous motor is supplied at a constant pressure of 200 volts per phase and a frequency of 50. The reactance per phase is 8 ohms and the motor has four poles. Find the necessary excitation if the motor develops 10 horse-power at unity power factor and draw the torque-Θ curve. What excitation will be necessary if the motor is to develop 10 horse-power at a power factor of 0·9 leading ? The open-circuit characteristic, that is the relationship between the induced e.m.f. E per phase and the excitation ampere-turns per pair of poles at constant speed is given in the accompanying diagram.



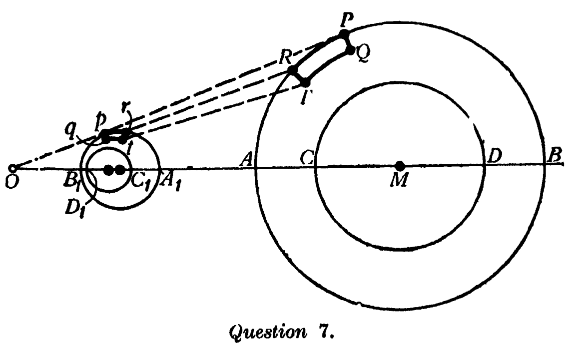
Question 5.

6. Obtain an expression for the natural frequency of oscillation of a three-phase synchronous machine.

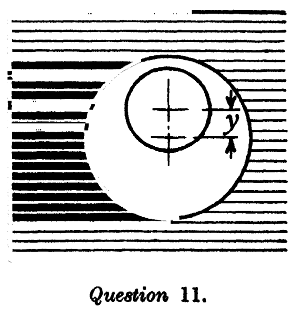
7. Show that, if there is a pair of curves such as the circles MCD and MAB in the accompanying diagram, and if these curves be inverted with reference to a point O, the ratio $\frac{RP}{RT}$ for an elementary area PQTR of the first pair of curves will be the same as the ratio $\frac{rp}{rt}$ for the area of inversion in the second pair of curves.

8. A long, straight horizontal wire of 0·8 cm. radius is supported within a cylindrical metal sheath of radius 3 cm. so that the axes are parallel and 1 cm. apart. Find the capacitance of the electrostatic field between the two conductors by the method of transforming this eccentric system into an equivalent system of a wire 0·8 cm. radius set at a definite height h above a flat metal plate parallel to the conductor.

9. Find the capacitance of the system of two eccentric cylindrical conductors defined



Question 7.



Question 11.

in Example 8 by the method of transforming the eccentric system into an equivalent system of two concentric conductors.

10. Apply the results obtained for the capacitance between two eccentric cylindrical conductors to determine the magnetic reluctance of the space between a cylindrical iron rod which is supported in a cylindrical tunnelled hole in a laminated iron block, the axes of the rod and the tunnelled hole being parallel and at a definite distance apart.

11. Obtain the expression for the magnetic reluctance of the air-space between the two eccentric cylindrical surfaces of the accompanying diagram, by inverting the two eccentric surfaces into two equivalent co-axial surfaces.

12. By means of the method of symmetrical component analysis, derive an expression for the current which will flow when one phase of a generator is short-circuited to the star point and the other two phases are open.

TEST PAPER ON CHAPTER XIII

1. The primary winding of a single-phase transformer is connected to a pressure wave of sinusoidal form and frequency f hz. Neglecting the resistance of the winding, obtain an expression for the induced back e.m.f.

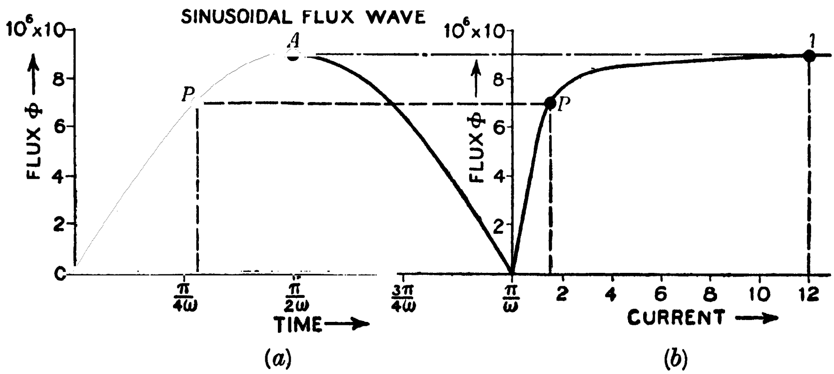
2. In the accompanying diagram (a) is shown a sine wave form of flux of peak value 9 = 10^6 c.g.s. lines, as a function of the circular frequency $\omega = 2\pi f$, and in the diagram (b) is shown the relationship between the magnetising current and the flux. Obtain the wave form of the magnetising current.

3. Analyse, as far as the seventh harmonic, the current wave form obtained in Example 2.

4. Currents of different frequencies are flowing in a circuit as follows:

Frequency : hertz	*0 (i.e., d.c.)*	*50*	*150*	*750*
Current in r.m.s. amperes	*10*	*10*	*5*	*3*

Find the r.m.s. value of the current indicated by an a.c. ammeter in series with the circuit.



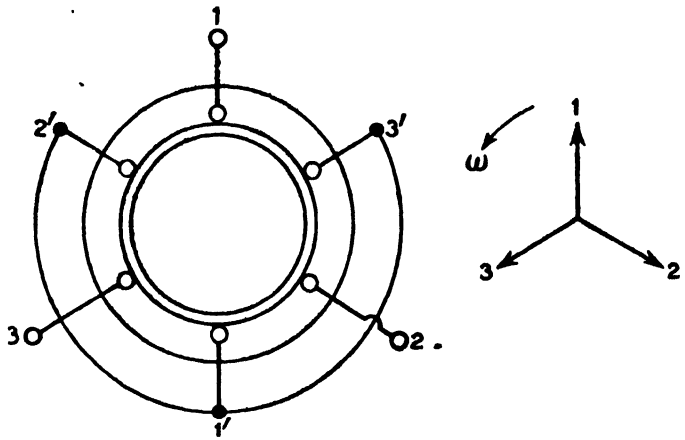
Question 2.

5. Since a third harmonic cannot flow in a three-phase supply system with an insulated star point, state what means are available for enabling the requisite third harmonic of the current wave to flow in the transformer winding.

6. What will be the effect on the harmonics of the magnetising current wave of a transformer if a third harmonic appears in the flux wave?

7. Draw a diagram showing the flux density distribution in the air-gap of a single-phase concentrated two-pole winding. The radial length of the air-gap is δ cm. and the air-gap surfaces of both stator and rotor cores are smooth cylindrical surfaces. Show in the diagram the fundamental sine wave and the third harmonic of flux density distribution in the air-gap.

8. A three-phase two-pole concentrated stator winding is shown in the accompanying diagram, together with the time vector diagram of the three-phase alternating current which is supplied to the winding. Construct a diagram showing the three fundamental



Question 8.

waves of magnetic flux density with the time as the abscissa axis, as well as the fifth harmonic waves associated with each of these fundamental waves. From this diagram derive the direction of rotation of the 10-pole magnetic field as referred to that of the 2-pole field. Find the speed of rotation of the 10-pole field in terms of the speed of rotation of the 2-pole field.

9. If a balanced three-phase star-connected inductive system is connected in parallel with a balanced three-phase mesh-connected inductive system, show that, when the cores of the inductances are operating at saturation values of the flux densities, the consequent fifth and seventh harmonics in the individual phases of each system can be eliminated from the line currents.

10. Describe some method by means of which the magnitudes of the individual harmonics in a given wave form of e.m.f. may be measured.

11. What is meant by the "frequency spectrum" of a non-sinusoidal wave form?

TEST PAPER ON CHAPTER XIV

1. *Find the diameter of the iron rod which will satisfy the following conditions, viz.:*

$$\frac{\text{Resistance to a.c.}}{\text{Resistance to d.c.}} = 6 : \text{ frequency } f = 2{,}000 \text{ hz.}:$$

$$\rho = 12 \times 10^{-6} \text{ ohm per cm./cm.}^2 ; \text{ permeability } \mu = 500.$$

2. *At what frequency will a straight copper wire, 1 mm. diameter, have a resistance 4 times its d.c. value?*

3. *An iron wire carries a.c. at a frequency of 5,000 hz. Find what the diameter will be if the resistance to a.c. is 4 times the d.c. value. The specific resistance is* $\rho = 12 \times 10^{-6}$ *ohm/cm./cm.*^{2}

4. *Find the current distribution over the section of a straight copper wire, 1 mm. diameter, for a frequency of* $0{\cdot}25 \times 10^6$ *hz., if* $\rho = 1{\cdot}77 \times 10^{-6}$ *ohm/cm./cm.*^{2} *Draw the vector diagram for the current distribution.*

5. *Alternating current at 5,000 frequency is passed through an iron lamination to which the following numerical data apply:*

$$\text{thickness } 2\Delta = 0{\cdot}5 \text{ mm.}: \ \mu = 2{,}000 : \ \rho = 10 \times 10^{-6} \text{ ohm/cm./cm.}^2$$

Plot a curve of current distribution for the following moments:

(*i*) *The current in the surface skin has its maximum value.*

(*ii*) *The current in the surface skin is zero.*

6. *Find the "penetration depth" of alternating magnetic flux in laminated conductors as defined by the following data:*

	ρ Ω/cm./cm.^{2}	μ
Dynamo stampings	$1^8\ 10^{-5}$	2,000
Permalloy	$0{\cdot}2 \times 10^{-4}$	10,000
Cast steel	$0{\cdot}2 \times 10^{-4}$	1,000

7. *Derive an expression for* σ, *the depth of penetration in a straight wire which is carrying high-frequency alternating current.*

8. *Give an account of some practical application of "skin-effect" at low frequencies.*

9. *Derive the conditions for which the damping of an iron-core, alternating-current inductance coil becomes a minimum.*

10. *Give an account of the characteristic features of a "pot" type oscillator or resonator for ultra-high-frequency systems.*

11. *A copper oscillation-pot of the form described in Example 10 carries an oscillatory current of 1 metre wavelength. Find the effective resistance and the power loss per square centimetre of the cylindrical wall. The specific resistance of copper is* $\rho = 1{\cdot}77 \times 10^{-6}$ *ohm per cm. cube.*

12. *Discuss the significance of the oscillation pot with respect to its "resonance quality"* $Q_{res.}$ *and its "resonance resistance"* $R_{res.}$.

TEST PAPER ON CHAPTER XV

1. ***If a three-phase transmission line is "compensated" by connecting a uniformly distributed reactance between each line and the star point, find what reactive volt-amperes will be taken by this reactance for any given value of the load at the consumer's terminals.***

2. ***If the line pressure at each end of an unloaded three-phase transmission line is***

maintained at the constant value of 132 kV., find: (i) What capacitance current must be supplied to each end of the line; (ii) what pressure rise will develop at the midpoint of the line. The surge-impedance is $Z_0 = 380$ ohms per phase, the supply frequency is 50 and the length of the line is $2s = 200$ km.

3. Referring to Example 2, one method by means of which the necessary capacitance current can be supplied to the receiver's terminals is to connect at that end of the line an inductance of value L_B henry, such that it will draw a current of 20·7 amperes from that end of the line. Explain how it is that an inductance can be said to supply a capacitance current to the line in this way.

4. What is the "Lecher Line", and what is its purpose?

5. Find what reactive kVA. per kilometre must be supplied to a three-phase overhead transmission line and show the results as a function of the line pressure for the following loads: (i) No load, (ii) 0·75 times the natural load, (iii) 1·25 times the natural load. The line constants are as follows:

$$\text{Capacitance } C_0 = 0{\cdot}0096 \times 10^{-6} \text{ farad per kilometre;}$$
$$\text{Inductance } L_0 = 0{\cdot}00127 \text{ henry per kilometre.}$$

The supply frequency is 50 hz.

6. Draw the vector diagram for one phase of a three-phase transmission line by making use of the equivalent 4-terminal π unit. From this diagram derive the relationship which defines the natural power of the line.

7. Show how the resistance of a transmission line may be taken into account in the equivalent 4-terminal π unit.

8. Explain how an arc to earth which starts on an overhead line may be automatically suppressed (Petersen coil).

9. Define the "bel": "decibel": "neper". From these definitions derive the numerical relationships between the three quantities.

10. A resistance of R ohms connects an underground cable of surge impedance $Z_{02} = 50$ ohms with an overhead line of surge impedance $Z_{01} = 500$ ohms. The resistance is of such a magnitude that it gives maximum efficiency of absorption of energy. Construct the graphical representations of the current and pressure for the following conditions, viz.: (i) When the surge passes from the overhead line into the cable; (ii) when the surge passes from the cable into the overhead line. Contrast these results with those obtained when the resistance $R = 0$.

11. Show how a 4-pole unit system may be used as a filter to block the passage of high-frequency currents.

12. Show how a 4-pole unit system may be used to block the passage of low-frequency currents.

13. Show how a 4-pole T system may be used as a filter to pass a selected band of frequencies.

14. A single-phase overhead line of bronze conductors 3 mm. diameter is used for high-frequency transmission. Calculate the damping factor as a function of the frequency.

15. How would you measure the damping factor of a high-frequency cable?

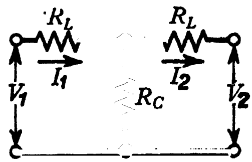
16. Discuss the significance of the damping factor as regards the question of the installation of amplifiers for the operation of a high-frequency transmission line.

17. For the transmission cable such as that shown in Fig. 29, Example 15, find what power is dissipated per metre length of the cable when the consumer is receiving 1 watt.

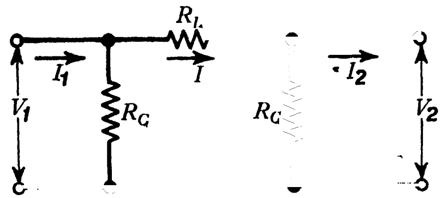
18. Show how: (1) the damping of a cable may be measured, and (2) how the damping of cable may be derived from the Q factor and the dielectric loss factor of the cable, viz., $\tan \delta$.

19. What is meant by the "attentuation" of a transmission line? Show by means of a numerical example the effect of the attentuation of the power transmitted.

20. A symmetrical 4-pole T system of non-inductive resistances is shown in the accompanying diagram. Obtain the two simultaneous equations which relate the input pressure and current respectively, with the output pressure and current.



Question 20.



Question 21.

21. A symmetrical 4-pole π system of non-inductive resistances is shown in the accompanying diagram. Obtain the two simultaneous equations relating the input pressure and current respectively with the output pressure and current.

TEST PAPER ON CHAPTER XVI

1. Show that when a quarter-wave transmission line of surge impedance Z_0 is connected to a resistance R_E ohms at the receiver's end, the input impedance is $\frac{Z_0^2}{R_E}$.

2. What methods are used for connecting a radio sending station to the antenna? What are the characteristic features of each method?

3. If a quarter-wave transmission line is open-circuited at the receiver's end, find what the input impedance will be if the supply frequency is slightly higher than the resonance value.

4. If a quarter-wave transmission line is short-circuited at the receiver's end, find what will be the input impedance when the frequency is slightly higher than the resonance value.

5. Compare the characteristics of the normal high-frequency cable and the co-axial cable.

6. Prove that the surge impedance of open space is 376·7 ohms.

7. Show how a transmission line may be matched to a load by means of a transformer.

8. What considerations of the damping characteristics of telephone lines govern the suitable spacing of the amplifiers?

9. In the development of the Maxwell-Hertz equations it was assumed that the current density was uniform throughout the dipole. For practical antennae, however, this assumption is not valid. Find the height of an antenna having uniformly distributed current density, which will be equivalent to the actual antenna having sinusoidally distributed current density.

10. Compare the characteristic features of high-frequency cables and of high-frequency radio for alternating-current transmission.

11. Prove that if a parallel resonance circuit of inductance L henry, capacitance C farad and non-inductive series resistance of R ohms is placed in each of the two parallel branches, then the input impedance of the system will be independent of the frequency if $R = \sqrt{\frac{L}{C}}$ (television aerial of the Empire State Building, New York).

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